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Research Article

Statistical Soft Wijsman Convergence

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ABSTRACT

The concept of convergence is a fundamental tool for building or understanding a mathematical structure. In particular, many applied areas of mathematics require the analysis of sets or set-based approximations. A notable example is Wijsman convergence, a type of set convergence defined as the distance from a point to a set as determined by a metric function. Another important concept in our study is soft set theory, which generalizes classical set theory and provides an effective approach to addressing uncertainties in a parametric manner. Building on this foundation, we introduce the concepts of statistical and lacunary statistical convergence in the sense of Wijsman for sequences of closed sets within the framework of soft metric spaces. We establish fundamental properties associated with these types of convergence and explore their interrelationships, resulting in several inclusion results. Furthermore, we utilize a generalized version of the natural density function, referred to as the ϕ-weighted density function, to strengthen our findings.

1. Introduction

Classical set theory is based on precise, binary definitions, determining whether an element belongs to a set or not. However, this approach is inadequate for the complexities of real-world situations where incomplete or ambiguous information is common. Understanding such uncertainty is essential for analyzing various events and systems. Soft sets provide the necessary flexibility to model these scenarios effectively. Soft set theory, introduced by Molodtsov [1], generalizes classical set theory and addresses uncertainty through a parametric approach. Although there is a substantial literature on soft sets and soft topological spaces, this study will focus on soft metric spaces. Das and Samanta [2, 3] first introduced the concept of extending metrics to soft set theory by defining a metric on the set of soft points. Their research established various topological concepts—such as soft open sets, soft closed sets, soft limit points, soft closure, and soft completeness—within the framework of soft metrics, while also addressing properties common in classical metric theory, resulting in significant findings. Additionally, Alcantud et al. [4] have recently provided a comprehensive and in-depth survey of soft set theory.

The concept of convergence, which is another key element of our study, has several different forms in both theoretical and applied fields. These types vary depending on the mathematical structure and the mathematical objects under investigation. Consequently, diverse convergence types deepen mathematical understanding by providing alternative approaches to various problems. In the literature, there exist different types of convergence for sequences of sets defined by various tools. One such type is the Wijsman convergence, initially defined by R.A. Wijsman for the convergence of convex sequences of sets in Euclidean spaces [5, 6].

Let (X, d) be a metric space, and let $\{A_k\}$ be a sequence of closed sets, with A being a non-empty closed set in X. The sequence $\{A_k\}$ is said to converge to A in the Wijsman sense if, for every $x \in X$, the sequence $(d(x, A_k))$ converges in the sense of d to $d(x, A)$. Here, $d(x, A)$ denotes the distance from the point x to the set A. As can be seen, Wijsman convergence is characterized by the pointwise convergence of the distance function. Several generalizations of this type of convergence, along with their applications to different mathematical structures and comparisons to other convergence types, have been studied (see [7-10]).

Another type of convergence is statistical convergence, which was initially defined for real sequences and is a generalization of topological convergence. Over time, it has become one of the most extensively studied areas, finding applications in various fields of mathematics. Statistical convergence was first introduced independently by Fast [11] and Steinhaus [12]. Among the extensive and comprehensive literature on this topic, foundational works on statistical convergence include those by Schoenberg [13], Maddox [14], Connor [15], Fridy [16], and Šalát [17]. Di Maio and Kočinać [18] extended the concept of topological convergence to statistical convergence in topological spaces.

Variations in density functions and sequences across different mathematical structures have led to the formulation of various generalizations of statistical convergence. A well-known generalization is lacunary statistical convergence, which was initially defined by Fridy and Orhan [19] using lacunary sequences. Over the years, lacunary statistical convergence has been examined from multiple perspectives, and extensions of this concept have been developed utilizing different density functions. Pehlivan and Fisher [20] introduced the notion of lacunary strong convergence by employing sequences of modulus functions within a Banach space. Additionally, Şengül and Et [21], as well as Bhardwaj and Dhawan [22], explored α-order lacunary statistical convergence and α-order strongly summable sequences.

Statistical convergence for sequences of sets was initially defined by Nuray and Rhoades [23]. Additionally, Ulusu and Nuray [24] introduced the notion of Wijsman lacunary statistical convergence for sequences of sets. Although there have been studies on statistical and lacunary statistical convergence of sequences of sets in the Wijsman sense, to the best of our knowledge, there is no existing research on Wijsman statistical convergence within the framework of soft set theory. Motivated by this gap, the present study aims to address this deficiency in the literature and provide a foundation for subsequent studies.

The structure of this paper is outlined as follows: Section 2 offers a concise overview of the key concepts and results associated with soft set theory and statistical convergence. Section 3, which details the primary findings, is divided into two subsections. The first subsection defines statistical soft Wijsman convergence and discusses its general properties. The second subsection introduces the concepts of lacunary statistical soft Wijsman convergence, highlighting their connections to statistical soft Wijsman convergence. The final section summarizes the main contributions and provides some recommendations for future research directions.

2. General Methods

As a preliminary remark for the reader's convenience, this section presents the essential definitions and results that will be utilized in soft set theory and statistical convergence.

Throughout this work, we denote the universe set by X and the set of all parameters by E.

Definition 2.1. [1] A soft set over X is denoted as F^E, where $F: E \to P(X)$ is a mapping. The set of all soft sets over X is represented by $SS(X)$.

In other words, the soft set is a parametrized family of subsets of X. For $e \in E$, every set $F(e)$ may be considered as the set of e-approximate elements of the soft set F^E. Considering $F(e) = \emptyset$ for every $e \in E \setminus A$, F^E can be written instead of F^A for a subset A of E .

Based on Molodtsov's definition [1], several key operations in soft set theory that are relevant to this study are outlined [2-4]. By selecting a specific function F that characterizes the soft set, we can derive various types of soft sets commonly discussed in the literature.

i. A soft set F^E is referred to as a *singleton soft set* if $F(g)$ is a singleton for every $g \in E$.

ii. More specifically, any singleton soft set F^E is called a *soft element*. If, for each $g \in E$, $F(g) = \{x_g\}$, this soft element is denoted as $\tilde{x}$. The set of all soft elements over X is denoted by $SE(X)$.

iii. A soft set F^E is termed a *soft point*, represented as P_e^x, if there exists an $g \in E$ such that $F(g) = \{x\}$ for some $x \in X$ and $F(e') = \varnothing$ for all $e' \in E - \{g\}$. The set of all soft points over X is denoted by $SP(X)$. Notably, every soft set can be expressed as a union of soft points.

iv. A soft set F^E is defined as a *soft real set* if F is a mapping given by $F: E \to B(\mathbb{R})$, where $B(\mathbb{R})$ is the set of all non-empty bounded subsets of $\mathbb{R}$.

v. If a soft real set F^E is a soft element, it is termed a *soft real number*. If, for each $g \in E$, there exists $r_g \in \mathbb{R}$ such that $F(g) = \{r_g\}$, this soft real number is denoted by $\tilde{r}$. The set of all soft real numbers is represented by $\mathbb{R}^E$.

vi. If $F(g) = \{r\}$ for every $g \in E$, then F^E is referred to as a *constant soft real number*, denoted by $\overline{r}$.

vii. For $\tilde{r}, \tilde{s} \in \mathbb{R}^E$, we express $\tilde{r} \leq_s \tilde{s}$ if and only if $r_g \leq s_g$ for every $g \in E$. Thus, $\leq$ establishes a partial order relation on $\mathbb{R}^E$. If $\tilde{r} \geq_s \overline{0}$, meaning $r_g \geq 0$ for all $g \in E$, then $\tilde{r}$ is classified as a non-negative soft real number. The set of all non-negative soft real numbers is denoted by $\mathbb{R}_+^E$. Similarly, we denote $\tilde{r} <_s \tilde{s}$ if $r_g < s_g$ for every $g \in E$ [2]. Consequently, we can define the supremum and infimum of any set in $\mathbb{R}_+^E(E)$, analogous to classical definitions.

The extension of the concept of metric to soft set theory was initially proposed by Das and Samanta [2, 3] through the definition of a metric on the set of soft points. In these works, various topological concepts—including soft open sets, soft closed sets, soft limit points, soft closure, and soft completeness—were defined in the context of soft metrics, in addition to the properties typically recognized in classical metric theory, leading to notable results.

Definition 2.2. [2] A function $\mathrm{d}: \mathrm{SP(X)} \times \mathrm{SP(X)} \to \mathbb{R}_+^{\mathrm{E}}$ is defined as a soft metric if it satisfies the following criteria. For all $\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}, \mathrm{P}_{\mathrm{e}_3}^{\mathrm{z}} \in \mathrm{SP(X)}$:

sm1. Non-negativity: $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}) \geq_{\mathrm{s}} \overline{0}$ (i.e., $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}})(\mathrm{e}) \geq 0$ for every $\mathrm{e} \in \mathrm{E}$).

sm2. Identity of Indiscernibles: $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}) =_{\mathrm{s}} \overline{0}$ if and only if $\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}} = \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}$ (i.e., $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}})(\mathrm{e}) = 0$ for every $\mathrm{e} \in \mathrm{E}$ if and only if $\mathrm{e}_1 = \mathrm{e}_2$ and $\mathrm{x} = \mathrm{y}$).

sm3. Symmetry: $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}) =_{\mathrm{s}} \mathrm{d}(\mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}, \mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}})$ (i.e., $\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}})(\mathrm{e}) = \mathrm{d}(\mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}, \mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}})(\mathrm{e})$ for every $\mathrm{e} \in \mathrm{E}$).

sm4. Triangle Inequality:

$\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}) \leq_{\mathrm{s}} \mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_3}^{\mathrm{z}}) + \mathrm{d}(\mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}, \mathrm{P}_{\mathrm{e}_3}^{\mathrm{z}})$ (i.e.,

$\mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}})(\mathrm{e}) \leq \mathrm{d}(\mathrm{P}_{\mathrm{e}_1}^{\mathrm{x}}, \mathrm{P}_{\mathrm{e}_3}^{\mathrm{z}})(\mathrm{e}) + \mathrm{d}(\mathrm{P}_{\mathrm{e}_2}^{\mathrm{y}}, \mathrm{P}_{\mathrm{e}_3}^{\mathrm{z}})(\mathrm{e})$ for every $\mathrm{e} \in \mathrm{E}$).

In this framework, the triplet $(\mathrm{X}, \mathrm{d}, \mathrm{E})$ is referred to as a soft metric space.

Wijsman convergence is characterized by the distance from a point to a set as determined by a metric function [5, 6]. Therefore, the distance of soft points from a soft set is a fundamental tool in this framework.

Definition 2.3. [2] Let (X, d) be a soft metric space and $\mathrm{F}^{\mathrm{E}}, \mathrm{G}^{\mathrm{E}} \in \mathrm{SS(X)}$. The distance of any soft point $\mathrm{P}_{\mathrm{e}}^{\mathrm{x}}$ to the soft set F^{E} is defined as the non-negative soft real number $\mathrm{D}(\mathrm{P}_{\mathrm{e}}^{\mathrm{x}}, \mathrm{F}^{\mathrm{E}})$, which is expressed as follows:

$$\mathrm{D}(\mathrm{P}_{\mathrm{e}}^{\mathrm{x}}, \mathrm{F}^{\mathrm{E}})(\mathrm{g}) = \inf\{\mathrm{d}(\mathrm{P}_{\mathrm{e}}^{\mathrm{x}}, \mathrm{P}_{\mathrm{h}}^{\mathrm{y}})(\mathrm{g}): \mathrm{P}_{\mathrm{h}}^{\mathrm{y}} \in \mathrm{SP}(\mathrm{F}^{\mathrm{E}})\} \quad \text{for every} \quad \mathrm{g} \in \mathrm{E}.$$

In particular, if P_e^x belongs to F^E, then $D(P_e^x, F^E) = \overline{0}$.

Definition 2.4. [2] Let (X, d) be a soft metric space.

i. The soft open ball centered at $P_f^a \in SP(X)$ of radius $\tilde{r} \in \mathbb{R}_+^E$ is the set

$$B(P_f^a, \tilde{r}) = \{P_e^x \in SP(X): d(P_f^a, P_e^x) <_s \tilde{r}\}.$$

ii. A soft point P_e^x is defined as a soft limit point of $F^E \in SS(X)$ if every soft open ball centered at P_e^x contains at least one soft point of F^E that is distinct from P_e^x.

iii. The soft set formed by the set of all soft points and soft limit points of F^E in the soft metric space (X, d) is referred to as the soft closure of F^E in (X, d). This set is denoted by $\overline{F^E}$.

iv. The soft set F^E is called closed provided that $F^E = \overline{F^E}$. The collection $\overline{SS_d(X)}$ represents all soft closed sets within the soft metric space (X, d).

The statistical convergence primarily depends on the natural (asymptotic) density of subsets of $\mathbb{N}$, defined by $\delta(K) = \lim_{n\to\infty} n^{-1}|\{k \le n: k \in K\}|$, where vertical bars denote the cardinality of the set K up to n [16].

A sequence of numbers (x_k) is said to be statistical convergent to a number L if, for every positive ε, $\delta(\{k \le n: |x_k - L| \ge \varepsilon\}) = 0$ [11].
Balcerzak et al. [25] proposed a generalized form of natural density by considering a weight function. The function $\phi: \mathbb{N} \to [0, \infty)$ is said to be weight function whenever it satisfies the following properties:

$$\lim_{n\to\infty} \phi(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{n}{\phi(n)} \neq 0.$$

Henceforth, the set of all weight functions that satisfy properties given above is denoted by $\mathcal{G}$.

Definition 2.5. [25] The density of a set $A \subseteq \mathbb{N}^+$ with respect to a weight function $\phi \in \mathcal{G}$ is defined by the following limit

$$\delta_\phi(A) := \lim_{n\to\infty} \frac{1}{\phi(n)} |\{k \in A: k \le n\}|.$$

whenever it exists. This type density is abbreviated as ϕ −weight density.

3. Results and Discussion

The results obtained in this section will be served in two parts.

3.1. Statistical Soft Wijsman Convergence

In this section, the concept of statistical convergence of sequences of closed soft sets is defined, and some properties are presented.
The consept of sof Wijsman convergence was given in [26] as follows.

Definition 3.1.1. [26] Let (X, d) be a soft metric space and a sequence of soft closed sets $\{A_k^E\}$ and A^E be non-empty soft closed sets in X_s. We say that the sequence $\{A_k^E\}$ is soft Wijsman convergent to A^E if, for every $g \in E$ and for all $P_e^x \in SP(X)$,

$$\lim_{k\to\infty} D(P_e^x, A_k^E)(g) = D(P_e^x, A^E)(g).$$

This convergence is denoted by $A_k^E \xrightarrow{sW} A^E$.
Naturally, the concept of soft Wijsman convergence, in the sense of statistical convergence, can be given as follows.

Definition 3.1.2. Let (X, d) be a soft metric space, and consider a sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ and $A^E \in \overline{SS_d(X)}$. We define the sequence $\{A_k^E\}$ to be ϕ-statistical soft Wijsman convergent (abbreviated as ssW_ϕ-convergent) to A^E (denoted $A_k^E \xrightarrow{ssW_\phi} A^E$) if, for every $\tilde{\varepsilon} >_s \overline{0}$ and for each $P_e^x \in SP(X)$, the following condition holds:
$\delta_\phi(\{n \in \mathbb{N} | d(D(P_e^x, A_k^E), D(P_e^x, A^E)) \ge_s \tilde{\varepsilon}\}) = 0$.
In other words, for every $g \in E$,

$$\lim_{n\to\infty}\frac{1}{\phi(n)}\left|\{k\le n\colon d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\ge\varepsilon_g\}\right|=0.$$

Remark 3.1.3. Accordingly, the necessary and sufficient condition for $A_k^E \xrightarrow{ssW_\phi} A^E$ is that for every $P_e^x \in SP(X)$ and for each $g \in E$, the sequence $\{D(P_e^x, A_k^E)(g)\}$ converges ϕ-statistically to the point $D(P_e^x, A^E)(g)$ in $\mathbb{R}$.

Theorem 3.1.4. For any soft metric space (X, d), soft Wijsman convergence implies ϕ-statistical soft Wijsman convergence. Specifically, this can be expressed as:

$$A_k^E \xrightarrow{sW} A^E \quad \Longrightarrow \quad A_k^E \xrightarrow{ssW_\phi} A^E.$$

Proof. Assume that $A_k^E \xrightarrow{sW} A^E$ is satisfied. Let $P_e^x \in SP(X)$ be chosen arbitrarily. Then, for every $\tilde{\varepsilon} >_s 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ and for every $g \in E$, the following holds:

$$d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) < \varepsilon_g.$$

Consequently, we can derive the inclusion:

$$\left\{k \le n\colon d\left(D\left(P_e^x, A_k^E\right), D(P_e^x, A^E)\right)(g) \ge \varepsilon_g\right\} \subseteq \{1,2,\dots,k_0\}.$$

By applying the monotonicity property of ϕ-density, we obtain the inequality:

$$\delta_\phi\left(\{n \in \mathbb{N} | d(D(P_e^x, A_k^E), D(P_e^x, A^E) \ge_s \tilde{\varepsilon}\}\right) \le \delta_\phi(\{1,2,\dots,k_0\}) = 0.$$

This establishes that $A_k^E \xrightarrow{ssW_\phi} A^E$.

However, converse of the above proposition may not generally be true.

Example 3.1.5. Let $E = \{g\}$, $X = \mathbb{R}$ and $\phi(n) = n$. We define a sequence of soft closed sets $\{A_k^E\}$ as follows:

$$A_k^E = \begin{cases} g = [2,k], & \text{if } k \ge 2 \text{ and } k \text{ is a perfect square,} \\ g = \{1\}, & \text{otherwise.} \end{cases}$$

If the soft metric is considered as $d(P_g^x, P_g^y) = \{g = \{|x-y|\}\}$, then the sequence $\{A_k^E\}$ is not soft Wijsman convergent to $A^E = \{g = \{1\}\}$. However, since the following inequality

$$\frac{1}{n}\left|\{k \le n\colon d(D(P_g^x, A_k^E), D(P_g^x, A^E))(g) \ge \varepsilon_g\}\right| \le \frac{1}{\sqrt{n}}$$

holds [23], the sequence $\{A_k^E\}$ is ϕ-statistical soft Wijsman convergent to $A^E = \{g = \{1\}\}$.

Note that soft metric spaces are Hausdorff spaces [2].

Theorem 3.1.6. Let (X, d) be a soft metric space, and let $\{A_k^E\} \subset \overline{SS_d(X)}$ be a sequence along with $A^E, B^E \in \overline{SS_d(X)}$. If the following conditions hold:

$$A_k^E \xrightarrow{ssW_\phi} A^E \quad \text{and} \quad A_k^E \xrightarrow{ssW_\phi} B^E \tag{1}$$

then it follows that $A^E = B^E$.

Proof. Let us assume that the convergences given in (1) are satisfied. In this case, for every $P_e^x \in SP(X)$ and for each $g \in E$, the sequence $\{D(P_e^x, A_k^E)(g)\}$ converges ϕ-statistically to the points $D(P_e^x, A^E)(g)$ and $D(P_e^x, B^E)(g)$ in $\mathbb{R}$. From the uniqueness of ϕ-statistical convergence in $\mathbb{R}$, it follows that $D(P_e^x, A^E)(g) = D(P_e^x, B^E)(g)$ for each $g \in E$. Consequently, for every $P_e^x \in SP(X)$, we have $D(P_e^x, A^E) = D(P_e^x, B^E)$. This demonstrates that the sets A^E and B^E are equal.

The following result establishes the relationships between the sequences under varying conditions on the growth rates of the density functions ϕ and ψ.

Theorem 3.1.7. Let (X, d) be a soft metric space, and let the sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ and $A^E \in \overline{SS_d(X)}$. For the unbounded functions $\phi, \psi \in \mathcal{G}$, the following results are hold:

i. If there exists a constant $M > 0$ and $n_0 \in \mathbb{N}$ such that $\frac{\phi(n)}{\psi(n)} \le M$ for all $n \ge n_0$, then

$$A_k^E \xrightarrow{ssW_\phi} A^E \quad \Rightarrow \quad A_k^E \xrightarrow{ssW_\psi} A^E.$$

ii. If there exists a constant $m > 0$ and $n_0 \in \mathbb{N}$ such that $m \leq \frac{\phi(n)}{\psi(n)}$ for all $n \geq n_0$, then

$$A_k^E \xrightarrow{ssW_\psi} A^E \quad \Rightarrow \quad A_k^E \xrightarrow{ssW_\phi} A^E.$$

iii. If there exist constants $m, M > 0$ and $n_0 \in \mathbb{N}$ such that $m \leq \frac{\phi(n)}{\psi(n)} \leq M$ for all $n \geq n_0$, then

$$A_k^E \xrightarrow{ssW_\phi} A^E \quad \Leftrightarrow \quad A_k^E \xrightarrow{ssW_\psi} A^E.$$

Proof. Assume that $A_k^E \xrightarrow{ssW_\phi} A^E$ holds. Let $P_e^x \in SP(X)$ be chosen arbitrarily. For every $\tilde{\varepsilon} >_s 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k > k_0$ and for every $g \in E$, we have:

$$\lim_{n\to\infty} \frac{1}{\phi(n)} \left|\{k \leq n: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right| = 0. \quad (2)$$

Moreover, given that there exists a constant $M > 0$ and $n_0 \in \mathbb{N}$ such that $\frac{\phi(n)}{\psi(n)} \leq M$ for all $n \geq n_0$, we can deduce:

$$\frac{1}{\psi(n)} \left|\left\{k \leq n: d\left(D(P_e^x, A_k^E), D(P_e^x, A^E)\right)(g) \geq \varepsilon_g\right\}\right| \leq M \frac{1}{\phi(n)} \left|\left\{k \leq n: d\left(D(P_e^x, A_k^E), D(P_e^x, A^E)\right)(g) \geq \varepsilon_g\right\}\right|.$$

Taking the limit as $n \to \infty$, we find that (2) implies the convergence $A_k^E \xrightarrow{ssW_\psi} A^E$. The proof of (ii) follows a similar approach, while (iii) results from (i) and (ii).

3.2. Lacunary Statistical Soft Wijsman Convergence

A lacunary sequence is defined as an increasing sequence of integers $\theta = (k_r)_{r\in\mathbb{N}}$ where $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In our context, we denote $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. In this section, we will employ these notations to avoid reiteration.

Fridy and Orhan introduced the concept of lacunary statistical convergence in relation to lacunary sequences [16]. A real sequence (x_k) is said to be lacunary statistical convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_{r\to\infty} h_r^{-1}|\{k \in I_r: |x_k - L| \geq \varepsilon\}| = 0.$$

In this context, the objective of this section is to define the lacunary statistical convergence of soft set sequences in soft metric spaces within the framework of Wijsman convergence.

Definition 3.2.1. Let (X, d) be a soft metric space, and let the sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ and $A^E \in \overline{SS_d(X)}$ be given. The sequence $\{A_k^E\}$ is said to be ϕ-lacunary statistical soft Wijsman convergent (abbreviated as $lssW_\phi^\theta$-convergent) to A^E (denoted $A_k^E \xrightarrow{lssW_\phi^\theta} A^E$) if, for every $\tilde{\varepsilon} >_s 0$ and for each $P_e^x \in SP(X)$, the following condition holds:

$$\lim_{r\to\infty} \frac{1}{\phi(h_r)} \left|\{k \in I_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E)) \geq_s \tilde{\varepsilon}\}\right| = 0.$$

In other words, for every $g \in E$,

$$\lim_{r\to\infty} \frac{1}{\phi(h_r)} \left|\{k \in I_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right| = 0.$$

Similarly to Example 1 in [24], the following example can be given.

Example 3.2.2. Let $E = \{g\}$, $X = \mathbb{R}$, $\phi(n) = n$, and $\theta = (k_r)$ be a lacunary sequence. We define a sequence of soft closed sets $\{A_k^E\}$ as follows:

$$A_k^E = \begin{cases} g = [2, k_r - k_{r-1}], & \text{if } k \geq 2 \text{ and k is a perfect square,} \\ g = \{1\}, & \text{otherwise} \end{cases}$$

Assuming the soft metric is given by $d(P_g^x, P_g^y) = \{g = \{|x - y|\}\}$, the sequence $\{A_k^E\}$ is ϕ-lacunary statistical soft Wijsman convergent to $A^E = \{g = \{1\}\}$, since the following inequality holds:

$$\frac{1}{h_r}\left|\{k \in I_r: d(D(P_g^x, A_k^E), D(P_g^x, A^E))(g) \geq \varepsilon_g\}\right| \leq \frac{\sqrt{k_r - k_{r-1}}}{h_r}$$

and $\lim_{r\to\infty} \frac{\sqrt{k_r - k_{r-1}}}{h_r} = 0$.

The following theorem presents a case in which soft Wijsman statistical convergence necessitates soft Wijsman lacunary convergence.

Theorem 3.2.3. Let (X, d) be a soft metric space, let the sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ and $A^E \in \overline{SS_d(X)}$ be given. If $\liminf_{r\to\infty} \frac{\phi(h_r)}{\phi(k_r)} > 0$ and $\{A_k^E\}$ converges to A^E in the ϕ −statistical soft Wijsman sense, then $\{A_k^E\}$ is also ϕ −lacunary statistically soft Wijsman convergent to A^E.

Proof. Let $P_e^x \in SP(X)$ be chosen arbitrarily. According to the hypothesis, there exists a real number $\delta > 0$ such that $\frac{\phi(h_r)}{\phi(k_r)} \geq \delta$. Consequently, for sufficiently large r, we have $\frac{1}{\phi(k_r)} \geq \delta \frac{1}{\phi(h_r)}$. Thus, for every $\tilde{\varepsilon} >_s 0$ and for every $g \in E$, the following inequalities hold:

$$\delta \frac{1}{\phi(h_r)}\left|\left\{k \in I_r: d\left(D(P_e^x, A_k^E), D(P_e^x, A^E)\right)(g) \geq \varepsilon_g\right\}\right|$$
$$\leq \frac{1}{\phi(k_r)}\left|\{k \in I_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right|$$
$$\leq \frac{1}{\phi(k_r)}\left|\{k \leq k_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right|.$$

Conversely, if $A_k^E \xrightarrow{ssW_\phi} A^E$, then for each $g \in E$, we have

$$\lim_{r\to\infty} \frac{1}{\phi(k_r)}\left|\{k \leq k_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right| = 0.$$

As a result of the inequalities above, it follows that

$$\lim_{r\to\infty} \frac{1}{\phi(h_r)}\left|\{k \in I_r: d(D(P_e^x, A_k^E), D(P_e^x, A^E))(g) \geq \varepsilon_g\}\right| = 0$$

for each $g \in E$. This indicates that $A_k^E \xrightarrow{lssW_\phi^\theta} A^E$. The proof of the following theorem is omitted, as it parallels that of Theorem 3.1.7.

Theorem 3.2.4. Let (X, d) be a soft metric space, and let the sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ and $A^E \in \overline{SS_d(X)}$ be given. For the unbounded functions $\phi, \psi \in \mathcal{G}$, the following statements are established:

i. If there exists a constant $M > 0$ and $n_0 \in \mathbb{N}$ such that $\frac{\phi(n)}{\psi(n)} \leq M$ for all $n \geq n_0$, then

$$A_k^E \xrightarrow{lssW_\phi^\theta} A^E \implies A_k^E \xrightarrow{lssW_\psi^\theta} A^E.$$

ii. If there exists a constant $m > 0$ and $n_0 \in \mathbb{N}$ such that $m \leq \frac{\phi(n)}{\psi(n)}$ for all $n \geq n_0$, then

$$A_k^E \xrightarrow{lssW_\psi^\theta} A^E \implies A_k^E \xrightarrow{lssW_\phi^\theta} A^E.$$

iii. If there exist constants $m, M > 0$ and $n_0 \in \mathbb{N}$ such that $m \leq \frac{\phi(n)}{\psi(n)} \leq M$ for all $n \geq n_0$, then

$$A_k^E \xrightarrow{lssW_\phi^\theta} A^E \iff A_k^E \xrightarrow{lssW_\psi^\theta} A^E.$$

Theorem 3.2.5. Let (X, d) be a soft metric space, and consider the sequence $\{A_k^E\} \subset \overline{SS_d(X)}$ along with $A^E \in \overline{SS_d(X)}$ and $\phi \in \mathcal{G}$. Suppose that $\theta = (k_r)$, $\beta = (w_r)$ are lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$ where $I_r = (k_{r-1}, k_r]$, $J_r = (w_{r-1}, w_r]$. Define $h_r = k_r - k_{r-1}$ and $v_r = w_r - w_{r-1}$. The following results hold:

i. If $\lim_{r\to\infty} \frac{(v_r - h_r)}{\phi(h_r)} = 0$, then the convergence $A_k^E \xrightarrow{lssW_\phi^\theta} A^E$ implies $A_k^E \xrightarrow{lssW_\phi^\beta} A^E$.

ii. If $\liminf \frac{\phi(h_r)}{\phi(v_r)} > 0$, then the convergence $A_k^E \xrightarrow{lssW_\phi^\beta} A^E$ implies $A_k^E \xrightarrow{lssW_\phi^\theta} A^E$.

iii. If both $\lim_{r\to\infty} \frac{(v_r - h_r)}{\phi(h_r)} = 0$ and $\liminf \frac{\phi(h_r)}{\phi(v_r)} > 0$ are satisfied, then the convergence $A_k^E \xrightarrow{lssW_\phi^\theta} A^E$ is equivalent to $A_k^E \xrightarrow{lssW_\phi^\beta} A^E$.

Proof. Let $P_e^x \in SP(X)$ be chosen arbitrarily.

i) Assume that $\lim_{r\to\infty}\frac{(v_r-h_r)}{\phi(h_r)}=0$. Under this condition, for every $g\in E$, the following inequalities hold:

$$\frac{1}{\phi(v_r)}\left|\left\{k\in J_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|$$
$$=\frac{1}{\phi(v_r)}\left|\left\{w_{r-1}<k\leq w_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|$$
$$\leq\frac{1}{\phi(v_r)}\left(\left|\left\{\begin{array}{c}w_{r-1}<k\leq k_{r-1}:\\ d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\end{array}\right\}\right|+\left|\left\{\begin{array}{c}k_{r-1}<k\leq k_r:\\ d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\end{array}\right\}\right|+\left|\left\{\begin{array}{c}k_r<k\leq w_r:\\ d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\end{array}\right\}\right|\right)$$
$$\leq\frac{1}{\phi(v_r)}\left[\begin{array}{c}(k_{r-1}-w_{r-1})+\\ \left|\left\{\begin{array}{c}k\in I_r:\\ d\left(D\left(P_e^x,A_k^E\right),D(P_e^x,A^E)\right)(g)\\ \geq\varepsilon_g\end{array}\right\}\right|\\ +(w_r-k_r)\end{array}\right]$$
$$\leq\frac{1}{\phi(v_r)}\left[(v_r-h_r)+\left|\left\{k\in I_r: P_{e_k}^{x_k}\notin F_E\right\}\right|\right]$$
$$\leq\frac{1}{\phi(h_r)}(v_r-h_r)+\frac{1}{\phi(h_r)}\left|\left\{k\in I_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|$$
$$\leq\frac{(v_r-h_r)}{\phi(h_r)}+\frac{1}{\phi(h_r)}\left|\left\{k\in I_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|.$$

Furthermore, base on the the hypothesis, for every $\tilde{\varepsilon}>_s 0$,

$$\lim_{r\to\infty}\frac{1}{\phi(h_r)}\left|\left\{k\in I_r: d\left(D\left(P_e^x,A_k^E\right),D(P_e^x,A^E)\right)\geq_s\tilde{\varepsilon}\right\}\right|=0.$$

Consequently, from the preceding inequality, it follows that for every $g\in E$,

$$\lim_{r\to\infty}\frac{1}{\phi(v_r)}\left|\left\{k\in J_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|=0,$$

indicating that $A_k^E\xrightarrow{lssW_\phi^\beta}A^E$.

ii) Assume that $\liminf\frac{\phi(h_r)}{\phi(v_r)}>0$. Thus, there exists a real number $\delta>0$ such that $\phi(h_r)\geq\delta\phi(v_r)$ for sufficiently large r. From the inclusion $I_r\subseteq J_r$, the following inequality holds, for every $g\in E$,

$$\delta\frac{1}{\phi(h_r)}\left|\left\{k\in I_r: d\left(D\left(P_e^x,A_k^E\right),D(P_e^x,A^E)\right)(g)\geq\varepsilon_g\right\}\right|$$
$$\leq\frac{1}{\phi(v_r)}\left|\left\{k\in J_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|.$$

According to the hypothesis, for every $\tilde{\varepsilon}>_s 0$,

$$\lim_{r\to\infty}\frac{1}{\phi(h_r)}\left|\left\{k\in J_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))\geq_s\tilde{\varepsilon}\right\}\right|=0.$$

Thus, it follows that

$$\frac{1}{\phi(h_r)}\left|\left\{k\in I_r: d(D(P_e^x,A_k^E),D(P_e^x,A^E))(g)\geq\varepsilon_g\right\}\right|=0.$$

This implies that $A_k^E\xrightarrow{lssW_\phi^\theta}A^E$.

iii) The conclusion follows readily from the results in parts i) and ii).

4. Conclusion

In this study, we introduced the concepts of ϕ-statistical soft Wijsman convergence and ϕ-lacunary statistical soft Wijsman convergence within the framework of soft metric spaces. By focusing on the convergence of sequences of closed sets, we established fundamental properties and definitions that diversify the understanding of convergence in soft set theory. Our findings confirm that soft Wijsman convergence leads to ϕ-statistical convergence. In the obtained results, for soft closed set sequences, we present how transitions between the concepts of statistical and lacunary statistical convergence occur and some conditions under which these transitions take place.

Looking forward, future research could enhance our findings by investigating the effects of these types of convergence in soft topological spaces and exploring the connections between different definitions of convergence. Additionally, further exploration of applications in areas dealing with uncertainty and vagueness, such as decision-making and fuzzy logic, may be beneficial for developing different approaches to problems. Moreover, similar results could be developed using different density functions.

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The author of the paper declare that they comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that they do not make any falsification on the data collected. In addition, he declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

References

[1] D. Molodtsov, “Soft set theory—first results,” Computers and Mathematics with Applications, vol. 37, no. 4-5, pp. 19-31, 1999.

[2] S. Das, S. Samanta, “Soft metric,” Annals of Fuzzy Mathematics and Informatics, vol. 6, no. 1, pp. 77-94, 2013.

[3] S. Das, S. Samanta, “On soft metric spaces,” The Journal of Fuzzy Mathematics, vol. 21, no. 3, pp. 707-734, 2013.

[4] J. C. R. Alcantud, A. Z. Khameneh, G. Santos-Garcia, M. Akram, “A systematic literature review of soft set theory,” Neural Computing and Applications, vol. 36, no. 16, pp. 8951-8975, 2024.

[5] R. A. Wijsman, “Convergence of sequences of convex sets, cones and functions,” Bulletin of the American Mathematical Society, vol. 70, pp. 186-188, 1964.

[6] R. A. Wijsman, “Convergence of sequences of convex sets, cones and functions. II,” Transactions of the American Mathematical Society, vol. 123, no. 1, pp. 32-45, 1966.

[7] G. Beer, Topologies on Closed and Closed Convex Sets, vol. 268 of Mathematics and its Applications. Dordrecht: Kluwer Academic Publishers Group, 1993.

[8] G. Beer, “Wijsman convergence: A survey,” Set-Valued Analysis, vol. 2, pp. 77-94, 1994.

[9] R. T. Rockafellar, R. J. B. Wets, Variational Analysis, Springer Berlin, Heidelberg, 1997.

[10] J. O. Royset, “Set-Convergence and Its Application: A Tutorial,” Set-Valued Variational Analysis, vol. 28, pp. 707–732, 2020.

[11] H. Fast, “Sur la convergence statistique,” Colloquium Mathematicae, vol. 2, no. 3-4, pp. 241-244, 1951.

[12] H. Steinhaus, “Sur la convergence ordinaire et la convergence asymptotique,”

Colloquium Mathematicae, vol. 2, pp. 73-74, 1951.

[13] I. J. Schoenberg, “The integrability of certain functions and related summability methods”, The American Mathematical Monthly, vol. 66, pp. 361-375, 1959.

[14] I. J. Maddox, Statistical convergence in a locally convex space, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 104, pp. 141–145, 1988.

[15] J. S. Connor, “The statistical and strong $p-$Cesàro convergence of sequences,” Analysis, vol. 8, no. 1-2, pp. 47-63, 1988.

[16] J. A. Fridy, “On statistical convergence,” Analysis, vol. 5, pp. 301-313, 1985.

[17] T. Šalát, “On statistically convergent sequences of real numbers,” Mathematica Slovaca, vol. 30, no. 2, pp. 139-150, 1980.

[18] G. Di Maio, Lj. A. D. Kočínac, “Statistical convergence in topology,” Topology and its Applications, vol. 156, no. 1, pp. 28-45, 2008.

[19] J. A. Fridy, C. Orhan, “Lacunary statistical convergence,” Pacific Journal of Mathematics, vol. 160, no. 1, pp. 43-51, 1993.

[20] S. Pehlivan, B. Fisher, “Lacunary strong convergence with respect to a sequence of modulus functions,” Commentationes Mathematicae Universitatis Carolinae, vol. 36, no. 1, pp. 69-76, 1995.

[21] H. Şengül, M. Et, “f-Lacunary Statistical Convergence and Strong f-Lacunary Summability of Order α”, Filomat, vol. 32, no. 13, pp. 4513-4521, 2018.

[22] V. K. Bhardwaj, S. Dhawan, “f-Statistical convergence of order α and strong Cesàro summability of order with respect to a modulus,” Journal of Inequalities and Applications, vol. 332, 2015.

[23] F. Nuray, B. E. Rhoades, “Statistical convergence of sequences of sets,” Fasciculi Mathematici, vol. 49, pp. 87–99, 2012.

[24] U. Ulusu, F. Nuray, “Lacunary statistical convergence of sequence of sets,” Progress in Applied Mathematics, vol. 4, no. 2, pp. 99–109, 2012.

[25] M. Balcerzak, P. Das, M. Filipczak, J. Swaczyna, “Generalized kinds of density and the associated ideals,” Acta Mathematica Hungarica, vol. 147, pp. 97-115, 2015.

[26] E. Bayram, M. D. Çınar, “Soft Wijsman convergence,” Maejo International Journal of Science and Technology, vol. 18, no. 3, pp.267-279, 2024.