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Trapezoid-type Inequalities Based on Generalized Conformable Integrals via Co-ordinated h-Convex Mappings

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Abstract

In this study, some new trapezoid type inequalities are generalized for $h-$ convex functions in coordinates by means of generalized conformable fractional integrals. For functions with $h-$convex absolute values of their partial derivatives, some new trapezoid type inequalities are obtained using the well-known Holder and Power Mean inequalities. In addition, some findings of this study include some results based on Riemann Liouville fractional integrals.

1. Introduction

Convexity theory represents a distinct and highly specialized field within the mathematical sciences. The field has attracted significant interest among researchers due to its extensive and diverse range of applications in fields such as engineering, optimization theory, energy systems, and physics. Over the years, many articles have been written on It is important to avoid making generalisations and to ensure that any new versions of existing theories are fully evidenced-based. some of its inequalities using different types of convex functions. This work aims to add a new one to the trapezoid type inequalities using $h-$convexity in co-ordinates. Some important definitions and theorems necessary for the main results of our work are given below:

Definition 1.1. *[1]Let I be convex set on $\mathbb{R}$. The function $\psi : I \to \mathbb{R}$ is said to be convex on I, if it is demonstrated that the following inequality is removed:*

$$\psi(\tau\chi + (1-\tau)\ \varphi) \leq \tau\psi(\chi) + (1-\tau)\psi(\varphi) \tag{1.1}$$

for all $(\chi,\varphi) \in I$ and $\tau \in [0,1]$. The mapping ψ is a concave on I if the inequality (1.1). In the event of a reversal, the aforementioned item should be held in the opposite direction for all $\tau \in [0,1]$ and $\chi,\varphi \in I$.

Let us begin by examining a rectangle positioned on a flat surface. $\Delta := [\sigma,\phi] \times [\varsigma,\rho]$ in $\mathbb{R}^2$. A mapping $\psi : \triangle \to \mathbb{R}$ the following definition of a mapping in coordinated convex is provided for the reader's convenience:

Definition 1.2. *[2]A function $\psi : \Delta \to \mathbb{R}$ The term "coordinated convex on" is used to describe this phenomenon. Δ, for all$(\chi,\varphi),(\upsilon,\omega) \in \Delta$ and $\tau,\xi \in [0,1]$. The following inequality must be satisfied:*

$$\psi(\tau\chi + (1-\tau)\ \varphi, \xi u + (1-\xi)\ \omega) \quad \leq \quad \tau\xi\ \psi(\chi,\upsilon) + \tau(1-\xi)\psi(\chi,\omega) + \xi(1-\tau)\psi(\varphi,\upsilon) + (1-\tau)(1-\xi)\psi(\varphi,\omega).$$

It is evident that all convex functions are convex with respect to the given coordinates. Nevertheless, it is not necessarily the case that every function which is convex in coordinates will necessarily be convex (see, [2]).

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Definition 1.3. *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $\psi : I \to \mathbb{R}$ is an h-convex function, or that ψ belongs to the class $SX(h,I)$, if ψ non-negative and for all $\chi, \varphi \in I, \tau \in (0,1)$ we have,*

$$\psi(\tau\chi + (1-\tau)\ \varphi) \leq h(\tau)\psi(\chi) + h(1-\tau)\psi(\varphi).$$

If this inequality is reversed, then ψ is said to be $h-$concave [3].

Definition 1.4. *A function $\psi : I \to \mathbb{R}$ is said to be $h-$convex on the coordinates on Δ, if the following inequality holds:*

$$\begin{aligned}\psi(\tau\chi + (1-\tau)\ \varphi, \xi u + (1-\xi)\ \omega) \ &\leq\ h(\tau)h(\xi)\ \psi(\chi,\upsilon) + h(\tau)h(1-\xi)\psi(\chi,\omega)\\ &+\ h(\xi)h(1-\tau)\psi(\varphi,\upsilon) + h(1-\tau)h(1-\xi)\psi(\varphi,\omega)\end{aligned}$$

holds for all $\forall(\tau,\xi) \in [0,1]$ and $(\chi,\upsilon),(\chi,\omega),(\varphi,\upsilon),(\varphi,\omega) \in \Delta$ [4].

Definition 1.5. *The gamma function and beta function are defined by*

$$\Gamma(\chi) := \int_0^\infty \tau^{x-1} e^{-\tau} d\tau$$

and

$$B(\chi,\varphi) := \int_0^1 \tau^{x-1}(1-\tau)^{\varphi-1} d\tau,$$

respectively. Here, $0 < \chi, \varphi < \infty$.

In the analysis of mathematics in convex mappings, integral inequalities are the most frequently used field. These inequalities, which were formulated by C. Hermite and J. Hadamard, are widely cited in the literature (see, e.g., [1], [5, p.137],[6]). The aforementioned inequalities posit that if $\psi : I \to \mathbb{R}$ The function is convex on the interval. I of real numbers and $\sigma, \phi \in I$ with $\tau \leqslant \xi$ then,

$$\psi\left(\frac{\sigma+\phi}{2}\right) \leq \frac{1}{\phi-\sigma}\int_\tau^\xi \psi(\chi)d\chi \leq \frac{\psi(\sigma)+\psi(\phi)}{2}.$$

The Hermite-Hadamard inequality is a celebrated result that elucidates the impact of a function's convexity on its mean value and integral value over a given interval. The inequality states that if a function ψ is convex on a real interval I and if a and b are two points in I, then the value of ψ at the midpoint of a and ϕ is less than or equal to the average value of f on the interval $[\sigma,\phi]$. Hermite Hadamard's inequality is employed to compare The mean value of a convex function is defined as the sum of the areas of the function's convex hulls, divided by the number of hulls. over a given interval with the values observed at the endpoints of the interval. This inequality can be extended to higher dimensions, such as the plane $\mathbb{R}^2$, where the interval is transformed into a rectangle. The inequality comprises two parts, one utilising the midpoint of the rectangle and the other employing the corners of the rectangle. The second part is referred to as the trapezoid-type inequality, due to its resemblance to the shape of a trapezoid. The trapezoid-type inequality, which forms part of the Hermite-Hadamard inequality, has been the subject of extensive research. The trapezoid-type inequalities for convex functions were initially formulated by Dragomir and Agarwal in [7]. In [8], Sarikaya et al. provided a generalisation of the inequalities for fractional integrals, and also demonstrated the validity of certain midpoint-type inequalities. The inequality has been the subject of extensive study and improvement, particularly in the context of fractional integrals, which represent a generalisation of ordinary integrals [2], [7], [8], [9], [10].

Theorem 1.6. *Suppose that $\psi : \Delta \to \mathbb{R}$ is co-ordinated convex. Then we have the following inequalities:*

$$\begin{aligned}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right) \ &\leq\ \frac{1}{2}\left[\frac{1}{\sigma-\phi}\int_\sigma^\phi \psi\left(\chi,\frac{\varsigma+\rho}{2}\right)d\chi + \frac{1}{\rho-c}\int_\varsigma^\rho \psi\left(\frac{\tau+\xi}{2},\varphi\right)d\varphi\right]\\ &\leq\ \frac{1}{(\phi-\sigma)(\rho-\varsigma)}\int_\sigma^\phi\int_\varsigma^\rho \psi(\chi,\varphi)d\varphi d\chi \qquad (1.2)\\ &\leq\ \frac{1}{4}\left[\frac{1}{\phi-\sigma}\int_\tau^\xi \psi(\chi,c)d\chi + \frac{1}{\phi-\sigma}\int_\tau^\xi \psi(\chi,\rho)d\chi + \frac{1}{\rho-\varsigma}\int_\varsigma^\rho \psi(\tau,\varphi)d\varphi + \frac{1}{\rho-\varsigma}\int_\varsigma^\rho \psi(\xi,\varphi)d\varphi\right]\\ &\leq\ \frac{\psi(\tau,\varsigma)+\psi(\tau,\rho)+\psi(\xi,\varsigma)+\psi(\xi,\rho)}{4}.\end{aligned}$$

The above inequalities are sharp. The inequalities in (1.2) hold in reverse direction if the mapping ψ is a co-ordinated concave mapping.

Fractional calculus describes a type of calculus that is capable of utilising any real or complex number as the power of the derivative or the integral. There are a number of different approaches to defining fractional calculus, including the Caputo, Riemann-Liouville, and Grünwall-Letnikov definitions. While these definitions offer certain advantages, they also present certain problems. To illustrate, the Riemann-Liouville definition does not always yield a derivative of zero for a constant. In the Caputo definition, the function f must be differentiable [11], [12], [13], [14], [15], [16]. Furthermore, a number of definitions deviate from the conventional tenets of calculus, including the operations of division, multiplication and composition between two functions. To address these and other issues, Khalil et al. proposed a novel definition of fractional calculus, termed the compatible fractional derivative. Furthermore, the compatible fractional integral for powers between $(0<\sigma\leq 1)$ was also introduced. The researchers demonstrated several significant findings, including methods for multiplying two functions and calculating the mean value of a function. Additionally, they solved equations involving fractional calculus and exponential functions (see, [17], [18], [19], [20], [21], [22], [23], [24], [25]).

The definitions and mathematical foundations of the principles of conformable fractional calculus that are employed in this study are set forth below:

Definition 1.7. *[26]For $\psi\in L_1[\sigma,\phi]$, the Riemann-Liouville integrals of order $\tau>0$ are given by*

$$J^{\sigma}_{\sigma+}\psi(\chi)=\frac{1}{\Gamma(\sigma)}\int_{\sigma}^{\chi}(\chi-\tau)^{\sigma-1}\psi(\tau)d\tau,\ \ \chi>\sigma$$

and

$$J^{\sigma}_{\phi-}\psi(\chi)=\frac{1}{\Gamma(\sigma)}\int_{\chi}^{\phi}(\tau-\chi)^{\sigma-1}\psi(\tau)d\tau,\ \ \chi<\phi.$$

In accordance with the aforementioned criteria, the respective values are as follows: it can be demonstrated that the Riemann-Liouville integrals are equal to their classical counterparts when the requisite condition is $\theta=1$.

Definition 1.8. *[27]Let $\psi\in L_1([\sigma,\phi]\times[\varsigma,\rho])$. The Riemann-Liouville integrals $J^{\theta,\vartheta}_{\sigma+,\varsigma^+}, J^{\theta,\vartheta}_{\sigma+,\rho^-}, J^{\theta,\vartheta}_{\phi^-,\varsigma^+}$ and $J^{\theta,\vartheta}_{\phi^-,\rho^-}$ of order $\theta,\vartheta>0$ with $\sigma,\varsigma\geq 0$ are defined by*

$$J^{\theta,\vartheta}_{\sigma+,\varsigma^+}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varsigma}^{\varphi}(\chi-\tau)^{\theta-1}(\varphi-\xi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi>\sigma,\varphi>\varsigma, \tag{1.3}$$

$$J^{\theta,\vartheta}_{\sigma+,\rho^-}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varphi}^{\rho}(\chi-\tau)^{\theta-1}(\xi-\varphi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi>\sigma,\varphi<\rho, \tag{1.4}$$

$$J^{\theta,\vartheta}_{\phi-,\varsigma^+}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\chi}^{\phi}\int_{\varsigma}^{\varphi}(\tau-\chi)^{\theta-1}(\varphi-\xi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi<\phi,\varphi>\varsigma, \tag{1.5}$$

and

$$J^{\theta,\vartheta}_{\phi-,\rho^-}\psi(\chi,\varphi)=\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\chi}^{\phi}\int_{\varphi}^{\rho}(\tau-\chi)^{\theta-1}(\xi-\varphi)^{\vartheta-1}\psi(\tau,\xi)d\xi d\tau,\ \ \chi<\phi,\varphi<\rho, \tag{1.6}$$

respectively. Here, Γ is the Gama function.

Definition 1.9. *[28] For $\psi\in L_1[\sigma,\phi]$, the fractional conformable integral operator ${}^{\vartheta}I^{\theta}_{\sigma+}\psi$and ${}^{\vartheta}I^{\theta}_{\phi-}\psi$ of order $\vartheta>0$ and $\sigma\in(0,1]$ are presented by*

$${}^{\vartheta}\mathfrak{J}^{\theta}_{\sigma+}\psi(\chi)=\frac{1}{\Gamma(\vartheta)}\int_{\sigma}^{\chi}\left(\frac{(\chi-\sigma)^{\theta}-(\tau-\sigma)^{\theta}}{\theta}\right)^{\vartheta-1}\frac{\psi(\tau)}{(\tau-\sigma)^{1-\theta}}d\tau,\quad \tau>\sigma \tag{1.7}$$

and

$${}^{\vartheta}\mathfrak{J}^{\theta}_{\phi-}\psi(\chi)=\frac{1}{\Gamma(\vartheta)}\int_{\chi}^{\phi}\left(\frac{(\phi-\chi)^{\theta}-(\phi-\tau)^{\theta}}{\theta}\right)^{\vartheta-1}\frac{\psi(\tau)}{(\phi-\tau)^{1-\theta}}d\tau,\quad \tau<\phi, \tag{1.8}$$

respectively.

Definition 1.10. *[29]Let $\psi\in L_1([\sigma,\phi]\times[\varsigma,\rho])$ and let $\gamma_1\neq 0$, $\gamma_2\neq 0$, $\theta,\vartheta\in\mathbf{C}$, $Re(\theta)>0$ and $Re(\vartheta)>0$. The generalized conformable integral of order θ,ϑ of $\psi(\chi,\varphi)$ is defined by;*

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\right)(\chi,\varphi)\quad=\quad\left[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_{\sigma}^{\chi}\int_{\varsigma}^{\varphi}\left(\frac{(\chi-\sigma)^{\gamma_1}-(\tau-\sigma)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}\right. \tag{1.9}$$

$$\times\left(\frac{(\varphi-\varsigma)^{\gamma_2}-(\xi-\varsigma)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau\Bigg],$$

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\chi^\phi\int_\varsigma^\varphi\left(\frac{(\phi-\chi)^{\gamma_1}-(\phi-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\varphi-\varsigma)^{\gamma_2}-(\xi-\varsigma)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.10}$$

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\sigma^\chi\int_\varphi^\rho\left(\frac{(\chi-\sigma)^{\gamma_1}-(\tau-\sigma)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\rho-\varphi)^{\gamma_2}-(\rho-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.11}$$

and

$$\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\right)(\chi,\varphi) = \Bigg[\frac{1}{\Gamma(\theta)\Gamma(\vartheta)}\int_\chi^\phi\int_\varphi^\rho\left(\frac{(\phi-\chi)^{\gamma_1}-(\phi-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1} \times\left(\frac{(\rho-\varphi)^{\gamma_2}-(\rho-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau\Bigg], \tag{1.12}$$

the generalized conformable integrals.

Remark 1.11. *[29] If $\gamma_1=\gamma_2=1$ in (1.9), (1.10),(1.11) and (1.12), we have (1.3)-(1.6) the Fractional integrals of the functions of two variables.*

Remark 1.12. *[29] If we consider $\theta=1$ and $\vartheta=1$ in (1.9), (1.10),(1.11) and (1.12), we have*

$$\left(I^{1,1}_{\sigma^+,\varsigma^+}\psi\right)(\chi,\varphi)=\int_\sigma^\chi\int_\varsigma^\varphi\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau \tag{1.13}$$

$$\left(I^{1,1}_{\phi^-,\varsigma^+}\psi\right)(\chi,\varphi)=\int_\chi^\phi\int_\varsigma^\varphi\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\xi-\varsigma)^{1-\gamma_2}}d\xi d\tau, \tag{1.14}$$

$$\left(I^{1,1}_{\sigma^+,\rho^-}\psi\right)(\chi,\varphi)=\int_\sigma^\chi\int_\varphi^\rho\frac{\psi(\tau,\xi)}{(\tau-\sigma)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau, \tag{1.15}$$

and

$$\left(I^{1,1}_{\phi^-,\rho^-}\psi\right)(\chi,\varphi)=\int_\chi^\phi\int_\varphi^\rho\frac{\psi(\tau,\xi)}{(\phi-\tau)^{1-\gamma_1}(\rho-\xi)^{1-\gamma_2}}d\xi d\tau. \tag{1.16}$$

the conformable fractional integrals for double integrals.

Theorem 1.13. *[30] In assume ψ is a co-ordinated convex function that goes from $[\sigma,\phi]\times[\varsigma,\rho]$ into $\mathbb{R}$ and let $\gamma_1\neq0,\ \gamma_2\neq0$, $\theta,\vartheta\in(0,1]$, $Re(\theta)>0$ and $Re(\vartheta)>0$. The following inequality holds for generalized conformable fractional integrals,*

$$\begin{aligned}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right) &\le \frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^\theta\gamma_2^\vartheta}{(\phi-\sigma)^{\gamma_1\alpha}(\rho-\varsigma)^{\gamma_2\beta}}\Bigg[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\\ &+ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\Bigg]\\ &\le \frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}.\end{aligned} \tag{1.17}$$

2. Trapezoid type inequalities for co-ordinated h-convex functions

Lemma 2.1. *[31]Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[\sigma,\phi]\times[\varsigma,\rho]$ in $\mathbb{R}^2$ with $0\leq\sigma<\phi$, $0\leq\varsigma\leq\rho$. If $\frac{\partial^2\psi}{\partial\tau\partial\xi}\in L_1(\Delta)$, then the following identity:*

$$\begin{aligned}
&\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}} \qquad (2.1)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
&\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\\
=\;&\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\\
&\times\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\right.\\
&-\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\\
&-\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
&\left.+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\right],
\end{aligned}$$

where

$$\begin{aligned}
A\;=\;&\frac{2^{\gamma_2\vartheta-2}\,\gamma_2^{\vartheta}\,\Gamma(\vartheta+1)}{(\rho-\varsigma)^{\gamma_2\vartheta}} \qquad (2.2)\\
&\left[{}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\left(\sigma,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{\rho^-}\psi\left(\sigma,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{\varsigma^+}\psi\left(\phi,\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_2}I^{\vartheta}_{d^-}\psi\left(\phi,\frac{\varsigma+\rho}{2}\right)\right]\\
&+\frac{2^{\gamma_1\theta-2}\,\gamma_1^{\theta}\,\Gamma(\theta+1)}{(\phi-\sigma)^{\gamma_1\theta}}\\
&\left[{}^{\gamma_1}I^{\theta}_{\sigma^+}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)+{}^{\gamma_1}I^{\theta}_{\sigma^+}\psi\left(\frac{\sigma+\phi}{2},\rho\right)^{\gamma_1}I^{\theta}_{\phi^-}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)+{}^{\gamma_1}I^{\theta}_{\phi^-}\psi\left(\frac{\sigma+\phi}{2},\rho\right)\right].
\end{aligned}$$

Proof. By employing the technique of integration by parts, we obtain the following result:

$$\begin{aligned}
I_1\;=\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
=\;&\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left\{\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\frac{-2}{(\phi-\sigma)}\frac{\partial\psi}{\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\Bigg|_0^1\right.\\
&\left.+\int_0^1\frac{2\theta}{(\phi-\sigma)}\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\frac{\partial\psi}{\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\tau\right\}d\xi\\
=\;&\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left\{\left(\frac{1}{\gamma_1}\right)^{\theta}\left(\frac{-2}{\phi-\sigma}\right)\frac{\partial\psi}{\partial\xi}\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right.\\
&\left.+\frac{2\theta}{(\phi-\sigma)}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\frac{\partial\psi}{\partial s}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\tau\right\}d\xi\\
=\;&\frac{-2}{(\phi-\sigma)\gamma_1^{\theta}}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial\psi}{\partial\xi}\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\theta}{(\phi-\sigma)}\left[\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\right.\\
&\left.\times\left\{\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial\psi}{\partial\varsigma}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right\}d\tau\right]\\
=\;&\frac{-2}{(\phi-\sigma)}\left(\frac{1}{\gamma_1}\right)^{\theta}\left[\left(\frac{1}{\gamma_2}\right)^{\vartheta}\frac{-2}{(\rho-\varsigma)}\psi(\sigma,\varsigma)\right.\\
&\left.+\frac{2\vartheta}{(\rho-\varsigma)}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right]\\
&+\frac{2\theta}{(\phi-\sigma)}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\left\{\left(\frac{1}{\gamma_2}\right)^{\beta}\frac{-2}{(\rho-\varsigma)}\psi\left(\frac{1+\tau}{2}+\frac{1-\tau}{2}\phi,\varsigma\right)\right.\\
&\left.+\frac{2\vartheta}{(\rho-\varsigma)}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\right\}d\tau\\
=\;&\frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\varsigma)\\
&-\frac{4\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\left(\frac{1}{\gamma_1}\right)^{\theta}\int_0^1\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\psi\left(\sigma,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi\\
&-\frac{4\theta}{(\phi-\sigma)(\rho-\varsigma)}\left(\frac{1}{\gamma_2}\right)^{\vartheta}\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\varsigma\right)d\tau\\
&+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta-1}(1-\tau)^{\gamma_1-1}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta-1}(1-\xi)^{\gamma_2-1}\right.\\
&\left.\times\,\psi\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\right].
\end{aligned}\tag{2.3}
$$

In (2.3), using the change of the variables $u=\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi$ and $v=\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho$, we can write,

$$
\begin{aligned}
I_1 =\;&\frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\varsigma)-\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I_{\varsigma^+}^{\vartheta}\psi\right)\left(\sigma,\frac{\varsigma+\rho}{2}\right)-\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)^{\gamma_1}I_{\sigma^+}^{\theta}\psi\left(\frac{\sigma+\phi}{2},\varsigma\right)\\
&+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I_{\sigma^+,\varsigma^+}^{\theta,\vartheta}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right).
\end{aligned}\tag{2.4}
$$

By employing the technique of integration by parts, the following result is obtained:

$$
\begin{aligned}
I_2 =\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau\\
=\;&\frac{-4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\sigma,\rho)+\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I_{\rho^-}^{\vartheta}\psi\right)\left(\sigma,\frac{\varsigma+\rho}{2}\right)+\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I_{\sigma^+}^{\theta}\psi\right)\left(\frac{\sigma+\phi}{2},\rho\right)\\
&-\frac{4\theta\beta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I_{\phi^-,\varsigma^+}^{\theta,\vartheta}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right),
\end{aligned}\tag{2.5}
$$

$$
\begin{aligned}
I_3 =\;&\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)d\xi d\tau\\
=\;&\frac{-4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\phi,\varsigma)+\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I_{\varsigma^+}^{\vartheta}\psi\right)\left(\phi,\frac{\varsigma+\rho}{2}\right)+\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I_{\phi^-}^{\theta}\psi\right)\left(\frac{\sigma+\phi}{2},\varsigma\right)\\
&-\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I_{\sigma^+,\rho^-}^{\theta,\vartheta}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right),
\end{aligned}\tag{2.6}
$$

and

$$\begin{aligned} I_4 &= \int_0^1\int_0^1 \left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)d\xi d\tau \\ &= \frac{4}{(\phi-\sigma)(\rho-\varsigma)}\frac{1}{\gamma_1^{\theta}\gamma_2^{\vartheta}}\psi(\phi,\rho)-\left(\frac{2}{\rho-\varsigma}\right)^{\gamma_2\vartheta}\Gamma(\vartheta)\left({}^{\gamma_2}I^{\vartheta}_{\rho^-}\psi\right)\left(\phi,\frac{\varsigma+\rho}{2}\right)-\left(\frac{2}{\phi-\sigma}\right)^{\gamma_1\theta}\Gamma(\theta)\left({}^{\gamma_1}I^{\theta}_{\phi^-}\psi\right)\left(\frac{\sigma+\phi}{2},\rho\right) \\ &+\frac{4\theta\vartheta}{(\phi-\sigma)(\rho-\varsigma)}\frac{2^{\gamma_1\theta}2^{\gamma_2\vartheta}\Gamma(\theta)\Gamma(\vartheta)}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\left({}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\right)\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right). \end{aligned} \tag{2.7}$$

By equalities from (2.4)-(2.7), we obtain

$$\begin{aligned} &\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\left[I_1-I_2-I_3+I_4\right] \\ =\ &\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}} \\ &\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\ &\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A. \end{aligned}$$

□

This constitutes the proof.

The following section presents the initial Theorem, which encompasses the Hermite-Hadamard-type inequality for generalized conformable fractional integrals.

Theorem 2.2. *Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0\le\sigma<\phi,\ 0\le\varsigma<\rho$. If $\frac{\partial^2\psi}{\partial t\partial s}$ is a $h-$convex function on the coordinates on Δ, then the inequality below holds:*

$$\begin{aligned} &\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \\ &\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\ &\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\ \le\ &\frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}\left[\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right] \\ &\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right], \end{aligned} \tag{2.8}$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function and

$$\begin{aligned} \int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\ \int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\ \int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \\ \int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau &= \Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h) \end{aligned}$$

equals were used.

Proof. From Lemma 1, we acquire

$$
\begin{aligned}
&\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \qquad (2.9)\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
&\left.\left.+\ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|\\
\leq\ &\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\\
&\times\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right.\\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\\
&\left.+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\right]
\end{aligned}
$$

Since $\frac{\partial^2\psi}{\partial\tau\partial\xi}$ is $h-$convex function on the co-ordinates on Δ, then one has:

$$
\begin{aligned}
&\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right.\\
&\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.\\
&\left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|\\
\leq\ &\frac{\gamma_1^{\theta}\gamma_2^{\vartheta}(\phi-\sigma)(\rho-\varsigma)}{16}\\
&\times\left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right.\\
&\left[h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right.\\
&\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau\\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\\
&\left[h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right.\\
&\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau\\
&+\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\\
&\left[h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right.\\
&\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau
\end{aligned}
$$

$$
\begin{aligned}
& +\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \\
& \left[h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|\right. \\
& \left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right]d\xi d\tau\Bigg\} \\
= \; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}\left[\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right. \\
& \left.+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\right]\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right],
\end{aligned}
$$

which finishes the proof. □

Remark 2.3. *In Theorem 2.2, if we choose $h(t)=t$, then we have,*

$$
\begin{aligned}
& \left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \\
& \times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
& \left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\leq\; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16\,\gamma_1\gamma_2}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right],
\end{aligned}
\tag{2.10}
$$

which is given by Kiris et al. in [32]

Remark 2.4. *In Remark 3 , if we choose $\gamma_1=1$ and $\gamma_2=1$, the following inequalities are achieved [31]*

$$
\begin{aligned}
& \left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\theta}(\rho-\varsigma)^{\vartheta}}\right. \\
& \times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
& \left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\leq\; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\frac{1}{\theta+1}\frac{1}{\vartheta+1}\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|\right].
\end{aligned}
\tag{2.11}
$$

Theorem 2.5. *Let $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on Δ with $0\leq\sigma<\phi,\ 0\leq\varsigma<\rho$. If $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q q>1$, is a $h-$convex function on the co-ordinates on Δ, then the inequality below holds.*

$$
\begin{aligned}
& \left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \\
& \times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right. \\
& \left.\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right| \\
\leq\; & \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{4(h_1+h_2)^2}{\gamma_1\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right]^{\frac{1}{p}} \\
& \times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}
\end{aligned}
\tag{2.12}
$$

where A is defined by (2.2) and $B(\cdot,\cdot)$ refers to the Beta function and $\frac{1}{p}=1-\frac{1}{q}$ and

$$
\int h\left(\frac{1+\tau}{2}\right)d\tau = \int h\left(\frac{1+\xi}{2}\right)d\xi = h_1
$$

$$\int h\left(\frac{1-\tau}{2}\right)d\tau = \int h\left(\frac{1-\xi}{2}\right)d\xi = h_2$$

equals were used.

Proof. From Lemma, we have inequality (2.9). In order to employ the well-known Hölder's inequality for double integrals, in I_5 and since $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q$ is The application of $h-$convex functions to the coordinates of the triangle yields the following result: on $\triangle$

$$\begin{aligned}
I_5 &= \left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right\}\\
&\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}} \qquad (2.13)\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\int_0^1\int_0^1(1-(1-\tau)^{\gamma_1})^{\theta p}(1-(1-\xi)^{\gamma_2})^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}\\
&\quad\times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.\\
&\quad\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left((h_1)^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+(h_2)^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

Where we take advantage of the fact:

$$(\varpi-\sigma)^j\leq\varpi^j-\sigma^j,$$

for any $\varpi>\sigma\geq 0$ and $j\geq 1$. And similarly,

$$\begin{aligned}
I_6 &= \left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\right\}\\
&\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}} \qquad (2.14)\\
&\quad\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}\\
&\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}\\
&\quad\times\left(h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}},
\end{aligned}$$

$$I_7=\left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta} \qquad (2.15)$$

$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\Bigg\}$$

$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$

$$\times\left(h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}$$

and

$$I_8=\left\{\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right. \tag{2.16}$$

$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau\Bigg\}$$

$$\leq \left(\int_0^1\int_0^1\left(\frac{1-(1-t)^{\gamma_1}}{\gamma_1}\right)^{\theta p}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta p}d\xi d\tau\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{\gamma_1^{\theta}}\frac{1}{\gamma_2^{\vartheta}}\left(\frac{1}{\gamma_1}\frac{1}{\gamma_2}B\left(\theta p+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right)^{\frac{1}{p}}$$

$$\times\left(h_2^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+h_1h_2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h_1^2\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right)^{\frac{1}{q}}.$$

If we substitute from (2.13)-(2.16) in (2.9), we obtain the first inequality of (2.12) is achieved. □

Remark 2.6. *If we set $h(t)=t$ in Theorem (2.5), then we have,*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\right. \tag{2.17}$$

$$\times\left[\gamma_1\gamma_2 I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+\gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+\gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$

$$\left.\left.+\ \gamma_1\gamma_2 I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\right|$$

$$\leq \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{16}{\gamma_1\gamma_2}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta p+1,\frac{1}{\gamma_2}\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}$$

which is given by Kiris et al. in [32].

Remark 2.7. *If we take $h(t)=t$, $\gamma_1=1$ and $\gamma_2=1$ in Theorem (2.5), the following inequalities are achieved*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)}{(\phi-\sigma)^{\alpha}(\rho-\varsigma)^{\beta}}\right. \tag{2.18}$$

$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$
$$\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\Bigg|$$
$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left[\frac{16}{(\theta p+1)(\vartheta p+1)}\right]^{\frac{1}{p}}$$
$$\times\left[\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right]^{\frac{1}{q}}$$

which is proven by Hyder et al. in [31].

Theorem 2.8. *Assume $\psi:\triangle\subset\mathbb{R}^2\to\mathbb{R}$ The mapping is partially differentiable with respect to Δ with $0\leq\sigma<\phi,\ 0\leq\varsigma<\rho$. If $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q q\geq 1$, is a h-convex function on the coordinates on Δ, then we have the following inequality:*

$$\Bigg|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\gamma_1\theta-1}2^{\gamma_2\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)\gamma_1^{\theta}\gamma_2^{\vartheta}}{(\phi-\sigma)^{\gamma_1\theta}(\rho-\varsigma)^{\gamma_2\vartheta}}\tag{2.19}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$
$$\left.+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right]-A\Bigg|$$
$$\leq\frac{(\phi-\sigma)(\rho-\varsigma)}{16\gamma_1\gamma_2}\left(B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\Bigg\{\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q$$
$$+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q$$
$$+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q$$
$$+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}$$
$$+\Bigg(\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q$$
$$+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}.$$

Here A is defined as in (2.2) and

$$\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau=\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)$$
$$\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau=\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)$$
$$\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)d\xi d\tau=\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)$$
$$\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)d\xi d\tau=\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h).$$

Proof. By employing the principle of equality (2.9) and the Power-Mean inequality, in I_9, we get

$$I_9=\left[\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau\right] \tag{2.20}$$

$$\le\left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|^q d\xi d\tau\right)^{\frac{1}{q}}$$

In light of the convexity observed in the h-convex function when expressed in co-ordinates, $\triangle$ of $\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\right|^q$, then we acquire

$$\le\left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}d\xi d\tau\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\right.$$
$$\times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$

In this inequality, the change of variables allows us to express it as follows:

$$\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau \tag{2.21}$$
$$\le\left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Psi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(a,c)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q)\right\}^{\frac{1}{q}}.$$

Similarly, we have

$$I_{10}=\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^{\theta}\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^{\vartheta}\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1+\tau}{2}\sigma+\frac{1-\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+s}{2}\rho\right)\right|d\xi d\tau \tag{2.22}$$
$$\times\left\{h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+s}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$

$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(a,d)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Psi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q)\right\}^{\frac{1}{q}}$$

and

$$I_{11}=\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^\theta\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^\vartheta \tag{2.23}$$
$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1+\xi}{2}\varsigma+\frac{1-\xi}{2}\rho\right)\right|d\xi d\tau$$
$$\times\left\{h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left\{\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Psi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q)\right\}^{\frac{1}{q}}.$$

Finally

$$I_{12}=\int_0^1\int_0^1\left(\frac{1-(1-\tau)^{\gamma_1}}{\gamma_1}\right)^\theta\left(\frac{1-(1-\xi)^{\gamma_2}}{\gamma_2}\right)^\vartheta \tag{2.24}$$
$$\times\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}\left(\frac{1-\tau}{2}\sigma+\frac{1+\tau}{2}\phi,\frac{1-\xi}{2}\varsigma+\frac{1+\xi}{2}\rho\right)\right|d\xi d\tau$$
$$\times\left\{h\left(\frac{1-\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+h\left(\frac{1-\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1-\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+h\left(\frac{1+\tau}{2}\right)h\left(\frac{1+\xi}{2}\right)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q d\xi d\tau\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}B\left(\theta+1,\frac{1}{\gamma_1}\right)B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right)^{1-\frac{1}{q}}\left(\frac{1}{4}\frac{1}{\gamma_1^{\theta+1}}\frac{1}{\gamma_2^{\vartheta+1}}\right.$$
$$\times\left\{\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\right.$$
$$\left.+\ \Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q+\Phi(\gamma_1,\theta,h)\Phi(\gamma_2,\vartheta,h)\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\right\}^{\frac{1}{q}}.$$

By considering (2.21)-(2.24) in (2.9), The desired inequality is thus obtained. (2.19). □

Remark 2.9. *If we assign $h(t)=t$ in Theorem 4, then we have following inequality [32] :*

$$\left|\frac{\psi(\sigma,\varsigma)+\psi(\sigma,\rho)+\psi(\phi,\varsigma)+\psi(\phi,\rho)}{4}+\frac{2^{\theta-1}2^{\vartheta-1}\Gamma(\theta+1)\Gamma(\vartheta+1)}{(\phi-\sigma)^\theta(\rho-\varsigma)^\vartheta}\right. \tag{2.25}$$
$$\times\left[{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\beta}_{\phi^-,\varsigma^+}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)+{}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\sigma^+,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\right.$$

$$
\begin{aligned}
&+ {}^{\gamma_1\gamma_2}I^{\theta,\vartheta}_{\phi^-,\rho^-}\psi\left(\frac{\sigma+\phi}{2},\frac{\varsigma+\rho}{2}\right)\Big]-A\Bigg| \\
\leq\ & \frac{(\phi-\sigma)(\rho-\varsigma)}{16}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left(\frac{1}{\theta+1}\frac{1}{\vartheta+1}\right)^{1-\frac{1}{q}} \\
&\times\Bigg\{\Bigg(\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q \\
&+\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{1}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
&+\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right] \\
&\times\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\rho)\right|^q\quad+\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
&+\left[B\left(\sigma+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
&+\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,d)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,\varsigma)\right|^q \\
&+\left[B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\sigma,d)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\varsigma)\right|^q \\
&+\left[2B\left(\theta+1,\frac{1}{\gamma_1}\right)-B\left(\theta+1,\frac{2}{\gamma_1}\right)\right]\left[2B\left(\vartheta+1,\frac{1}{\gamma_2}\right)-B\left(\vartheta+1,\frac{2}{\gamma_2}\right)\right]\left|\frac{\partial^2\psi}{\partial\tau\partial\xi}(\phi,\rho)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}
$$

3. Conclusion

In this study, we derived some inequalities of trapezoid type for coordinated h-convex functions by means of conformable fractional integrals. To obtain new inequalities and to generalize the obtained inequalities, it would be useful for future work to use different types of convex maps or different types of fractional integral operators.

Declarations

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