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# A new perspective to using statistical convergence with respect to power series method in measure

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ABSTRACT. In this paper, we study the concept of statistical convergence with respect to the power series method in measure, which is a new modification of the power series method. Also, using this convergence, we obtain the Lebesgue bounded convergence theorem and the fundamental theorems of measure theory. Finally, we prove Korovkin's theorem as an application using this notion of convergence.

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## 1. Introduction

Fast and Steinhauss independently introduced the concept of statistical convergence for sequences of real numbers [8,15]. The concept of statistical convergence has always been a popular topic and there are many studies in the literature. Recently, a new variant of statistical convergence, statistical convergence with power series method, was introduced by Ünver and Orhan [16]. Since this new convergence cannot be compared with statistical convergence, it has become the subject of study in many fields [2,4,5,7,17,18].

The convergence theorems in its simplest form states that integrability is preserved under taking limits. These theorems can be used to construct an integrable function. It also has an important place in measure theory. An essential and well-known one is the Lebesgue bounded convergence theorem. Recently, Kişi and Güler presented a statistical version of this theorem [10].

The aim of this paper is to construct convergence theorems using the concept of statistical convergence with respect to power series method in measure. We also analyse the fundamental theorems of measure theory according to this notion of convergence. In the last part, we aim to present a different perspective by proving Korovkin's theorem as an application.

Let us now give the basic definitions and notations necessary for our study.

Let $E$ be a subset of $\mathbb{N}$, the set natural numbers. Then the natural density of $E$, denoted by $\delta(E)$, is given by:

$$\delta(E) := \lim \frac{1}{n} \left|\{k \leq n : \ k \in E\}\right|$$

whenever this limit exists, where $|.|$ denotes the cardinality of the set [13].

A sequence $x = \{x_k\}$ of real numbers is statistically convergent to $L$ provided that, for every $\varepsilon > 0$,

$$\lim \frac{1}{n} \left|\{k \leq n : |x_k - L| \geq \varepsilon\}\right| = 0$$

that is,

$$E := E_n(\varepsilon) := \{k \leq n : \ |x_k - L| \geq \varepsilon\}$$

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has natural density zero. This is denoted by $st-\lim x_k = L$ [8,15]. It is clear from the definition that every convergent sequence (in the usual sense) is statistically convergent to the same number. However, statistically convergent sequence need not to be convergent.

We now start with some definitions and notations that are essential for power series methods.

Let $\{p_k\}$ be a non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(t) := \sum_{k=0}^{\infty} p_k t^k$$

has radius of convergence $R$ with $0 < R \leq \infty$. Consider a sequence of real numbers $x = \{x_k\}$. Then $x = \{x_k\}$ is convergent to $L$ in the sense of power series method if

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k=0}^{\infty} p_k t^k x_k = L.$$

This is denoted by $P_p - \lim x_k = L$ [12, 14]. For this study it should be noted that the power series method is regular if and only if

$$\lim_{0<t\to R^-} \frac{p_k t^k}{p(t)} = 0$$

for every $k \in \mathbb{N}_0$ [14].

Let us now give the concept of $P-$statistical convergence and some of its properties expressed by Ünver and Orhan [16].

**Definition 1.** *[16] Let $E \subset \mathbb{N}_0$. If the limit*

$$\delta_p(E) := \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k\in E} p_k t^k$$

*exists, then $\delta_p(E)$ is called the $P-$density of $E$. Note that from the definition of the power series method and $P-$density it is obvious that $0 \leq \delta_p(E) \leq 1$ whenever it exists.*

Let us now state below some properties of the concept of $P-$density:

i) $\delta_p(\mathbb{N}) = 1$,

ii) if $E \subset F$ then $\delta_p(E) \leq \delta_p(F)$,

iii) if $E$ has $P-$density then $\delta_p(\mathbb{N}/E) = 1 - \delta_p(E)$,

iv) if $E, F$ have $P-$density then $\delta_p(E \cup F) \leq \delta_p(E) + \delta_p(F)$.

**Definition 2.** *[16] Let $x = \{x_k\}$ be a sequence. Then $x$ is said to be statistically convergent with respect to power series method ($P-$statistically convergent) to $L$ if for any $\varepsilon > 0$,*

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k\in E_\varepsilon} p_k t^k = 0$$

*where $E_\varepsilon = \{k \in \mathbb{N}_0 : |x_k - L| \geq \varepsilon\}$. This is denoted by $st_P - \lim x_k = L$.*

Let us show with the following example that the concepts of statistical convergence and $P-$statistical convergence are not comparable.

**Example 1.** *Let $\{p_k\}$ be defined as follows*

$$p_k = \begin{cases} 1, & k = m^2 \\ 0, & otherwise \end{cases}, \quad m = 1, 2, ...$$

*and take the sequence $\{x_k\}$ defined by*

$$x_k = \begin{cases} 1, & k = m^2 \\ k, & otherwise \end{cases}, \quad m = 1, 2, ....$$

*We calculate that, since for any $\varepsilon > 0$ we have,*

$$\lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{k:|x_k-1|\geq\varepsilon} p_k t^k = 0,$$

*that is* $st_P - \lim x_k = 1$. *However, the sequence* $\{x_k\}$ *is not statistically convergent to* $1$. *If we consider the following sequence* $\{s_k\}$ *defined by*

$$s_k = \begin{cases} k, & k = m^2 \\ 1, & otherwise \end{cases}, \ m = 1, 2, ...,$$

*then* $st - \lim s_k = 1$, *but* $\{s_k\}$ *is not* $P-$*statistically convergent to* $1$.

**Definition 3.** *[6] A sequence of real numbers* $x = \{x_k\}$ *is said to be* $P-$*statistically bounded if for some* $M > 0$ , $\delta_p (\{k : |x_k| > M\}) = 0$.

**Theorem 1.** *[1] Let* $x = \{x_k\}$ *be a sequence. Then, the following expressions are equivalent:*
*(i)* $st_P - \lim x_k = L$,
*(ii) There it exists a subset* $F$ *of* $\mathbb{N}_0$ *such that* $\delta_p (F) = 1$ *and* $\lim_{k \in F} x_k = L$,
*(iii) There exist two sequences* $y = \{y_k\}$ *and* $z = \{z_k\}$ *such that* $x = y + z$ *and* $\lim y_k = L$ *and* $st_P - \lim z_k = 0$.

Throughout this paper $(X, \mathcal{A}, \mu)$ will denote a measure space, $f_n$, $f : X \to \mathbb{R}$, $n \in \mathbb{N}$ , measurable functions. Let us recall the following theorem and give the definitions and notations related to our work.

**Theorem 2.** *[9] Let a sequence of measurable functions* $\{f_k\}$ *converge in measure to* $f$. *Then one can select from* $\{f_k\}$ *a subsequence* $\{f_{k_n}\}$ *which converges to* $f$ *almost everywhere.*

**Definition 4.** *Let* $f_k$, $f$ *be measurable functions* $(k = 1, 2, ...)$ *on* $X$. *Then,* $\{f_k\}$ *is* $P-$*statistically convergent in measure to* $f$ *if for each* $\varepsilon > 0$, $\eta > 0$,

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k \in E_k(\varepsilon)} p_k t^k = 0,$$

*where* $E_k (\varepsilon) := \{k \in \mathbb{N}_0 : \mu (\{x \in X : |f_k (x) - f (x)| \geq \varepsilon\}) \geq \eta \}$. *Then we write* $f_k \overset{st_P - \mu}{\to} f$.

## 2. Main Results

In this section, we first construct the convergence theorem using the concept of statistical convergence with respect to the power series method in measure. Then, the theorems that are important in measure theory will be analyzed with this type of convergence.

**Theorem 3.** *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space with* $\mu (X) < \infty$ *and* $f_k$, $F$ *are bounded measurable functions. Assume that*

$$f_k \overset{st_P - \mu}{\to} F.$$

*If there exists a constant* $M$ *such that for almost all* $x$,

$$\delta_P (\{k \in \mathbb{N}_0 : \ |f_k| \geq M\}) = 0, \tag{1}$$

*then*

$$st_P - \lim \int_X f_k dx = \int_X F dx.$$

*Proof.* Let $f_k \overset{st_P - \mu}{\to} F$. In this case, for every $\varepsilon > 0$,

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k \in E_k(\varepsilon)} p_k t^k = 0,$$

where $E_k (\varepsilon) := \{k \in \mathbb{N}_0 : \mu (\{x \in X : |f_k - F| \geq \varepsilon\}) \geq \eta \}$.
According to Theorem 1, if $K = \{k_n : k_n \leq k_{n+1}, \ n \in \mathbb{N}\}$, with $\delta_p (K) = 1$, then

$$\lim \mu (x \in X : \ |f_{k_n} - F| \geq \varepsilon).$$

In this case, $f_{k_n} \overset{\mu}{\to} F$.
According to Theorem 2, we can write

$$|f_{k_n} - F| < \varepsilon$$

for almost all $x \in X$. From the above inequality and since the power series method is regular, we can write the following

$$\delta_P\left(n \in \mathbb{N}_0 : |f_{k_n} - F| < \varepsilon\right) = 1$$

and

$$\delta_P\left(n \in \mathbb{N}_0 : |f_{k_n}| < M\right) = 1.$$

Since, for each $n \in \mathbb{N}_0$,

$$|F| \leq |F - f_{k_n}| + |f_{k_n}| < \varepsilon + M,$$

we have $|F| \leq \varepsilon + M$.

Hence, we get $|F| \leq M$.

Now let $\lambda > 0$ be a positive number. Set

$$A_k(\lambda) = \{x \in X : |f_k - F| \geq \lambda\}, \; B_k(\lambda) = \{x \in X : |f_k - F| < \lambda\}.$$

Then

$$\begin{aligned} & \left\{k \in \mathbb{N}_0 : \left|\int_X f_k dx - \int_X F dx\right| \geq \lambda\right\} \\ \subset \quad & \left\{k \in \mathbb{N}_0 : \int_X |f_k - F| \, dx \geq \lambda\right\} \\ = \quad & \left\{k \in \mathbb{N}_0 : \int_{A_k(\lambda)} |f_k - F| \, dx \geq \frac{\lambda}{2}\right\} \\ & \cup \left\{k \in \mathbb{N}_0 : \int_{B_k(\lambda)} |f_k - F| \, dx \geq \frac{\lambda}{2}\right\}. \end{aligned} \tag{2}$$

for almost all $x \in X$. Then it follows from that

$$\{k \in \mathbb{N}_0 : |f_k - F| \geq 2M\} \subset \{k \in \mathbb{N}_0 : |f_k| \geq M\},$$

which gives that for almost all $x \in A_k(\lambda)$

$$\delta_P\left(k \in \mathbb{N}_0 : |f_k - F| \geq 2M\right) \leq \delta_P\left(k \in \mathbb{N}_0 : |f_k| \geq M\right).$$

Using (1), we immediately get that

$$\delta_P\left(k \in \mathbb{N}_0 : \; |f_k - F| \geq 2M\right) = 0.$$

We can state the following

$$\int_{A_k(\lambda)} |f_k - F| \, dx \leq 2M.\mu\left(A_k(\lambda)\right).$$

Also it is easy to verify

$$\left\{k \in \mathbb{N}_0 : \int_{A_k(\lambda)} |f_k - F| \, dx \geq \frac{\lambda}{2}\right\} \subset \left\{k \in \mathbb{N}_0 : \mu\left(A_k(\lambda)\right) \geq \frac{\lambda}{4M}\right\}$$

for almost all $x \in X$ and $\lambda > 0$. Since

$$\delta_P\left\{k \in \mathbb{N}_0 : \mu\left(A_k(\lambda)\right) \geq \frac{\lambda}{4M}\right\} = 0,$$

then, we have

$$\delta_P\left\{k \in \mathbb{N}_0 : \int_{A_k(\lambda)} |f_k - F| \, dx \geq \frac{\lambda}{2}\right\} = 0. \tag{3}$$

On the other hand, observe the following for almost all $x \in B_k(\lambda)$

$$\int\limits_{B_k(\lambda)} |f_k - F|\, dx \leq \lambda.\mu\left(B_k(\lambda)\right) \leq \lambda.\mu(X).$$

Again it is easy to verify

$$\left\{ k \in \mathbb{N}_0 : \int\limits_{B_k(\lambda)} |f_k - F|\, dx \geq \lambda.\mu(X) \right\} \subset \left\{ k \in \mathbb{N}_0 : \mu\left(B_k(\lambda)\right) \geq \mu(X) \right\}$$

for almost all $x \in X$ and $\lambda > 0$. Since

$$\delta_P \left\{ k \in \mathbb{N}_0 : \int\limits_{B_k(\lambda)} |f_k - F|\, dx \geq \lambda.\mu(X) \right\} \leq \delta_P \left\{ k \in \mathbb{N}_0 : \mu\left(B_k(\lambda)\right) \geq \mu(X) \right\} = 0,$$

then, we have

$$\delta_P \left\{ k \in \mathbb{N}_0 : \int\limits_{B_k(\lambda)} |f_k - F|\, dx \geq \lambda.\mu(X) \right\} = 0. \tag{4}$$

Given (2), we can consider the following sets for $\lambda > 0$,

$$\begin{aligned}
\mathcal{R} &:= \left\{ k \in \mathbb{N}_0 : \left| \int\limits_X f_k dx - \int\limits_X F dx \right| \geq \lambda \right\}, \\
\mathcal{R}_1 &:= \left\{ k \in \mathbb{N}_0 : \int\limits_{A_k(\lambda)} |f_k - F|\, dx \geq \frac{\lambda}{2} \right\}, \\
\mathcal{R}_2 &:= \left\{ k \in \mathbb{N}_0 : \int\limits_{B_k(\lambda)} |f_k - F|\, dx \geq \frac{\lambda}{2} \right\}.
\end{aligned}$$

Then, it follows from that

$$\mathcal{R} \subset \mathcal{R}_1 \cup \mathcal{R}_2$$

which gives that

$$\frac{1}{p(t)} \sum_{k \in \mathcal{R}} p_k t^k \leq \frac{1}{p(t)} \sum_{k \in \mathcal{R}_1} p_k t^k + \frac{1}{p(t)} \sum_{k \in \mathcal{R}_2} p_k t^k. \tag{5}$$

Now letting $0 < t < R^-$ in the both sides of (5) and using (3) and (4), we immediately get that

$$\lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k \in \mathcal{R}} p_k t^k \leq \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k \in \mathcal{R}_1} p_k t^k + \lim_{0<t \to R^-} \frac{1}{p(t)} \sum_{k \in \mathcal{R}_2} p_k t^k.$$

In this case we obtain the following

$$\delta_P \left\{ k \in \mathbb{N}_0 : \left| \int\limits_X f_k dx - \int\limits_X F dx \right| \geq \varepsilon \right\} = 0,$$

which proves the theorem. □

**Theorem 4.** *If a sequence of functions $\{f_k\}$ converges $P-$statistically in measure to the functions $f$ and $g$, then these limit functions are equal.*

*Proof.* Suppose $f_k \overset{st_P - \mu}{\rightarrow} f$ and $f_k \overset{st_P - \mu}{\rightarrow} g$. Then for every $\varepsilon > 0$, we have

$$\begin{aligned}
st_P - \lim \mu\left(x : \ |f_k - f| > \varepsilon\right) &= 0, \\
st_P - \lim \mu\left(x : \ |f_k - g| > \varepsilon\right) &= 0.
\end{aligned} \tag{6}$$

To show that $f$ and $g$ are equal a.e., on $X$, let us assume the contrary, that is $\mu\left(x:\ f\neq g\right)>0$. Then since $f\neq g$ if and only if $|f-g|>0$, we have

$$\mu\left(x\in X:|\ f-g|>0\right)>0.$$

Now, since

$$\{x\in X:|\ f-g|>0\ \}\subset\bigcup_{k=1}^{\infty}\left\{x:|\ f-g|\geq\frac{1}{k}\ \right\},\tag{7}$$

we have

$$\mu\left(\{x\in X:|\ f-g|>0\ \}\right)\leq\sum_{k=1}^{\infty}\mu\left(\left\{x:|\ f-g|\geq\frac{1}{k}\ \right\}\right).\tag{8}$$

By (7), the left side of (8) is positive. Then not all of terms on the right side are equal to 0. Thus there exists some $k_0\in\mathbb{N}_0$ such that

$$\mu\left(\left\{x\in X:|\ f-g|\geq\frac{1}{k_0}\ \right\}\right)>0.$$

For every $k\in\mathbb{N}_0$ we have

$$\begin{aligned}\mu\left(\left\{x\in X:|\ f-g|\geq\frac{1}{k_0}\ \right\}\right)\ &\leq\ \mu\left(\left\{x\in X:|\ f-f_k|\geq\frac{1}{2k_0}\ \right\}\right)\\&\quad+\mu\left(\left\{x\in X:|f_k-g|\geq\frac{1}{2k_0}\ \right\}\right).\end{aligned}$$

We can consider the following sets for $\eta>0$:

$$\begin{aligned}\mathcal{V}\ &:=\left\{k\in\mathbb{N}_0:\mu\left(\left\{x:|\ f-g|\geq\frac{1}{k_0}\ \right\}\right)\geq\eta\ \right\},\\\mathcal{V}_1\ &:=\left\{k\in\mathbb{N}_0:\mu\left(\left\{x:|\ f-f_k|\geq\frac{1}{2k_0}\ \right\}\right)\geq\eta\ \right\},\\\mathcal{V}_2\ &:=\left\{k\in\mathbb{N}_0:\mu\left(\left\{x:|f_k-g|\geq\frac{1}{2k_0}\ \right\}\right)\geq\eta\ \right\}.\end{aligned}$$

Then it follows from that

$$\mathcal{V}\subset\mathcal{V}_1\cup\mathcal{V}_2$$

which gives that

$$\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}}p_kt^k\leq\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}_1}p_kt^k+\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}_2}p_kt^k.\tag{9}$$

Now letting $0<t<R^-$ in the both sides of (9) and using (6), we immediately get that

$$\lim_{0<t\to R^-}\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}}p_kt^k\leq\lim_{0<t\to R^-}\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}_1}p_kt^k+\lim_{0<t\to R^-}\frac{1}{p\left(t\right)}\sum_{k\in\mathcal{V}_2}p_kt^k=0.$$

We obtain the following

$$\delta_P\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{x\in X:|\ f-g|\geq\frac{1}{k_0}\ \right\}\right)\geq\eta\ \right\}\right)=0$$

which proves the theorem. □

**Theorem 5.** *Let the sequence of functions $\{f_k\}$ converge $P-$statistically in measure to $f$ on $X$. Assume that $\Psi$ is a real function that satisfies Lipschitz condition on $\mathbb{R}$. If under these conditions, for each $k\in\mathbb{N}_0$, the sequence $\{\Psi\circ f_k\}$ is given on $X$ then, $\{\Psi\circ f_k\}$ converges $P-$statistically in measure to the functions $\Psi\circ f$.*

*Proof.* Let $\varepsilon>0$ and the sequence of functions $\{f_k\}$ converge $P-$ statistically in measure to the function $f$, then

$$st_P-\lim\mu\left(\{x\in X:\ \ |f_k-f|>\varepsilon\}\right)=0$$

and also let $\Psi:\mathbb{R}\to\mathbb{R}$ satisfies Lipschitz condition. Hence there is $L>0$ such that

$$|\Psi(x)-\Psi(y)|\leq L|x-y|$$

for each $x,y\in\mathbb{R}$. Observe this

$$\{x\in X:\ |\Psi\circ f_k-\Psi\circ f|>\varepsilon\}\subset\left\{x\in X:\ |f_k-f|>\frac{\varepsilon}{L}\right\}.$$

We can consider the following sets for $\eta>0$

$$\begin{aligned}E_1&=\{k\in\mathbb{N}_0:\ \mu(\{x\in X:\ |\Psi\circ f_k-\Psi\circ f|>\varepsilon\})\geq\eta\},\\ E_2&=\left\{k\in\mathbb{N}_0:\ \mu\left(\left\{x\in X:|f_k-f|>\frac{\varepsilon}{L}\right\}\right)\geq\eta\right\}.\end{aligned}$$

Since

$$E_1\subset E_2$$

and also, the sequence of functions $\{f_k\}$ converges $P-$statistically in measure to the function $f$, then it holds that

$$\delta_P\left(\left\{k\in\mathbb{N}_0:\mu\left(\left\{x:|f_k-f|>\frac{\varepsilon}{L}\right\}\right)\geq\eta\ \right\}\right)=0.$$

By monotonicity of the measure,

$$\delta_P(\{k\in\mathbb{N}_0:\mu(\{x:\ |\Psi\circ f_k-\Psi\circ f|>\varepsilon\})\geq\eta\ \})=0.$$

Hence the sequence of functions $\{\Psi\circ f_k\}$ converges $P-$statistically in measure to the function $\Psi\circ f$. □

## 3. Further Results

In this section, the concept of convergence of the power series method in measure will be introduced. Also, the relationship between it and the convergence we have examined in the previous section will be discussed. A version of the theorems given in the main results section will be stated.

Similar to Definition 4, let the concept of the power series method in measure be presented as follows:

Let $f_k$, $f$ be measurable functions $(k=1,2,...)$ on $X$. Then, $f_k$ is convergent in the sense of power series method in measure to $f$ if for each $\varepsilon>0$,

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_kt^ke_k=0,$$

where $e_k:=\mu(\{x\in X:|f_k-f|\geq\varepsilon\})$ . This is denoted by $f_k\overset{P_p-\mu}{\to}f$.

The following theorem shows the relation between these two convergences.

**Theorem 6.** *If* $f_k\overset{P_p-\mu}{\to}f$, *then* $f_k\overset{st_P-\mu}{\to}f$.

*Proof.* Let $f_k\overset{P_p-\mu}{\to}f$. Then for $\varepsilon>0$, we have

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{k=0}^{\infty}p_kt^ke_k=0,\tag{10}$$

where $e_k:=\mu(\{x\in X:|f_k(x)-f(x)|\geq\varepsilon\})$.

Now, $E_k(\varepsilon):=\{k\in\mathbb{N}_0:\mu(\{x\in X:|f_k(x)-f(x)|\geq\varepsilon\})\geq\eta\ \}$.

This implies

$$\frac{1}{p(t)}\sum_{k\in E_k(\varepsilon)}p_kt^k\leq\frac{1}{p(t)}\sum_{k=0}^{\infty}p_kt^ke_k.$$

Using (10), we get

$$\lim_{0<t\to R^-}\frac{1}{p(t)}\sum_{k\in E_k(\varepsilon)}p_kt^k=0$$

and hence, the result. □

Now, the following results are stated. Their proofs are obtained using a technique similar to the theorems in the main results section.

**Corollary 1.** *Let $(X,\mathcal{A},\mu)$ be a finite measure space with $\mu(X)<\infty$ and $f_k$, $f$ are bounded measurable functions. Assume that*

$$f_k \overset{P_p-\mu}{\to} F.$$

*If there exists a constant $M$ such that for almost all $x$,*

$$\delta_P\left(\{k\in\mathbb{N}_0:\ |f_k(x)|\geq M\}\right)=0,$$

*then*

$$P_p-\lim\int\limits_X f_k dx=\int\limits_X F dx.$$

**Corollary 2.** *If a sequence of functions $\{f_k\}$ converges the in the sense of power series method in measure to the functions $f$ and $g$, then these limit functions are equal.*

**Corollary 3.** *Let the sequence of functions $\{f_k\}$ converge in the sense of power series method in measure to $f$ on $X$. Assume that $\Psi$ is a real function that satisfies Lipschitz condition on $\mathbb{R}$. If under these conditions, for each $k\in\mathbb{N}_0$, the sequence $\{\Psi\circ f_k\}$ is given on $X$ then, $\{\Psi\circ f_k\}$ converges in the sense of power series method in measure to the function $\Psi\circ f$.*

## 4. An Application

In this section, we will prove the classical Korovkin theorem [11], which has an important place in approximation theory, using the statistical convergence with the power series method in measure. Let $K$ be a compact subset of the real numbers and $C(K)$ be the space of all real-valued continuous functions on $K$. Then $C(K)$ is a Banach space with the norm defined by $\|f\|=\sup\limits_{t\in[a,b]}|f(t)|,\ f\in C(K).$

**Theorem 7.** *Let $P_p$ be a regular power series method and let $\{L_k\}$ be a sequence of positive linear operators from $C(K)$ into $C(K)$. If*

$$L_k f_i \overset{st_P-\mu}{\to} f_i \text{ on } K \tag{11}$$

*where $f_i(x)=x^i$ for each $i=0,1,2,$ then*

$$L_k f \overset{st_P-\mu}{\to} f \text{ on } K, \text{ for all } f\in C(K). \tag{12}$$

*Proof.* Let $f\in C(K)$ and $t\in K$. Then there exists $M>0$ such that $|f(t)|\leq M$ for all $t\in K$. Also, for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-f(t)|<\varepsilon$ whenever $|x-t|<\delta$ for all $x,t\in K$. Let $K_\delta:=([t-\delta,t+\delta])\cap K$. Therefore

$$\begin{aligned} |f(x)-f(t)| &= |f(x)-f(t)|\chi_{K_\delta}(x)+|f(x)-f(t)|\chi_{K\backslash K_\delta}(x) \\ &\leq \varepsilon+2M\chi_{K\backslash K_\delta}(x) \end{aligned} \tag{13}$$

where $\chi_A$ is the characteristic function of the set $A$. We have

$$\chi_{K\backslash K_\delta}(x)\leq\frac{(x-t)^2}{\delta^2}. \tag{14}$$

From (13) and (14), we get

$$|f(x)-f(t)|\leq\varepsilon+\frac{2M}{\delta^2}(x-t)^2. \tag{15}$$

By using monotonicity and linearity of $L_k(f;t)$, we get

$$|L_k(f;t)-f(t)|\leq L_k(|f(x)-f(t)|;t)+|f(t)|\,|L_k(f_0;t)-f_0(t)|. \tag{16}$$

Using (15) in (16), we obtain

$$\begin{aligned} |L_k(f;t)-f(t)| &\leq L_k\left(\left(\varepsilon+\frac{2M}{\delta^2}(x-t)^2\right);t\right)+|f(t)|\,|L_k(f_0;t)-f_0(t)| \\ &\leq \varepsilon+(\varepsilon+M)\,|L_k(f_0;t)-f_0(t)|+\frac{2M}{\delta^2}L_k\left((x-t)^2;t\right). \end{aligned} \tag{17}$$

So we get

$$
\begin{aligned}
|L_k(f;t) - f(t)| \quad &\leq \quad \varepsilon + \left(\varepsilon + M + \frac{2M\|f_2\|}{\delta^2}\right)|L_k(f_0;t) - f_0(t)| \\
&\quad + \frac{4M\|f_1\|}{\delta^2}|L_k(f_1;t) - f_1(t)| \\
&\quad + \frac{2M}{\delta^2}|L_k(f_2;t) - f_2(t)|,
\end{aligned}
$$

which implies that

$$
|L_k(f;t) - f(t)| \leq \varepsilon + H\{|L_k(f_0;t) - f_0(t)| + |L_k(f_1;t) - f_1(t)| + |L_k(f_2;t) - f_2(t)|\} \tag{18}
$$

where $H = \max\left\{\varepsilon + M + \frac{2M\|f_2\|}{\delta^2},\ \frac{4M\|f_1\|}{\delta^2},\ \frac{2M}{\delta^2}\right\}$.

Now, for a $r > 0$, there exists $\delta, \varepsilon > 0$ such that $\varepsilon < r$. Then let

$$
\mathcal{T}_k = \{k \in \mathbb{N}_0 : \mu(\{x : |L_k(f;t) - f(t)| \geq r\}) \geq \delta\}
$$

and for $i = 0, 1, 2$,

$$
\mathcal{T}_{k,i} = \left\{k \in \mathbb{N}_0 : \mu\left(\left\{x : |L_k(f_i;t) - f_i(t)| \geq \frac{r - \varepsilon}{3H}\right\}\right) \geq \delta\right\}.
$$

From (18), we obtain

$$
\mathcal{T}_k \subset \bigcup_{i=0}^{2} \mathcal{T}_{k,i}
$$

and so we get

$$
\frac{1}{p(t)}\sum_{k \in \mathcal{T}_k} p_k t^k \leq \sum_{i=0}^{2} \frac{1}{p(t)} \sum_{k \in \mathcal{T}_{k,i}} p_k t^k.
$$

Hence, using (11), we get

$$
L_k f \overset{st_P - \mu}{\rightarrow} f \text{ on } K, \text{ for all } f \in C(K). \tag{19}
$$

This completes the proof. □

Now the following theorem follows immediately from Theorem 6.

**Theorem 8.** *Let $P_p$ be a regular power series method and let $\{L_k\}$ be a sequence of positive linear operators from $C(K)$ into $C(K)$. If*

$$
L_k f_i \overset{P_p - \mu}{\rightarrow} f_i \text{ on } K \tag{20}
$$

*where $f_i(x) = x^i$ for each $i = 0, 1, 2$, then*

$$
L_k f \overset{P_p - \mu}{\rightarrow} f \text{ on } K, \text{ for all } f \in C(K). \tag{21}
$$

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## References


[1] Bayram Şahin, N., Criteria for statistical convergence with respect to power series methods, *Positivity*, 25(3) (2021), 1097–1105. https://doi.org/10.1007/s11117-020-00801-6.

[2] Bayram Şahin, N., Yıldız, S., Approximation by statistical convergence with respect to power series methods, *Hacet. J. Math. Stat.*, 51(4), 51–64. https://doi.org/10.15672/hujms.1022072.

[3] Boos, J., Classical and Modern Methods in Summability, Oxford University Press, 2000.

[4] Çınar, S., Yıldız, S., $P-$statistical summation process of sequences of convolution operators, *Indian J. Pure Appl. Math.*, 53(3) (2022), 648–659. https://doi.org/10.1007/s13226-021-00156-y.

[5] Demirci, K., Đurčić, D., Kočinac, Lj. D. R., Yıldız, S., A theory of variations via P-statistical convergence, *Publ. Inst. Math. (Beograd) (N.S.)*, 110(124) (2021), 11–27. https://doi.org/10.2298/PIM2123011D.

[6] Demirci, K., Yıldız, S., Dirik, F., Statistical convergence with respect to power series method on time scales, *Filomat*, 38(13) (2024), 4775—4789. https://doi.org/10.2298/FIL2413775D.

[7] Demirci, K., Yıldız, S., Dirik, F., Approximation via power series method in two-dimensional weighted spaces, *Bull. Malays. Math. Sci. Soc.*, 43 (2020), 3871–3883. https://doi.org/10.1007/s40840-020-00902-1.

[8] Fast, H., Sur la convergence statistique, *Colloq. Math.*, 2 (1951), 241–244.

[9] Federer, H., Geometric Measure Theory, Springer, New York, 1969.
[10] Kişi, O., Guler, E., On statistical convergence with respect to measure, *J. Class. Anal*,10(1) (2017), 77–85. https://doi.org/10.7153/jca-10-08.
[11] Korovkin, P.P., Linear operators and approximation theory, Hindustan Publ. Co., Delhi, 1960.
[12] Kratz, W., Stadtmüller, U., Tauberian theorems for $J_p$-summability, *J. Math. Anal. Appl.*, 139 (1989), 362–371.
[13] Niven, I., Zuckerman H.S., An Introduction to the theory of numbers, John Wiley and Sons Fourt Ed., New York, 1980.
[14] Stadtmüller, U., Tali, A., On certain families of generalized Nörlund methods and power series methods, *J. Math. Anal. Appl.*, 238 (1999), 44–66.
[15] Steinhaus, H., Sur la convergence ordinaire et la convergence asymtotique, *Colloq. Math.*, 2 (1951), 73–74.
[16] Ünver, M., Orhan, C., Statistical convergence with respect to power series methods and applications to approximation theory, *Numer. Funct. Anal. Optim.*, 40(5) (2019), 535–547. https://doi.org/10.1080/01630563.2018.1561467.
[17] Ünver, M., Bayram Şahin, N., On statistical convergence with respect to power series methods, *Positivity*, 26(3) (2022), 123–139. https://doi.org/10.1007/s11117-022-00921-1.
[18] Yıldız, S., Bayram Şahin, N., Approximation theorems via $Pp-$statistical convergence on weighted spaces, *Math. Slovaca*, 74(3) (2024), 665–678. https://doi.org/10.1515/ms-2024-0050.