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Research Article

Some characterizations of BMO spaces via commutators of maximal functions on Morrey-Lorentz spaces

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Abstract

In this paper, we investigate the commutators of the fractional maximal function and the sharp maximal function on Morrey-Lorentz spaces. Furthermore, we present some new characterizations of BMO spaces.

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1. Introduction and main results

Let T be the classical singular integral operator, the commutator $[b, T]$ generated by T and a suitable function b is given as

$$[b, T]f(x) = bTf(x) - T(bf)(x).$$

A significant conclusion of Coifman, Rochberg and Weiss[5] showed that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson[16] gave some characterizations of the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ via the commutator $[b, T]$ and proved that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ $(0 < \beta < 1)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{n}$. Recently, many authors have conducted extensive studies on the theory of commutators, as it plays a important role in harmonic analysis and partial differential equations, see for example [6, 8, 19, 20, 23].

As usual, let $B := B(x, r)$ denote the ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We define $|B|$ as the Lebesgue measure of the ball B and let χ_B represent the characteristic function of the ball B. Define $L^1_{\text{loc}}(\mathbb{R}^n)$ as the set of all locally integrable functions on $\mathbb{R}^n$. For $1 \leq p < \infty$, we define the conjugate index of p as $p' = \frac{p}{p-1}$. We will use the symbol C to refer to a positive constant that is independent of the main parameters, but it may vary from line to line. The notation $f \lesssim g$ indicates that $f \leq Cg$. If $f \lesssim g$ and $g \lesssim f$ we write $f \approx g$.

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Let $0 \le \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the fractional maximal function $M_\alpha(f)$ is defined as follows:

$$M_\alpha(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

When $\alpha = 0$, $M_0(f)$ corresponds to the classical Hardy-Littlewood maximal function. For $0 < \alpha < n$, $M_\alpha(f)$ represents the classical fractional maximal function.

The sharp maximal function $M^\sharp(f)$ was introduced by Fefferman and Stein [9] and is defined as follows:

$$M^\sharp(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x and $f_B := \frac{1}{|B|}\int_B f(x)dx$.

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the maximal commutator of the fractional maximal function $M_\alpha(f)$ is defined by

$$M_{\alpha,b}(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

The nonlinear commutator of fractional maximal function $M_\alpha(f)$ is given as

$$[b, M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x).$$

For $\alpha = 0$, we simply write by $M_b = M_{0,b}$ and $[b, M] = [b, M_0]$.

For a function b defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0, & \text{if } b(x) \ge 0. \\ |b(x)|, & \text{if } b(x) < 0. \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Clearly, $b(x) = b^+(x) - b^-(x)$.

Let $b \ge 0$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\begin{aligned} |[b, M_\alpha] f(x)| &= |b(x)M_\alpha f(x) - M_\alpha(bf)(x)| \\ &= \left| b(x) \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|dy - \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(y)f(y)|dy \right| \\ &\le \sup_{B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |b(x) - b(y)||f(y)|dy \\ &= M_{b,\alpha}(f)(x). \end{aligned}$$

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then, for $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$|[b, M_\alpha] f(x)| \le M_{b,\alpha}(f)(x) + 2b^-(x)M_\alpha f(x) \tag{1.1}$$

holds (see, for example, [28]). Indeed, the commutators $M_{\alpha,b}$ and $[b, M_\alpha]$ evidently differ from each other. The maximal commutator $M_{\alpha,b}$ is both positive and sublinear, while the nonlinear commutator $[b, M_\alpha]$ does not possess either property. Many authors have intensively studied the mapping properties of commutators of maximal functions, we refer the readers to see [1–4, 11–14, 22, 24, 25] and therein references.

For a given ball B and $0 \le \alpha < n$, the fractional maximal function with respect to B of a function f is defined as follows:

$$M_{\alpha,B}(f)(x) = \sup_{B \supseteq B_0 \ni x} \frac{1}{|B_0|^{1-\frac{\alpha}{n}}} \int_{B_0} |f(y)|dy,$$

where the supremum is taken over all balls B_0 with $B_0 \subseteq B$ and $B_0 \ni x$. Also, we define $M_B = M_{0,B}$ for $\alpha = 0$.

The space of functions with bounded mean oscillation, denoted as $BMO(\mathbb{R}^n)$, was introduced by John and Nirenberg [17].

Definition 1.1. The space $BMO(\mathbb{R}^n)$ consists of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_B \frac{1}{|B|}\int_B |f(x) - f_B|dx < \infty,$$

where the supremum is taken over all balls in $\mathbb{R}^n$.

Let $0 < p < \infty$, the Lebesgue space $L^p(\mathbb{R}^n)$ consists of all functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ that satisfy the following condition:

$$\|f\|_{L^p(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |f(x)|^p \, \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

We also need to review the decreasing rearrangement of a real function f. For $s > 0$ and $t > 0$, we define the distribution function d_f and the rearrangement function f^* as follows:

$$d_f(s) = |\{x \in \mathbb{R}^n : |f(x)| > s\}|, \quad f^*(t) = \inf\{s > 0 : d_f(s) \le t\}.$$

We will now revisit the definition of Lorentz spaces.

Definition 1.2 ([18])**.** Given a measurable function f on $\mathbb{R}^n$ and $0 < q, r \le \infty$, we define

$$\|f\|_{L^{q,r}(\mathbb{R}^n)} := \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{q}} f^*(t)\right)^r \frac{dt}{t}\right)^{\frac{1}{r}}, & \text{if } r < \infty, \\ \sup_{t>0} t^{\frac{1}{q}} f^*(t), & \text{if } r = \infty. \end{cases}$$

Thus, the Lorentz space $L^{q,r}(\mathbb{R}^n)$ consists of all functions f for which $\|f\|_{L^{q,r}(\mathbb{R}^n)} < \infty$.

Remark 1.3. If we set $r = q$, then the Lorentz space $L^{q,r}(\mathbb{R}^n)$ corresponds to the Lebesgue space $L^q(\mathbb{R}^n)$. For a ball B, we define $\|f\|_{L^{q,r}(B)} = \|f\chi_B\|_{L^{q,r}(\mathbb{R}^n)}$.

The Morrey-Lorentz spaces are defined as follows.

Definition 1.4 ([21])**.** Let $1 < q < \infty, 1 \le r \le \infty$ and $0 < \lambda \le \frac{n}{q}$. For any measurable function f, we define the Morrey-Lorentz space $L^{q,r}_\lambda(\mathbb{R}^n)$ as follows:

$$L^{q,r}_\lambda(\mathbb{R}^n) = \left\{f : \|f\|_{L^{q,r}_\lambda(\mathbb{R}^n)} = \sup_B |B|^{\frac{\lambda}{n} - \frac{1}{q}} \|f\|_{L^{q,r}(B)} < \infty\right\}.$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.

Remark 1.5. If we set $r = q$, then the Morrey-Lorentz $L^{q,r}_\lambda(\mathbb{R}^n)$ becomes the Morrey space $L^q_\lambda(\mathbb{R}^n)$. When $\lambda = \frac{n}{q}$, then the Morrey-Lorentz $L^{q,r}_\lambda(\mathbb{R}^n)$ corresponds to the Lorentz space $L^{q,r}(\mathbb{R}^n)$.

We can express our first result as follows.

Theorem 1.6. *Let $0 \le \alpha < n$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $0 < r, u \le \infty$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{r}{u} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(T1) $b \in BMO(\mathbb{R}^n)$.

(T2) *$M_{\alpha,b}$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$.*

(T3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C. \tag{1.2}$$

(T4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - b_B|dx \le C. \tag{1.3}$$

If we choose $r = q$, then the following corollary can be derived.

Corollary 1.7. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$.*

(C2) *$M_{\alpha,b}$ is bounded from $L^q_\lambda(\mathbb{R}^n)$ to $L^t_\mu(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx \le C.$$

If we set $\lambda = \frac{n}{q}$, then we arrive at the following conclusion.

Corollary 1.8. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty$, $0 < r, u \le \infty$, $\frac{1}{q} - \frac{1}{t} = \frac{\alpha}{n}$ and $\frac{q}{t} = \frac{r}{u}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$.*

(C2) *$M_{\alpha,b}$ is bounded from $L^{q,r}(\mathbb{R}^n)$ to $L^{t,u}(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - b_B)\chi_B\|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - b_B| dx \le C.$$

Here, we present our second result.

Theorem 1.9. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $0 < r, u \le \infty$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{r}{u} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(T1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(T2) *$[b, M_\alpha]$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$.*

(T3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|\left(b - M_B(b)\right)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C. \tag{1.4}$$

(T4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)| dx \le C. \tag{1.5}$$

If we take $r = q$, then we can get the following conclusion.

Corollary 1.10. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty, 0 < \lambda \le \frac{n}{q}, 0 < \mu \le \frac{n}{t}$, $\lambda - \alpha = \mu$ and $\frac{q}{t} = \frac{\mu}{\lambda}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M_\alpha]$ is bounded from $L^q_\lambda(\mathbb{R}^n)$ to $L^t_\mu(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|\left(b - M_B(b)\right)\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)| dx \le C.$$

If we take $\lambda = \frac{n}{q}$, then the following result holds.

Corollary 1.11. *Let $0 \le \alpha < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < q, t < \infty$, $0 < r, u \le \infty$, $\frac{1}{q} - \frac{1}{t} = \frac{\alpha}{n}$ and $\frac{q}{t} = \frac{r}{u}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M_\alpha]$ is bounded from $L^{q,r}(\mathbb{R}^n)$ to $L^{t,u}(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|\left(b - M_B(b)\right)\chi_B\|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - M_B(b)(x)|dx \le C.$$

Next, our third result is as follows.

Theorem 1.12. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $0 < u \le \infty$, $1 < t < \infty$ and $0 < \mu \le \frac{n}{t}$, then the subsequent statements hold equivalently:*

(T1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(T2) *$[b, M^\sharp]$ is bounded on $L^{t,u}_\mu(\mathbb{R}^n)$.*

(T3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C. \tag{1.6}$$

(T4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - 2M^\sharp(b\chi_B)(x)|dx \le C. \tag{1.7}$$

If we take $r = q$, then the following conclusion holds.

Corollary 1.13. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < t < \infty, 0 < \mu \le \frac{n}{t}$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M^\sharp]$ is bounded on $L^t_\mu(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^t_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^t_\mu(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - 2M^\sharp(b\chi_B)(x)|dx \le C.$$

If we take $\lambda = \frac{n}{q}$, then the following result can be obtained.

Corollary 1.14. *Let $b \in L^1_{\rm loc}(\mathbb{R}^n)$. If $1 < t < \infty$, $0 < u \le \infty$, then the subsequent statements hold equivalently:*

(C1) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.*

(C2) *$[b, M^\sharp]$ is bounded on $L^{t,u}(\mathbb{R}^n)$.*

(C3) *There is a constant $C > 0$ such that*

$$\sup_B \frac{\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}(\mathbb{R}^n)}} \le C.$$

(C4) *There is a constant $C > 0$ such that*

$$\sup_B \frac{1}{|B|}\int_B |b(x) - 2M^\sharp(b\chi_B)(x)|dx \le C.$$

2. Preliminaries

To demonstrate our main results, we will present several important notions and known results in the section.

First, we must introduce the predual spaces of Morrey-Lorentz spaces.

Definition 2.1 ([7]). Let $1<q<\infty, 1 \leq r \leq \infty$ and $\beta>0$. A function $b(x)$ is called a (q, r, β)-block, if there exists a ball B in $\mathbb{R}^n$, such that

$$\operatorname{supp}(b) \subset B(x_0, r), \quad \|b\|_{L^{q, r}(B)} \leq |B|^{\frac{1}{q}-\frac{\beta}{n}}$$

Next, we define the space $\mathscr{B}_\beta^{q, r}\left(\mathbb{R}^n\right)$ using (q, r, β)-blocks.

Definition 2.2 ([7]). Let $1<q<\infty, 1 \leq r \leq \infty$ and $\frac{n}{q} \leq \beta<n$. The space $\mathscr{B}_\beta^{q, r}\left(\mathbb{R}^n\right)$ is defined as follows:

$$\mathscr{B}_\beta^{q, r}\left(\mathbb{R}^n\right)=\left\{g \in L_{\text{loc}}^1\left(\mathbb{R}^n\right): g=\sum_{j=1}^{\infty} m_j b_j,\left\{b_j\right\}_{j \geq 1} \text{ are } (q, r, \beta)\text{-blocks and } \sum_{j=1}^{\infty}\left|m_j\right|<\infty\right\}.$$

Lemma 2.3 ([7]). *Let $1<q<\infty, 1 \leq r \leq \infty$, and $0<\lambda \leq \frac{n}{q}$. Then*

$$L_\lambda^{q, r}(\mathbb{R}^n)=\left(\mathscr{B}_{n-\lambda}^{q', r'}(\mathbb{R}^n)\right)' \text{ and } L_\lambda^{q, r}(\mathbb{R}^n)'=\mathscr{B}_{n-\lambda}^{q', r'}(\mathbb{R}^n)$$

Lemma 2.4 ([7]). *Let $1<q<\infty$, $1 \leq r \leq \infty$, $0<\lambda \leq \frac{n}{q}$ and $\frac{n}{q} \leq \beta<n$. Then*

$$\|\chi_B\|_{L_\lambda^{q, r}(\mathbb{R}^n)} \approx|B|^{\frac{\lambda}{n}} \text{ and } \|\chi_B\|_{\mathscr{B}_\beta^{q, r}(\mathbb{R}^n)} \approx|B|^{\frac{\beta}{n}}.$$

Lemma 2.5 ([7]). *Let $1<q, q', r, r'<\infty$ and $0<\lambda \leq \frac{n}{q}$. Assume that $f \in L_\lambda^{q, r}(\mathbb{R}^n)$ and $g \in \mathscr{B}_{n-\lambda}^{q', r'}(\mathbb{R}^n)$. Then the following statement is true:*

$$\int_{\mathbb{R}^n}|f(x) g(x)| d x \lesssim\|f\|_{L_\lambda^{q, r}(\mathbb{R}^n)}\|g\|_{\mathscr{B}_{n-\lambda}^{q', r'}(\mathbb{R}^n)}$$

Similarly to [15, Proposition 3], we obtain the following conclusion, the proof of which requires only slight modifications; thus, we omit the details.

Lemma 2.6. *Let $0 \leq \alpha<n$, $0<r, u \leq \infty$, $1<q, t<\infty, 0<\lambda \leq \frac{n}{q}$ and $0<\mu \leq \frac{n}{t}$. Suppose that $\lambda-\alpha=\mu$ and $\frac{q}{t}=\frac{r}{u}=\frac{\mu}{\lambda}$. Then for $f \in L_\lambda^{q, r}(\mathbb{R}^n)$,*

$$\|M_\alpha f\|_{L_\mu^{t, u}(\mathbb{R}^n)} \lesssim\|f\|_{L_\lambda^{q, r}(\mathbb{R}^n)}.$$

Lemma 2.7 ([10]). *Let $0 \leq \alpha<n$ and $b \in BMO(\mathbb{R}^n)$. Then, for $f \in L_{\text{loc}}^1\left(\mathbb{R}^n\right)$ and $x \in \mathbb{R}^n$, there is a constant C such that*

$$M_{b, \alpha} f(x) \leq C\|b\|_{BMO(\mathbb{R}^n)}(M(M_\alpha f)(x)+M_\alpha(M f)(x)).$$

Lemma 2.8 ([26]). *Let $0 \leq \alpha<n$, B be a ball in $\mathbb{R}^n$ and $f \in L_{\text{loc}}^1\left(\mathbb{R}^n\right)$. Then, for any $x \in B$, it holds that:*

$$M_\alpha\left(f \chi_B\right)(x)=M_{\alpha, B}(f)(x).$$

3. Proofs of main results

Proof of Theorem 1.6. (T1) $\Rightarrow$ (T2): Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Lemma 2.6 with Lemma 2.7 deduces that

$$\begin{aligned}
\|M_{\alpha, b}(f)\|_{L_\mu^{t, u}(\mathbb{R}^n)} & \leq C\|b\|_{BMO(\mathbb{R}^n)}\|(M(M_\alpha f)(x)+M_\alpha(M f)(x))\|_{L_\mu^{t, u}(\mathbb{R}^n)} \\
& \leq C\|b\|_{BMO(\mathbb{R}^n)}(\|M_\alpha f\|_{L_\mu^{t, u}(\mathbb{R}^n)}+\|M f\|_{L_\lambda^{q, r}(\mathbb{R}^n)}) \\
& \leq C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L_\lambda^{q, r}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, we conclude that $M_{\alpha,b}$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$.

(T2) ⇒ (T3): For a given ball $B \subset \mathbb{R}^n$ and $x \in B$, we obtain

$$\begin{aligned} |b(x) - b_B| &\leq \frac{1}{|B|}\int_B |b(x)-b(y)|dy \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B |b(x)-b(y)|\chi_B(y)dy \\ &\leq |B|^{-\frac{\alpha}{n}} M_{\alpha,b}(\chi_B)(x). \end{aligned}$$

Since $M_{\alpha,b}$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$, then using Lemma 2.4 and the condition $\lambda - \alpha = \mu$, we conclude that

$$\begin{aligned} \frac{\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n))}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &\leq \frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\|M_{\alpha,b}(\chi_B)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C\frac{1}{|B|^{\frac{\alpha}{n}}}\frac{\|\chi_B\|_{L^{q,r}_{\lambda}(\mathbb{R}^n))}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C, \end{aligned}$$

which deduces that (1.2) holds since the ball $B \subset \mathbb{R}^n$ is arbitrary.

(T3) ⇒ (T4): Assume that (1.2) is true, we will show (1.3). For a given ball B, by applying Lemma 2.4 and Lemma 2.5, we can derive

$$\begin{aligned} \frac{1}{|B|}\int_B |b(x)-b_B|dx &\leq C\frac{1}{|B|}\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\leq C\frac{\|(b-b_B)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\leq C. \end{aligned}$$

(T4) ⇒ (T1): It follows from Definition 1.1 directly, thus we omit the details.

This finishes the proof of Theorem 1.6. □

Proof of Theorem 1.9. (T1) ⇒ (T2): Suppose that $b \in BMO(\mathbb{R}^n)$ and $b^- \in L^{\infty}(\mathbb{R}^n)$. By (1.1), Lemma 2.6 and Lemma 2.7, we have

$$\begin{aligned} \|[b,M_\alpha](f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} &\leq \|M_{b,\alpha}(f) + 2b^- M_\alpha(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} \\ &\leq \|M_{b,\alpha}(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} + \|2b^- M_\alpha(f)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)} \\ &\lesssim \|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)} + \|b^-\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)} \\ &\lesssim \|f\|_{L^{q,r}_{\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Thus, we show that $[b, M_\alpha]$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$.

(T2) ⇒ (T3): We will divide the proof into two cases depending on the value of α.

Case 1. Let $0 < \alpha < n$. For a given ball B,

$$\begin{aligned} \frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &\leq \frac{\|(b-|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b)\,)\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\quad + \frac{\|(|B|^{-\frac{\alpha}{n}}M_{\alpha,B}(b) - M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &:= I + II. \end{aligned}$$

For I. For any $x \in B$, the definition of $M_{\alpha,B}$ implies that

$$M_{\alpha,B}(\chi_B)(x) = |B|^{\frac{\alpha}{n}}. \tag{3.1}$$

For any $x \in B$, Lemma 2.8 indicates that,

$$M_\alpha(\chi_B)(x) = M_{\alpha,B}(\chi_B)(x) = |B|^{\frac{\alpha}{n}} \text{ and } M_\alpha(b\chi_B)(x) = M_{\alpha,B}(b)(x).$$

Therefore, we have

$$\begin{aligned} b(x) - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(x) &= |B|^{-\frac{\alpha}{n}} (b(x)|B|^{\frac{\alpha}{n}} - M_{\alpha,B}(b)(x)) \\ &= |B|^{-\frac{\alpha}{n}} (b(x) M_\alpha(\chi_B)(x) - M_\alpha(b\chi_B)(x)) \\ &= |B|^{-\frac{\alpha}{n}} [b, M_\alpha](\chi_B)(x). \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$, then combining Lemma 2.4 with the condition $\lambda - \alpha = \mu$ deduces that

$$\begin{aligned} I &= \frac{\|(b - |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &= \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|[b, M_\alpha](\chi_B)\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|\chi_B\|_{L^{q,r}_\lambda(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C. \end{aligned}$$

For II. Similar to (3.1), by using Lemma 2.8 and for any $x \in B$,

$$M_B(\chi_B)(x) = \chi_B(x),$$

we deduce that

$$M(\chi_B)(x) = \chi_B(x) \text{ and } M(b\chi_B)(x) = M_B(b)(x). \tag{3.2}$$

Thus, Combining (3.1) with (3.2) implies that

$$\begin{aligned} \left| |B|^{-\frac{\alpha}{n}} M_{\alpha,B}(b)(x) - M_B(b)(x) \right| &\le |B|^{-\frac{\alpha}{n}} \left| M_\alpha(b\chi_B)(x) - |b(x)| M_\alpha(\chi_B)(x) \right| \\ &\quad + |B|^{-\frac{\alpha}{n}} \left| |b(x)| M_\alpha(\chi_B)(x) - M_\alpha(\chi_B)(x) M(b\chi_B)(x) \right| \\ &= |B|^{-\frac{\alpha}{n}} \left| M_\alpha(|b|\chi_B)(x) - |b(x)| M_\alpha(\chi_B)(x) \right| \\ &\quad + |B|^{-\frac{\alpha}{n}} M_\alpha(\chi_B)(x) \left| |b(x)| M(\chi_B)(x) - M(b\chi_B)(x) \right| \\ &= |B|^{-\frac{\alpha}{n}} \left| [|b|, M_\alpha](\chi_B)(x) \right| + \left| [|b|, M](\chi_B)(x) \right|. \end{aligned}$$

Since $[b, M_\alpha]$ is bounded from $L^{q,r}_\lambda(\mathbb{R}^n)$ to $L^{t,u}_\mu(\mathbb{R}^n)$. Then, by applying Lemma 2.4, we get

$$\begin{aligned} II &\le \frac{\|(|B|^{-\frac{\alpha}{n}} |[|b|, M_\alpha](\chi_B)| + |[|b|, M](\chi_B)|)\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\lesssim \frac{1}{|B|^{\frac{\alpha}{n}}} \frac{\|\chi_B\|_{L^{q,r}_\lambda(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} + \frac{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \\ &\le C. \end{aligned}$$

This deduces that the desired estimate

$$\frac{\|(b - M_B(b))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}} \le C,$$

which concludes that (1.4) holds.

Case 2. Let $\alpha = 0$. For a given ball B and $x \in B$, using (3.2), we obtain

$$b(x) - M_B(b)(x) = b(x) M(\chi_B)(x) - M(b\chi_B)(x) = [b, M](\chi_B)(x).$$

Suppose that $[b, M]$ is bounded from $L^{q,r}_{\lambda}(\mathbb{R}^n)$ to $L^{t,u}_{\mu}(\mathbb{R}^n)$, then by applying Lemma 2.4, we have

$$\begin{aligned}\frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} &= \frac{\|[b,M](\chi_B)\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\le C\frac{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\le C,\end{aligned}$$

which implies that (1.4).

(T3) $\Rightarrow$ (T4): Assume that (1.4) holds, then for a given ball B, by Lemma 2.5, we have

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-M_B(b)(x)|\,dx &\le C\frac{1}{|B|}\,\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}\,\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\le C\frac{1}{|B|}\frac{\|(b-M_B(b))\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{t,u}_{\mu}(\mathbb{R}^n)}} \\ &\le C,\end{aligned}$$

where the constant C does not depend on B. This deduces that (1.5).

(T4) $\Rightarrow$ (T1): To prove $b \in BMO(\mathbb{R}^n)$, we only need to demonstrate that there exists a constant $C > 0$ such that, for a given ball B,

$$\frac{1}{|B|}\int_B |b(x)-b_B|dx \le C.$$

For a given ball B, let $E = \{x \in B : b(x) \le b_B\}$ and $F = \{x \in B : b(x) > b_B\}$, then we get

$$\int_E |b(x)-b_B|dx = \int_F |b(x)-b_B|dx. \tag{3.3}$$

As $b(x) \le b_B \le M_B(b)(x)$ for any $x \in E$, we obtain

$$|b(x)-b_B| \le |b(x)-M_B(b)(x)|. \tag{3.4}$$

Combining (3.3) with (3.4) deduces that

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dx &= \frac{2}{|B|}\int_E |b(x)-b_B|dx \\ &\le \frac{2}{|B|}\int_E |b(x)-M_B(b)(x)|dx \\ &\le \frac{2}{|B|}\int_B |b(x)-M_B(b)(x)|dx \\ &\le C.\end{aligned}$$

Thus, we deduce that $b \in BMO(\mathbb{R}^n)$.

Next, we aim to prove that $b^- \in L^{\infty}(\mathbb{R}^n)$. Note that for any $y \in B$, we have $0 \le b^+(y) \le |b(y)| \le M_B(b)(y)$, then

$$0 \le b^-(y) \le M_B(b)(y) - b^+(y) + b^-(y) = M_B(b)(y) - b(y).$$

Furthermore, for a given ball B, we get

$$\begin{aligned}\frac{1}{|B|}\int_B b^-(y)dy &\le \frac{1}{|B|}\int_B (M_B(b)(y)-b(y))\,dy \\ &= \frac{1}{|B|}\int_B |b(y)-M_B(b)(y)|\,dy \\ &\le C.\end{aligned}$$

Let $|B| \to 0$ with $x \in B$. By applying Lebesgue's differentiation theorem, we deduce that

$$0 \le b^-(x) = \lim_{|B|\to 0} \frac{1}{|B|} \int_B b^-(y)dy \le C.$$

Hence, we establish that $b^- \in L^\infty(\mathbb{R}^n)$.

We have now completed the proof of Theorem 1.9. □

Proof of Theorem 1.12. (T1) ⇒ (T2): Suppose that $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$, for a given ball $B \subset \mathbb{R}^n$, the estimate below was established in [27]:

$$|[|b|, M^\sharp]f(x)| \le 2M_{|b|}f(x).$$

Noting that $|b| - b = 2b^-$, it follows from the definition of $[b, M^\sharp]$ that,

$$\begin{aligned}
&|[b, M^\sharp]f(x) - [|b|, M^\sharp]f(x)| \\
&\le |M^\sharp(bf)(x) - M^\sharp(|b|f)(x)| + ||b(x)|M^\sharp(f)(x) - b(x)M^\sharp f(x)| \\
&\le |M^\sharp((b - |b|)f)(x)| + 2b^-(x)M^\sharp f(x) \\
&\le M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp f(x).
\end{aligned}$$

Combined with previous estimates and $M^\sharp(f) \le 2M(f)$, for any $x \in \mathbb{R}^n$, we obtain

$$\begin{aligned}
|[b, M^\sharp](f)(x)| &\le |[b, M^\sharp]f(x) - [|b|, M^\sharp]f(x)| + |[|b|, M^\sharp]f(x)| \\
&\le M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp(f)(x) + |[|b|, M^\sharp]f(x)|, \\
&\le 2M(2b^- f)(x) + 4b^-(x)M(f)(x) + 2M_{|b|}f(x).
\end{aligned}$$

Since $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $|b| \in \mathrm{BMO}(\mathbb{R}^n)$. Based on Lemma 2.6 and Theorem 1.6, we find that

$$\|[b, M^\sharp](f)\|_{L^{t,u}_\mu(\mathbb{R}^n)} \le C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{t,u}_\mu(\mathbb{R}^n)},$$

which implies that $[b, M^\sharp]$ is bounded on $L^{t,u}_\mu(\mathbb{R}^n)$.

(T2) ⇒ (T3): Take B as a fixed ball and B_1 as a different ball. By the inequality $4ac \le (a+c)^2$, we can see that

$$\begin{aligned}
&\frac{1}{|B_1|}\int_{B_1} |\chi_B(x) - (\chi_B)_{B_1}|dx \\
&= \frac{1}{|B_1|}\left\{\int_{B_1\setminus B} |\chi_B(x) - (\chi_B)_{B_1}|dx + \int_{B_1\cap B} |\chi_B(x) - (\chi_B)_{B_1}|dx\right\} \\
&= \frac{1}{|B_1|}\left\{\int_{B_1\setminus B} |(\chi_B)_{B_1}|dx + \int_{B_1\cap B} |1 - (\chi_B)_{B_1}|dx\right\} \\
&= \frac{1}{|B_1|}\left\{\int_{B_1\setminus B} \left|\frac{1}{|B_1|}\int_{B_1\cap B} \chi_B(y)dy\right|dx\right. \\
&\left.+ \int_{B_1\cap B} \left|\frac{1}{|B_1|}\int_{B_1} \chi_{B_1}(y)dy - \frac{1}{|B_1|}\int_{B_1} \chi_B(y)\cdot\chi_{B_1}(y)dy\right| dx\right\} \\
&= \frac{1}{|B_1|}\left\{\frac{|B_1\cap B||B_1\setminus B|}{|B_1|} + \frac{1}{|B_1|}\int_{B_1\cap B}\left|\int_{B_1} \chi_{B_1}(y)(1-\chi_B(y))dy\right| dx\right\} \\
&= \frac{1}{|B_1|^2}\left\{|B_1\cap B||B_1\setminus B| + |B_1\cap B||B_1\setminus B|\right\} \\
&= \frac{2|B_1\cap B||B_1\setminus B|}{(|B_1\cap B| + |B_1\setminus B|)^2} \le \frac{1}{2}.
\end{aligned} \tag{3.5}$$

Moreover, for $x \in B$, we can find a ball B_0 that contains B and satisfies $|B_0| = 2|B|$. Then, using (3.5) and $|B_0 \setminus B| = |B_0 \cap B| = |B|$, we conclude that

$$\frac{1}{|B_0|}\int_{B_0} |\chi_B(x) - (\chi_B)_{B_0}|dx = \frac{2|B_0 \cap B||B_0 \setminus B|}{(|B_0 \cap B| + |B_0 \setminus B|)^2} = \frac{1}{2}.$$

Furthermore, we have

$$(M^\sharp(\chi_B)\chi_B)(x) = \sup_{B_1 \ni x} \frac{1}{|B_1|}\int\limits_{B_1} |\chi_B(y) - (\chi_B)_{B_1}|dy = \frac{1}{2} = \frac{1}{2}\chi_B(x).$$

Then, we can get

$$\begin{aligned}
\|(b - 2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} &= \Big\|2\Big(\frac{1}{2}b\chi_B - M^\sharp(b\chi_B)\Big)\chi_B\Big\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&= \|2(bM^\sharp(\chi_B)\chi_B - M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&= \|2(bM^\sharp(\chi_B) - M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&\le \|2[b, M^\sharp](\chi_B)\|_{L^{t,u}_\mu(\mathbb{R}^n)} \\
&\le C\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)},
\end{aligned}$$

where the constant C does not depend on B. This deduces that (1.6).

(T3) $\Rightarrow$ (T4): For a ball $B \subset \mathbb{R}^n$ and $x \in B$, we will show that $|b_B| \le 2M^\sharp(b\chi_B)(x)$. Take $x \in B$ and select a ball B_1 that includes B with the property that $|B_1| = 2|B|$. Thus,

$$\begin{aligned}
\frac{1}{2|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{4}|b_B| &= \frac{1}{2|B|}\left(\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{2}|B_1 \setminus B||b_B|\right) \\
&= \frac{1}{|B_1|}\int_{B_1} |b\chi_B(y) - (b\chi_B)_{B_1}|dy \\
&\le M^\sharp(b\chi_B)(x).
\end{aligned}$$

Moreover,

$$\begin{aligned}
|b_B| &\le \frac{1}{|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{|B|}\int_B |\frac{1}{2}b_B|dy \\
&= \frac{1}{|B|}\int_B |b(y) - \frac{1}{2}b_B|dy + \frac{1}{2}|b_B|.
\end{aligned}$$

Thus, for $x \in B$, we obtain

$$|b_B| \le 2M^\sharp(b\chi_B)(x). \tag{3.6}$$

Next, we will show that $b \in BMO(\mathbb{R}^n)$. To do this, let $E = \{x \in B : b(x) \le b_B\}$ and $F = \{x \in B : b(x) > b_B\}$, we then obtain

$$\int_E |b(x) - b_B|dx = \int_F |b(x) - b_B|dx.$$

Since $b(x) \le b_B \le |b_B| \le 2M^\sharp(b\chi_B)(x)$ for any $x \in E$, then

$$|b(x) - b_B| \le |b(x) - 2M^\sharp(b\chi_B)(x)|.$$

Using Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}\frac{1}{|B|}\int_B |b(x)-b_B|dx &= \frac{1}{|B|}\int_{E\cup F} |b(x)-b_B|dx \\ &= \frac{2}{|B|}\int_E |b(x)-b_B|dx \\ &\le \frac{2}{|B|}\int_E |b(x)-2M^\sharp(b\chi_B)(x)|dx \\ &\le \frac{2}{|B|}\int_B |b(x)-2M^\sharp(b\chi_B)(x)|dx \\ &\le \frac{C}{|B|}\|(b-2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\le \frac{C}{|B|}\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\le C,\end{aligned}$$

which deduces that $b\in BMO(\mathbb{R}^n)$. We shall now prove that b^- is in $L^\infty(\mathbb{R}^n)$. By (3.6), for $x\in B$, we have

$$|b_B|-b^+(x)+b^-(x)=|b_B|-b(x)\le 2M^\sharp(b\chi_B)(x)-b(x).$$

Therefore,

$$\begin{aligned}|b_B|-\frac{1}{|B|}\int_B b^+(x)dx+\frac{1}{|B|}\int_B b^-(x)dx &= \frac{1}{|B|}\int_B(|b_B|-b^+(x)+b^-(x))dx \\ &\le \frac{1}{|B|}\int_B(2M^\sharp(b\chi_B)(x)-b(x))dx \qquad (3.7) \\ &\le \frac{1}{|B|}\int_B|b(x)-2M^\sharp(b\chi_B)(x)|dx.\end{aligned}$$

Besides, combining Lemma 2.4 with Lemma 2.5, we can get

$$\begin{aligned}&\frac{1}{|B|}\int_B |b(x)-2M^\sharp(b\chi_B)(x)|dx \\ &\le \frac{C}{|B|}\|(b-2M^\sharp(b\chi_B))\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\le \frac{C}{|B|}\|\chi_B\|_{L^{t,u}_\mu(\mathbb{R}^n)}\|\chi_B\|_{\mathscr{B}^{t',u'}_{n-\mu}(\mathbb{R}^n)} \\ &\le C.\end{aligned}$$

Combining this inequality with (3.7), we deduce that

$$|b_B|-\frac{1}{|B|}\int_B b^+(x)dx+\frac{1}{|B|}\int_B b^-(x)dx\le C.$$

Let $|B|$ tend to 0 with $x\in B$, it follows from Lebesgue's differentiation theorem that,

$$2|b^-(x)|=2b^-(x)=|b(x)|-b^+(x)+b^-(x)\le C.$$

This implies that $b^-\in L^\infty(\mathbb{R}^n)$.

Therefore, we complete the proof of Theorem 1.12. □

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References

[1] M. Agcayazi, A. Gogatishvili, K. Koca and R. Mustafayev, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan **67** (2), 581–593, 2015.

[2] M. Agcayazi, A. Gogatishvili and R. Mustafayev, *Weak-type estimates in Morrey spaces for maximal commutator and commutator of maximal function*, Tokyo J. Math. **41** (1), 193–218, 2018.

[3] C. Aykol, H. Armutcu and M. N. Omarova, *Maximal commutator and commutator of maximal function on modified Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **36** (1), 29–35, 2016.

[4] J. Bastero, M. Milman and F. J. Ruiz, *Commutators for the maximal and sharp functions*, Proc. Am. Math. Soc. **128** (11), 3329–3334, 2000.

[5] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. Math. **103** (3), 611–635, 1976.

[6] N. A. Dao and S. G. Krantz, *Lorentz boundedness and compactness characterization of integral commutators on spaces of homogeneous type*, Nonlinear Anal. **203**, 112162, 2021.

[7] N. A. Dao and S. G. Krantz, *On the predual of a Morrey-Lorentz space and its applications to the linear Calderón-Zygmund operators*, Front. Math. **19** (3), 385–418, 2024.

[8] I. Ekincioglu, J. J. Hasanov and C. Keskin, *On the boundedness of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces*, Hacet. J. Math. Stat. **53** (2), 321-332, 2024.

[9] C. Fefferman and E. M. Stein, *H_p spaces of several variables*, Acta Math. **129**, 137–193, 1972.

[10] V. S. Guliyev, *Commutators of the fractional maximal function in generalized Morrey spaces on Carnot groups*, Complex Var. Elliptic **66** (6), 893–909, 2021.

[11] V. S. Guliyev, *Some characterizations of BMO spaces via commutators in Orlicz spaces on stratified Lie groups*, Results Math. **77** (1), 1–18, 2022.

[12] V. S. Guliyev, *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal, **16** (4), 1509-1524, 2022.

[13] V. S. Guliyev, *Maximal commutator and commutator of maximal operator on Lorentz spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **44** (4), 1–7, 2024.

[14] V. S. Guliyev, F. Deringoz and S. G. Hasanov, *Fractional maximal function and its commutators on Orlicz spaces*, Anal. Math. Phys. **9**, 165–179, 2019.

[15] N. Hatano, *Fractional operators on Morrey-Lorentz spaces and the Olsen inequality*, Math. Notes **107**, 63–79, 2020.

[16] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. **16** (1), 263–270, 1978.

[17] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (3), 415–426, 1961.

[18] G. G. Lorentz, *On the theory of spaces* Λ, Pacific J. Math. **1**, 411–429, 1951.

[19] G. Lu, *Bilinear Calderón-Zygmund operator and its commutator on some variable exponent spaces of homogeneous type*, Hacet. J. Math. Stat. **53** (2), 433-456, 2024.

[20] M. Milman and T. Schonbek, *Second order estimates in interpolation theory and applications*, Proc. Am. Math. Soc. **110** (4), 961–969, 1990.

[21] M. A. Ragusa, *Embeddings for Morrey-Lorentz spaces*, J. Optim. Theory Appl. **154**, 491–499, 2021.

[22] H. Yang and J. Zhou, *Some characterizations of Lipschitz spaces via commutators of the Hardy-Littlewood maximal operator on slice spaces*, Proc. Ro. Acad. Ser. A **24** (3), 223–230, 2023.

[23] H. Yang and J. Zhou, *Compactness of commutators of fractional integral operators on ball Banach function spaces*, AIMS Math. **9** (2), 3126–3149, 2024.
[24] H. Yang and J. Zhou, *Commutators of some maximal functions with Lipschitz functions on mixed Morrey spaces*, Filomat **38** (31), 11031–11043, 2024.
[25] X. Yang, Z. Yang and B. Li, *Characterization of Lipschitz space via the commutators of fractional maximal functions on variable lebesgue spaces*, Potential Anal. **60** (2), 703–720, 2024.
[26] P. Zhang and J. Wu, *Commutators of the fractional maximal functions*, Acta Math. Sin. **52** (6), 1235–1238, 2009.
[27] P. Zhang and J. Wu, *Commutators of the fractional maximal function on variable exponent Lebesgue spaces*, Czech. Math. J. **64** (1), 183–197, 2014.
[28] P. Zhang, J. Wu and J. Sun, *Commutators of some maximal functions with Lipschitz function on Orlicz spaces*, Mediterr. J. Math. **15**, 1–13, 2018.