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Research Article

On the numerical radii of certain operator forms

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Abstract

The main goal of this paper is to present new bounds for the norms and numerical radii of certain Hilbert space operators, which involve the sums of operators' products. The obtained result will be utilized to obtain some equivalent conditions regarding certain operator identities, to find upper bounds for the sum of two operators, to obtain certain refinements of celebrated numerical radius bounds, and to present a new reverse for the triangle inequality when positive operators are treated.

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1. Introduction

In the sequel, $\mathbb{H}$ is a complex Hilbert space, endowed with inner product $\langle \cdot, \cdot \rangle$, and induced norm $\| \cdot \|$. The set of all bounded linear operators from $\mathbb{H}$ to $\mathbb{H}$ will be denoted by $\mathbb{B}(\mathbb{H})$, with zero element O and identity I. Upper case letters through this paper will be used to denote elements of $\mathbb{B}(\mathbb{H})$.

Associated with $A \in \mathbb{B}(\mathbb{H})$, many scalar values have received the attention of specialists through the literature. Among those quantities, the operator norm, the numerical radius, and the spectral radius have received researchers' attention.

These three quantities are defined, respectively, for an arbitrary $A \in \mathbb{B}(\mathbb{H})$ as follows

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \omega(A) = \sup_{\|x\|=1} |\langle Ax, x\rangle| \text{ and } r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\},$$

where $\sigma(A)$ denotes the spectrum of A, defined as $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$.

It is well known that $\|A\| = \sup_{\|x\|=\|y\|=1} |\langle Ax, y\rangle|$, which makes the relation $\omega(A) \leq \|A\|$ clear, for any $A \in \mathbb{B}(\mathbb{H})$.

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Among those interesting investigations in the literature is the investigation of possible relations that govern the quantities $\|\cdot\|, \omega(\cdot)$ and $r(\cdot)$. It is well established that for any $A \in \mathbb{B}(\mathbb{H})$, one has [19, Theorem 1.3-1 and Theorem 1.4-4]

$$r(A) \leq \omega(A) \leq \|A\|. \tag{1.1}$$

However, discussing sharper forms of (1.1) has been an exciting topic. This interest has received considerable attention, and hundreds of research papers have been published. We refer the interested reader to [2–8, 10, 12–15, 18, 20, 22, 23, 30, 31, 33–38] and the references therein, as a sample of such work.

Letting A^* denote the adjoint of $A \in \mathbb{B}(\mathbb{H})$, we say that A is normal if $A^*A = AA^*$. The class of normal operators fulfills sharper bounds than general operators. For example, both inequalities in (1.1) become equalities [19, Theorem 1.4-2].

While simple forms are desired to describe such relations, the literature has witnessed more elaborated forms that had some influence on the advancement of this field. In particular, operator matrices have played a vital role in understanding some basic properties of the numerical radius and its relation with the operator norm and spectral radius. We recall here that if $A, B, C, D \in \mathbb{B}(\mathbb{H})$, then the operator matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{B}(\mathbb{H} \oplus \mathbb{H})$, where $\oplus$ denotes the direct sum. We refer the reader to [1, 9, 14, 29, 32] as a sample of how operator matrices influenced and enriched the study of the above interest.

In this paper, we discuss possible bounds in this context, where sums of products of certain operators are treated. We refer the reader to [21], where the quantity $\omega(AXB^* + BYA^*)$ has been studied, for $A, B, X, Y \in \mathbb{B}(\mathbb{H})$. It is interesting that the upper bound for this quantity is found in terms of an operator matrix.

Given $A_1, A_2, B_1, B_2 \in \mathbb{B}(\mathbb{H})$, we will discuss possible bounds for the operator norm and the numerical radius of the operator $A_1B_1 + A_2B_2$. The obtained bounds are used to obtain certain bounds for the norm of the sum of two operators and the norm of an operator's Cartesian parts. Here, we recall that if $A \in \mathbb{B}(\mathbb{H})$, the real and imaginary parts of A are defined, respectively, as

$$\Re A = \frac{A + A^*}{2} \text{ and } \Im A = \frac{A - A^*}{2\mathrm{i}}.$$

The real and imaginary parts have strong relations to the numerical radius, as one can verify that

$$\omega(A) = \sup_{\theta \in \mathbb{R}} \left\| \Re\left(e^{\mathrm{i}\theta} A\right) \right\| = \sup_{\theta \in \mathbb{R}} \left\| \Im\left(e^{\mathrm{i}\theta} A\right) \right\|.$$

We will need some lemmas from the literature to obtain our results, as follows.
For the first lemma, we recall that an operator $A \in \mathbb{B}(\mathbb{H})$ is said to be positive if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{H}$.

Lemma 1.1. [25] Let $A, B \in \mathbb{B}(\mathbb{H})$ be two positive operators. Then

$$\|A + B\| \leq \max\{\|A\|, \|B\|\} + \left\| A^{\frac{1}{2}} B^{\frac{1}{2}} \right\|.$$

The following is an arithmetic-geometric mean inequality; see [11].

Lemma 1.2. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$\|AB^*\| \leq \frac{1}{2} \left\| |A|^2 + |B|^2 \right\|,$$

where $|X|^2 = X^*X$ for $X \in \mathbb{B}(\mathbb{H})$.

Lemma 1.3. [16, p. 58] Let $A = U|A|$ be the polar decomposition of A. Then for any $q > 0$,

$$U|A|^q U^* = |A^*|^q.$$

Lemma 1.4. [24] Let $\mathbb{H}_1, \mathbb{H}_2, \ldots, \mathbb{H}_n$ be Hilbert spaces, and let $\mathbb{T} = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathbb{H}_i, \mathbb{H}_j)$. Then

$$\|\mathbb{T}\| \leq \|[\|T_{ij}\|]\|,$$

2. Main results

We present our results in two subsections, treating related bounds for the operator norm and the numerical radius.

2.1. Upper bounds for the operator norm

The first result gives an elaborated upper bound for $\|A_1B_1 + A_2B_2\|$. Some applications will follow.

Theorem 2.1. *Let $A_1, A_2, B_1, B_2 \in \mathbb{B}(\mathbb{H})$. Then*

$$\begin{aligned}\|A_1B_1 + A_2B_2\| \leq \frac{1}{2}\sqrt{\|A_1A_1^* + A_2A_2^*\| \, \|B_1^*B_1 + B_2^*B_2\|} \\ + \frac{1}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} A_1^*A_1B_1B_1^* + A_1^*A_2B_2B_1^* & A_1^*A_1B_1B_2^* + A_1^*A_2B_2B_2^* \\ A_2^*A_1B_1B_1^* + A_2^*A_2B_2B_1^* & A_2^*A_1B_1B_2^* + A_2^*A_2B_2B_2^* \end{bmatrix}\right).\end{aligned}$$

Proof. For any $\mathbf{A}, \mathbf{B} \in \mathbb{B}(\mathbb{H})$, Lemma 1.2 implies

$$\|\mathbf{A}\mathbf{B}^*\| \leq \frac{1}{2}\left\||\mathbf{A}|^2 + |\mathbf{B}|^2\right\|. \tag{2.1}$$

For nonzero operators $\mathbf{A}$ and $\mathbf{B}$, if we replace $\mathbf{A}$ and $\mathbf{B}$ by $\sqrt{\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}}\mathbf{A}$ and $\sqrt{\frac{\|\mathbf{A}\|}{\|\mathbf{B}\|}}\mathbf{B}$, in (2.1), we infer that

$$\begin{aligned}\|\mathbf{A}\mathbf{B}^*\| &\leq \frac{1}{2}\left\|\frac{\|\mathbf{B}\|}{\|\mathbf{A}\|}|\mathbf{A}|^2 + \frac{\|\mathbf{A}\|}{\|\mathbf{B}\|}|\mathbf{B}|^2\right\| \\ &\leq \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \|\,|\mathbf{A}|\,|\mathbf{B}|\,\|\right) \quad \text{(by Lemma 1.1)} \\ &= \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \left\||\mathbf{A}|\,|\mathbf{B}|^2\,|\mathbf{A}|\right\|^{\frac{1}{2}}\right) \\ &= \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \sqrt{r\left(|\mathbf{A}|\,|\mathbf{B}|^2\,|\mathbf{A}|\right)}\right) \\ &= \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \sqrt{r\left(|\mathbf{A}|^2|\mathbf{B}|^2\right)}\right) \\ &\leq \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \omega^{\frac{1}{2}}\left(|\mathbf{A}|^2|\mathbf{B}|^2\right)\right),\end{aligned}$$

where we have used the fact that $\|X\|^2 = \|X^*\| = \|\,|X|^2\|$ to obtain the first equality, that $r(X) = \|X\|$ when X is Hermitian to obtain the second equality, that $r(\cdot)$ is commutative to obtain the third equality and that $r(\cdot) \leq \omega(\cdot)$ to obtain the last inequality. Thus, we have shown that for all $\mathbf{A}, \mathbf{B} \in \mathbb{B}(\mathbb{H})$,

$$\|\mathbf{A}\mathbf{B}^*\| \leq \frac{1}{2}\left(\|\mathbf{A}\|\,\|\mathbf{B}\| + \omega^{\frac{1}{2}}\left(|\mathbf{A}|^2|\mathbf{B}|^2\right)\right). \tag{2.2}$$

Now for the given A_i, B_i, let $\mathbf{A} = \begin{bmatrix} A_1 & A_2 \\ O & O \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} B_1^* & B_2^* \\ O & O \end{bmatrix}$. Simple calculations reveal that

$$\|\mathbf{A}\mathbf{B}^*\| = \left\|\begin{bmatrix} A_1B_1 + A_2B_2 & O \\ O & O \end{bmatrix}\right\| = \|A_1B_1 + A_2B_2\|,$$

$$\|\mathbf{A}\| = \|\mathbf{A}\mathbf{A}^*\|^{\frac{1}{2}} = \left\| \begin{bmatrix} A_1A_1^* + A_2A_2^* & O \\ O & O \end{bmatrix} \right\|^{\frac{1}{2}} = \|A_1A_1^* + A_2A_2^*\|^{\frac{1}{2}},$$

$$\|\mathbf{B}\| = \|\mathbf{B}\mathbf{B}^*\|^{\frac{1}{2}} = \left\| \begin{bmatrix} B_1^*B_1 + B_2^*B_2 & O \\ O & O \end{bmatrix} \right\|^{\frac{1}{2}} = \|B_1^*B_1 + B_2^*B_2\|^{\frac{1}{2}},$$

and

$$\begin{aligned} &\omega^{\frac{1}{2}}\left(|\mathbf{A}|^2|\mathbf{B}|^2\right) \\ &= \omega^{\frac{1}{2}}\left(\begin{bmatrix} A_1^*A_1B_1B_1^* + A_1^*A_2B_2B_1^* & A_1^*A_1B_1B_2^* + A_1^*A_2B_2B_2^* \\ A_2^*A_1B_1B_1^* + A_2^*A_2B_2B_1^* & A_2^*A_1B_1B_2^* + A_2^*A_2B_2B_2^* \end{bmatrix} \right). \end{aligned}$$

We obtain the desired result by taking into account (2.2). This completes the proof for the nonzero operators case. The zero case is trivial. □

Remark 2.2. For any $A_1, A_2, B_1, B_2 \in \mathbb{B}(\mathbb{H})$, we have

$$\begin{aligned} &\omega\left(\begin{bmatrix} A_1^*A_1B_1B_1^* + A_1^*A_2B_2B_1^* & A_1^*A_1B_1B_2^* + A_1^*A_2B_2B_2^* \\ A_2^*A_1B_1B_1^* + A_2^*A_2B_2B_1^* & A_2^*A_1B_1B_2^* + A_2^*A_2B_2B_2^* \end{bmatrix} \right) \\ &= \omega\left(|\mathbf{A}|^{\mathbf{2}}|\mathbf{B}|^{\mathbf{2}}\right) \le \left\| |\mathbf{A}|^{\mathbf{2}}|\mathbf{B}|^{\mathbf{2}} \right\| \le \|\mathbf{A}\|^2 \ \|\mathbf{B}\|^2 \\ &= \|A_1A_1^* + A_2A_2^*\| \, \|B_1^*B_1 + B_2^*B_2\|. \end{aligned}$$

Consequently, Theorem 2.1 implies

$$\|A_1B_1 + A_2B_2\| \le \sqrt{\|A_1A_1^* + A_2A_2^*\| \, \|B_1^*B_1 + B_2^*B_2\|}.$$

Following the proof of Theorem 2.1, we may state the following result, which gives multiple equivalent statements that involve certain identities, which are inequalities in general. One can refer to [26] for a similar discussion.

Theorem 2.3. *Let $A, B \in \mathbb{B}(\mathbb{H})$. Then the following statements are equivalent:*

(i) $\|AB\| = \|A\| \, \|B\|$;
(ii) $\omega^{\frac{1}{2}}\left(|A|^2|B^*|^2\right) = \|A\| \, \|B\|$;
(iii) $\left\| |A|^2 + |B^*|^2 \right\| = \|A\|^2 + \|B\|^2$;
(iv) $\left\| |A|^2|B^*|^2 \right\| = \|A\|^2\|B\|^2$;
(v) $\sqrt{r\left(|A|^2|B^*|^2\right)} = \|A\| \, \|B\|$.

Proof. If $A = O$ or $B = O$, the result follows immediately. So, without loss of generality, we may assume that A and B are both nonzero.
$(i) \Rightarrow (ii)$ If $\|AB\| = \|A\| \, \|B\|$, then by (2.2), we get

$$\|A\| \, \|B\| \le \omega^{\frac{1}{2}}\left(|A|^2|B^*|^2\right). \tag{2.3}$$

On the other hand, we know that

$$\omega^{\frac{1}{2}}\left(|A|^2|B^*|^2\right) \le \left\| |A|^2|B^*|^2 \right\|^{\frac{1}{2}} \le \left\| |A|^2 \right\|^{\frac{1}{2}} \left\| |B^*|^2 \right\|^{\frac{1}{2}} = \| \, |A| \, \| \, \| \, |B^*| \, \| = \|A\| \, \|B\|. \tag{2.4}$$

Inequalities (2.3) and (2.4) imply $\omega^{\frac{1}{2}}\left(|A|^2|B^*|^2\right) = \|A\| \, \|B\|$.

$(ii) \Rightarrow (i)$ Assume that $\omega^{\frac{1}{2}}\left(|A|^2|B^*|^2\right) = \|A\| \, \|B\|$. Then then there exists a sequence of unit vectors $\{x_n\}$ such that $\|A\|^2\|B\|^2 = \lim_{n\to\infty} |\langle A^*ABB^*x_n, x_n\rangle|$, and so

$$\|A\|^2\|B\|^2 = \lim_{n\to\infty} |\langle ABB^*x_n, Ax_n\rangle| \le \|AB\| \limsup_{n\to\infty} \|Ax_n\| \, \|B^*x_n\| \le \|AB\| \, \|A\| \, \|B\|.$$

Hence, $\|A\| \, \|B\| \le \|AB\|$. But the inequality $\|AB\| \le \|A\| \, \|B\|$ is always true. These two inequalities imply $\|AB\| = \|A\| \, \|B\|$.

$(i) \Leftrightarrow (iii)$ See [26].
$(i) \Leftrightarrow (iv)$ See [26].
$(i) \Leftrightarrow (v)$ This follows from the following observation:

$$\|AB\| = \| \, |A| \, |B^*| \, \| = \|(|A| \, |B^*|) \, (|A| \, |B^*|)^*\|^{\frac{1}{2}} = \sqrt{r\left(|A|^2 |B^*|^2\right)}.$$

□

Now, as an application of Theorem 2.1, we present the following upper bound for the norm of the sum of two operators.

Corollary 2.4. *Let $S, T \in \mathbb{B}(\mathbb{H})$ be invertible. Then for any $0 \le t \le 1$,*

$$\begin{aligned}\|S+T\| &\le \frac{1}{2}\sqrt{\left\||S|^{2t} + |T|^{2(1-t)}\right\| \left\||S^*|^{2(1-t)} + |T^*|^{2t}\right\|} \\ &\quad + \frac{1}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |S|^2 + |S|^{-t}S^*T|S|^t & |S|^{2-t}|T|^{1-t} + |S|^{-t}S^*T|T|^{1-t} \\ |T|^{1+t}|S|^t + |T|^{t-1}T^*S|S|^t & |T|^2 + |T|^{t-1}T^*S|T|^{1-t} \end{bmatrix}\right) \\ &\le \sqrt{\left\||S|^{2t} + |T|^{2(1-t)}\right\| \left\||S^*|^{2(1-t)} + |T^*|^{2t}\right\|}.\end{aligned}$$

In particular,

$$\begin{aligned}\|S+T\| &\le \frac{1}{2}\sqrt{\| \, |S| + |T| \, \| \, \| \, |S^*| + |T^*| \, \|} \\ &\quad + \frac{1}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |S|^2 + |S|^{-\frac{1}{2}}S^*T|S|^{\frac{1}{2}} & |S|^{\frac{3}{2}}|T|^{\frac{1}{2}} + |S|^{-\frac{1}{2}}S^*T|T|^{\frac{1}{2}} \\ |T|^{\frac{3}{2}}|S|^{\frac{1}{2}} + |T|^{-\frac{1}{2}}T^*S|S|^{\frac{1}{2}} & |T|^2 + |T|^{-\frac{1}{2}}T^*S|T|^{\frac{1}{2}} \end{bmatrix}\right) \\ &\le \sqrt{\| \, |S| + |T| \, \| \, \| \, |S^*| + |T^*| \, \|}.\end{aligned}$$

Proof. Let $S = U\,|S|$ and $T = V\,|T|$ be the polar decompositions of S and T, respectively. Let $A_1 = U|S|^{1-t}$, $B_1 = |S|^t$, $A_2 = V|T|^t$, and $B_2 = |T|^{1-t}$. Then we observe that

$$\begin{aligned}A_1A_1^* + A_2A_2^* &= U|S|^{1-t}|S|^{1-t}U^* + V|T|^t|T|^tV^* \\ &= U|S|^{2(1-t)}U^* + V|T|^{2t}V^* \\ &= |S^*|^{2(1-t)} + |T^*|^{2t} \quad \text{(by Lemma 1.3)},\end{aligned}$$

$$\begin{aligned}&B_1^*B_1 + B_2^*B_2 = |S|^t|S|^t + |T|^{1-t}|T|^{1-t} = |S|^{2t} + |T|^{2(1-t)}, \\ &A_1^*A_1B_1B_1^* + A_1^*A_2B_2B_1^* \\ &= |S|^{1-t}U^*U|S|^{1-t}|S|^t|S|^t + |S|^{1-t}U^*V|T|^t|T|^{1-t}|S|^t \\ &= |S|^{1-t}U^*U|S|^{1-t}|S|^{2t} + |S|^{1-t}U^*V\,|T|\,|S|^t \\ &= |S|^2 + |S|^{1-t}U^*V\,|T|\,|S|^t \quad \text{(since } U^*U \text{ is a projection)} \\ &= |S|^2 + |S|^{-t}\,|S|\,U^*V\,|T|\,|S|^t \\ &= |S|^2 + |S|^{-t}S^*T|S|^t \quad \text{(since } S^* = |S|\,U^* \text{ and } T = V\,|T|),\end{aligned}$$

$$\begin{aligned}&A_1^*A_1B_1B_2^* + A_1^*A_2B_2B_2^* \\ &= |S|^{1-t}U^*U|S|^{1-t}|S|^t|T|^{1-t} + |S|^{1-t}U^*V|T|^t|T|^{1-t}|T|^{1-t} \\ &= |S|^{2-t}|T|^{1-t} + |S|^{1-t}U^*V\,|T|\,|T|^{1-t} \quad \text{(since } U^*U \text{ is projection)} \\ &= |S|^{2-t}|T|^{1-t} + |S|^{-t}\,|S|\,U^*V\,|T|\,|T|^{1-t} \\ &= |S|^{2-t}|T|^{1-t} + |S|^{-t}S^*T|T|^{1-t} \quad \text{(since } S^* = |S|\,U^* \text{ and } T = V\,|T|),\end{aligned}$$

$$\begin{aligned}
&A_2^*A_1B_1B_1^* + A_2^*A_2B_2B_1^* \\
&= |T|^tV^*U|S|^{1-t}|S|^t|S|^t + |T|^tV^*V|T|^t|T|^{1-t}|S|^t \\
&= |T|^tV^*U\,|S|\,|S|^t + |T|^{1+t}|S|^t \quad \text{(since } V^*V \text{ is projection)} \\
&= |T|^{t-1}\,|T|\,V^*U\,|S|\,|S|^t + |T|^{1+t}|S|^t \\
&= |T|^{t-1}T^*S|S|^t + |T|^{1+t}|S|^t \quad \text{(since } T^* = |T|\,V^* \text{ and } S = U\,|S|\text{)},
\end{aligned}$$

and

$$\begin{aligned}
&A_2^*A_1B_1B_2^* + A_2^*A_2B_2B_2^* \\
&= |T|^tV^*U|S|^{1-t}|S|^t|T|^{1-t} + |T|^tV^*V|T|^t|T|^{1-t}|T|^{1-t} \\
&= |T|^tV^*U\,|S|\,|T|^{1-t} + |T|^2 \quad \text{(since } V^*V \text{ is a projection)} \\
&= |T|^{t-1}\,|T|\,V^*U\,|S|\,|T|^{1-t} + |T|^2 \\
&= |T|^{t-1}T^*S|T|^{1-t} + |T|^2 \quad \text{(since } T^* = |T|\,V^* \text{ and } S = U\,|S|\text{)}.
\end{aligned}$$

The result follows by Theorem 2.1, noting Remark 2.2. □

Remark 2.5. Let $T \in \mathbb{B}(\mathbb{H})$.

(i) Replacing S by T^* in Corollary 2.4 yields

$$\begin{aligned}
\|\Re T\| &\le \frac{1}{4}\left\||T|^{2(1-t)} + |T^*|^{2t}\right\| \\
&\quad + \frac{1}{4}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |T^*|^2 + |T^*|^{-t}T^2|T^*|^t & |T^*|^{2-t}|T|^{1-t} + |T^*|^{-t}T^2|T|^{1-t} \\ |T|^{1+t}|T^*|^t + |T|^{t-1}(T^*)^2|T^*|^t & |T|^2 + |T|^{t-1}(T^*)^2|T|^{1-t} \end{bmatrix}\right) \\
&\le \frac{1}{2}\left\||T|^{2(1-t)} + |T^*|^{2t}\right\|.
\end{aligned}$$

(ii) Replacing T by T^* and S by $-T$ in Corollary 2.4 yield

$$\begin{aligned}
\|\Im T\| &\le \frac{1}{4}\left\||T^*|^{2(1-t)} + |T|^{2t}\right\| \\
&\quad + \frac{1}{4}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |T|^2 - |T|^{-t}(T^*)^2|T|^t & |T|^{2-t}|T^*|^{1-t} - |T|^{-t}(T^*)^2|T^*|^{1-t} \\ |T^*|^{1+t}|T|^t - |T^*|^{t-1}T^2|T|^t & |T^*|^2 - |T^*|^{t-1}T^2|T^*|^{1-t} \end{bmatrix}\right) \\
&\le \frac{1}{2}\left\||T^*|^{2(1-t)} + |T|^{2t}\right\|.
\end{aligned}$$

In particular,

(iii)

$$\begin{aligned}
\|\Re T\| &\le \frac{1}{4}\|\,|T| + |T^*|\,\| \\
&\quad + \frac{1}{4}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |T^*|^2 + |T^*|^{-\frac{1}{2}}T^2|T^*|^{\frac{1}{2}} & |T^*|^{\frac{3}{2}}|T|^{\frac{1}{2}} + |T^*|^{-\frac{1}{2}}T^2|T|^{\frac{1}{2}} \\ |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}} + |T|^{-\frac{1}{2}}(T^*)^2|T^*|^{\frac{1}{2}} & |T|^2 + |T|^{-\frac{1}{2}}(T^*)^2|T|^{\frac{1}{2}} \end{bmatrix}\right) \\
&\le \frac{1}{2}\|\,|T| + |T^*|\,\|.
\end{aligned}$$

(iv)

$$\begin{aligned}
\|\Im T\| &\le \frac{1}{4}\|\,|T| + |T^*|\,\| \\
&\quad + \frac{1}{4}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |T|^2 - |T|^{-\frac{1}{2}}(T^*)^2|T|^{\frac{1}{2}} & |T|^{\frac{3}{2}}|T^*|^{\frac{1}{2}} - |T|^{-\frac{1}{2}}(T^*)^2|T^*|^{\frac{1}{2}} \\ |T^*|^{\frac{3}{2}}|T|^{\frac{1}{2}} - |T^*|^{-\frac{1}{2}}T^2|T|^{\frac{1}{2}} & |T^*|^2 - |T^*|^{-\frac{1}{2}}T^2|T^*|^{\frac{1}{2}} \end{bmatrix}\right) \\
&\le \frac{1}{2}\|\,|T| + |T^*|\,\|.
\end{aligned}$$

Remark 2.6. If S, T are normal operators, then Corollary 2.4 ensures that

$$\begin{aligned}\|S+T\| &\leq \frac{1}{2}\|\,|S|+|T|\,\| \\ &\quad+\frac{1}{2}\omega^{\frac{1}{2}}\left(\begin{bmatrix} |S|^2+|S|^{-\frac{1}{2}}S^*T|S|^{\frac{1}{2}} & |S|^{\frac{3}{2}}|T|^{\frac{1}{2}}+|S|^{-\frac{1}{2}}S^*T|T|^{\frac{1}{2}} \\ |T|^{\frac{3}{2}}|S|^{\frac{1}{2}}+|T|^{-\frac{1}{2}}T^*S|S|^{\frac{1}{2}} & |T|^2+|T|^{-\frac{1}{2}}T^*S|T|^{\frac{1}{2}} \end{bmatrix}\right) \\ &\leq \|\,|S|+|T|\,\|.\end{aligned}$$

Thus, if S, T are positive operators, then

$$\begin{aligned}\|S+T\| &= \omega^{\frac{1}{2}}\left(\begin{bmatrix} S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}} & S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}} \\ T^{\frac{3}{2}}S^{\frac{1}{2}}+T^{\frac{1}{2}}S^{\frac{3}{2}} & T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}} \end{bmatrix}\right) \\ &= \left\|\begin{bmatrix} S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}} & S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}} \\ T^{\frac{3}{2}}S^{\frac{1}{2}}+T^{\frac{1}{2}}S^{\frac{3}{2}} & T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}} \end{bmatrix}\right\|^{\frac{1}{2}}.\end{aligned} \tag{2.5}$$

For positive invertible operators, (2.5) may be utilized to obtain the following upper bound for the norm of the sum of two such operators.

Corollary 2.7. *Let $S, T \in \mathbb{B}(\mathbb{H})$ be two positive operators. Then*

$$\begin{aligned}\|S+T\|^2 &\leq \frac{1}{2}\left(\left\|S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right\|+\left\|T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}}\right\|\right) \\ &\quad+\frac{1}{2}\sqrt{\left(\left\|S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right\|-\left\|T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}}\right\|\right)^2+4\left\|S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}}\right\|^2}.\end{aligned}$$

Proof. It follows from (2.5) and Lemma 1.4 that

$$\begin{aligned}&\|S+T\|^2 \\ &= \left\|\begin{bmatrix} S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}} & S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}} \\ T^{\frac{3}{2}}S^{\frac{1}{2}}+T^{\frac{1}{2}}S^{\frac{3}{2}} & T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}} \end{bmatrix}\right\| \\ &\leq \left\|\begin{bmatrix} \left\|S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right\| & \left\|S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}}\right\| \\ \left\|T^{\frac{3}{2}}S^{\frac{1}{2}}+T^{\frac{1}{2}}S^{\frac{3}{2}}\right\| & \left\|T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}}\right\| \end{bmatrix}\right\| \\ &= \frac{1}{2}\left(\left\|S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right\|+\left\|T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}}\right\|\right) \\ &\quad+\frac{1}{2}\sqrt{\left(\left\|S^2+S^{\frac{1}{2}}TS^{\frac{1}{2}}\right\|-\left\|T^2+T^{\frac{1}{2}}ST^{\frac{1}{2}}\right\|\right)^2+4\left\|S^{\frac{3}{2}}T^{\frac{1}{2}}+S^{\frac{1}{2}}T^{\frac{3}{2}}\right\|^2},\end{aligned}$$

where the last equality is obtained by direct calculations of the norm as the largest singular value of the matrix in context. This completes the proof. □

2.2. Upper bounds for the numerical radius

Related to the above discussion of the operator norm, we discuss some bounds for the numerical radius in the same context. In this section, it will be implicitly understood that the denominators are not zero by assumption.

Theorem 2.8. *Let $A_1, A_2, B_1, B_2 \in \mathbb{B}(\mathbb{H})$. Then*

$$\begin{aligned}&\omega(B_1A_1+B_2A_2) \\ &\leq \sqrt{\frac{\|A_1^*A_1+A_2^*A_2\|\,\|B_1B_1^*+B_2B_2^*\|}{\|A_1^*A_1+A_2^*A_2\|+\|B_1B_1^*+B_2B_2^*\|}}\,\|A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^*\|^{\frac{1}{2}}.\end{aligned}$$

Proof. Let $0<\nu<1$. For any unit vector $x\in\mathbb{H}$, we have

$$\begin{aligned}
\left|\left\langle \mathbf{B}^*\mathbf{A}x,x\right\rangle\right| &= \left|\left\langle \mathbf{A}x,\mathbf{B}x\right\rangle\right| \\
&\le \left\|\mathbf{A}x\right\|\left\|\mathbf{B}x\right\| \quad \text{(by the Cauchy-Schwarz inequality)} \\
&= \sqrt{\left\langle |\mathbf{A}|^2x,x\right\rangle\left\langle |\mathbf{B}|^2x,x\right\rangle} \\
&= \sqrt{\left\langle |\mathbf{A}|^{2\frac{\nu}{\nu}}x,x\right\rangle\left\langle |\mathbf{B}|^{2\frac{1-\nu}{1-\nu}}x,x\right\rangle} \\
&\le \sqrt{\left\langle |\mathbf{A}|^{\frac{2}{\nu}}x,x\right\rangle^{\nu}\left\langle |\mathbf{B}|^{\frac{2}{1-\nu}}x,x\right\rangle^{1-\nu}} \quad \text{(by the Hölder-McCarthy inequality)} \\
&\le \sqrt{\nu\left\langle |\mathbf{A}|^{\frac{2}{\nu}}x,x\right\rangle+(1-\nu)\left\langle |\mathbf{B}|^{\frac{2}{1-\nu}}x,x\right\rangle} \quad \text{(by the Young inequality)} \\
&= \sqrt{\left\langle \left(\nu|\mathbf{A}|^{\frac{2}{\nu}}+(1-\nu)\,|\mathbf{B}|^{\frac{2}{1-\nu}}\right)x,x\right\rangle} \\
&\le \left\|\nu|\mathbf{A}|^{\frac{2}{\nu}}+(1-\nu)\,|\mathbf{B}|^{\frac{2}{1-\nu}}\right\|^{\frac{1}{2}}.
\end{aligned}$$

That is,

$$\left|\left\langle \mathbf{B}^*\mathbf{A}x,x\right\rangle\right| \le \left\|\nu|\mathbf{A}|^{\frac{2}{\nu}}+(1-\nu)\,|\mathbf{B}|^{\frac{2}{1-\nu}}\right\|^{\frac{1}{2}}.$$

By taking the supremum over $x\in\mathbb{H}$ with $\|x\|=1$, we get

$$\omega\left(\mathbf{B}^*\mathbf{A}\right) \le \left\|\nu|\mathbf{A}|^{\frac{2}{\nu}}+(1-\nu)\,|\mathbf{B}|^{\frac{2}{1-\nu}}\right\|^{\frac{1}{2}}. \tag{2.6}$$

If we replace $\mathbf{A}$ and $\mathbf{B}$ by $\frac{\mathbf{A}}{\|\mathbf{A}\|}$ and $\frac{\mathbf{B}}{\|\mathbf{B}\|}$, in (2.6), we infer that

$$\omega\left(\frac{\mathbf{B}^*\mathbf{A}}{\|\mathbf{A}\|\,\|\mathbf{B}\|}\right) \le \left\|\nu\left(\frac{|\mathbf{A}|^2}{\|\mathbf{A}\|^2}\right)^{\frac{1}{\nu}}+(1-\nu)\left(\frac{|\mathbf{B}|^2}{\|\mathbf{B}\|^2}\right)^{\frac{1}{1-\nu}}\right\|^{\frac{1}{2}}. \tag{2.7}$$

If we use the fact that if $O\le X\le I$, then $X^p\le X$ for $p\ge 1$ (see [17]), we obtain from (2.7)

$$\omega\left(\frac{\mathbf{B}^*\mathbf{A}}{\|\mathbf{A}\|\,\|\mathbf{B}\|}\right) \le \left\|\nu\frac{|\mathbf{A}|^2}{\|\mathbf{A}\|^2}+(1-\nu)\,\frac{|\mathbf{B}|^2}{\|\mathbf{B}\|^2}\right\|^{\frac{1}{2}}. \tag{2.8}$$

Now putting $\nu=\frac{\|\mathbf{A}\|^2}{\|\mathbf{A}\|^2+\|\mathbf{B}\|^2}$ and $1-\nu=\frac{\|\mathbf{B}\|^2}{\|\mathbf{A}\|^2+\|\mathbf{B}\|^2}$, in (2.8), we deduce that

$$\omega\left(\frac{\mathbf{B}^*\mathbf{A}}{\|\mathbf{A}\|\,\|\mathbf{B}\|}\right) \le \left\|\frac{\|\mathbf{A}\|^2}{\|\mathbf{A}\|^2+\|\mathbf{B}\|^2}\frac{|\mathbf{A}|^2}{\|\mathbf{A}\|^2}+\frac{\|\mathbf{B}\|^2}{\|\mathbf{A}\|^2+\|\mathbf{B}\|^2}\frac{|\mathbf{B}|^2}{\|\mathbf{B}\|^2}\right\|^{\frac{1}{2}},$$

which is equivalent to

$$\omega\left(\mathbf{B}^*\mathbf{A}\right) \le \frac{\|\mathbf{A}\|\,\|\mathbf{B}\|}{\sqrt{\|\mathbf{A}\|^2+\|\mathbf{B}\|^2}}\left\||\mathbf{A}|^2+|\mathbf{B}|^2\right\|^{\frac{1}{2}}. \tag{2.9}$$

Let $\mathbf{A}=\begin{bmatrix} A_1 & O \\ A_2 & O \end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix} B_1^* & O \\ B_2^* & O \end{bmatrix}$. Simple calculations reveal that

$$\omega\left(\mathbf{B}^*\mathbf{A}\right)=\omega\left(\left[\begin{bmatrix} B_1 & B_2 \\ O & O \end{bmatrix}\begin{bmatrix} A_1 & O \\ A_2 & O \end{bmatrix}\right]\right)=\omega\left(\begin{bmatrix} B_1A_1+B_2A_2 & O \\ O & O \end{bmatrix}\right)=\omega\left(B_1A_1+B_2A_2\right),$$

$$\begin{aligned}\left\||\mathbf{A}|^2+|\mathbf{B}|^2\right\| &= \left\|\begin{bmatrix} A_1^* & A_2^* \\ O & O\end{bmatrix}\begin{bmatrix} A_1 & O \\ A_2 & O\end{bmatrix}+\begin{bmatrix} B_1 & B_2 \\ O & O\end{bmatrix}\begin{bmatrix} B_1^* & O \\ B_2^* & O\end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} A_1^*A_1+A_2^*A_2 & O \\ O & O\end{bmatrix}+\begin{bmatrix} B_1B_1^*+B_2B_2^* & O \\ O & O\end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^* & O \\ O & O\end{bmatrix}\right\| \\ &= \left\|A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^*\right\|,\end{aligned}$$

$$\begin{aligned}\|\mathbf{A}\| &= \left\|\begin{bmatrix} A_1 & O \\ A_2 & O\end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} A_1^* & A_2^* \\ O & O\end{bmatrix}\begin{bmatrix} A_1 & O \\ A_2 & O\end{bmatrix}\right\|^{\frac{1}{2}} \\ &= \left\|\begin{bmatrix} A_1^*A_1+A_2^*A_2 & O \\ O & O\end{bmatrix}\right\|^{\frac{1}{2}} \\ &= \|A_1^*A_1+A_2^*A_2\|^{\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\|\mathbf{B}\| &= \left\|\begin{bmatrix} B_1^* & O \\ B_2^* & O\end{bmatrix}\right\| \\ &= \left\|\begin{bmatrix} B_1 & B_2 \\ O & O\end{bmatrix}\begin{bmatrix} B_1^* & O \\ B_2^* & O\end{bmatrix}\right\|^{\frac{1}{2}} \\ &= \left\|\begin{bmatrix} B_1B_1^*+B_2B_2^* & O \\ O & O\end{bmatrix}\right\|^{\frac{1}{2}} \\ &= \|B_1B_1^*+B_2B_2^*\|^{\frac{1}{2}}.\end{aligned}$$

Now, the result follows directly from (2.9). □

The following corollary is an immediate consequence of Theorem 2.8, which improves the sub-multiplicative property of the operator norm.

Corollary 2.9. *Let $A,B\in\mathbb{B}(\mathbb{H})$ be nonzero operators. If B^*A is normal, then*

$$\|B^*A\|\le\frac{\left\||A|^2+|B|^2\right\|^{\frac{1}{2}}}{\sqrt{\|A\|^2+\|B\|^2}}\ \|A\|\,\|B\|\,.$$

Remark 2.10. By the triangle inequality for the operator norm, we have

$$\left\||A|^2+|B|^2\right\|^{\frac{1}{2}}\le\sqrt{\left\||A|^2\right\|+\left\||B|^2\right\|}=\sqrt{\|\,|A|\,\|^2+\|\,|B|\,\|^2}=\sqrt{\|A\|^2+\|B\|^2},\qquad(2.10)$$

i.e., $\frac{\left\||A|^2+|B|^2\right\|^{\frac{1}{2}}}{\sqrt{\|A\|^2+\|B\|^2}}\le 1$. Therefore, Corollary 2.9 provides an improvement of the well-known inequality $\|B^*A\|\le\|A\|\,\|B\|$.

Remark 2.11. Inequality (2.9) can also be written in the following arrangement, provided that $B^*A\ne O$,

$$\sqrt{\|A\|^2+\|B\|^2}\le\frac{\|A\|\,\|B\|}{\omega\left(B^*A\right)}\left\||A|^2+|B|^2\right\|^{\frac{1}{2}},$$

which delivers a reverse for the inequality (2.10). Of course, if A and B are positive operators such that $B^{\frac{1}{2}}A^{\frac{1}{2}}\ne O$, then

$$\|A\|+\|B\|\le\frac{\|A\|\,\|B\|}{\omega^2\left(B^{\frac{1}{2}}A^{\frac{1}{2}}\right)}\ \|A+B\|\,,$$

which presents a reverse for the triangle inequality.

Corollary 2.12. *Let* $A_1, A_2, B_1, B_2 \in \mathbb{B}(\mathbb{H})$. *Then*

$$\begin{aligned}&\omega\left(B_1A_1+B_2A_2\right)\\&\le\sqrt{\frac{\left\|A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^*\right\|}{\left\|A_1^*A_1+A_2^*A_2\right\|+\left\|B_1B_1^*+B_2B_2^*\right\|}}\sqrt{\left\|A_1^*A_1+A_2^*A_2\right\|\left\|B_1B_1^*+B_2B_2^*\right\|}.\end{aligned}$$

Proof. It follows from Theorem 2.8 that

$$\begin{aligned}&\left\|A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^*\right\|\\&\le\left\|A_1^*A_1+A_2^*A_2\right\|+\left\|B_1B_1^*+B_2B_2^*\right\|\\&\quad\text{(by the triangle inequality for the operator norm)}\\&\le\frac{\left\|A_1^*A_1+A_2^*A_2\right\|\left\|B_1B_1^*+B_2B_2^*\right\|}{\omega^2\left(B_1A_1+B_2A_2\right)}\left\|A_1^*A_1+A_2^*A_2+B_1B_1^*+B_2B_2^*\right\|.\end{aligned}$$

The result follows directly from the above inequality. ∎

Remark 2.13. If we replace A_1, B_1, A_2, B_2 by $\sqrt{\frac{\|B_1\|}{\|A_1\|}}A_1$, $\sqrt{\frac{\|A_1\|}{\|B_1\|}}B_1$, $\sqrt{\frac{\|B_2\|}{\|A_2\|}}A_2$, and $\sqrt{\frac{\|A_2\|}{\|B_2\|}}B_2$, in Corollary 2.12, we obtain

$$\begin{aligned}&\omega\left(B_1A_1+B_2A_2\right)\\&\le\sqrt{\frac{\left\|\frac{\|B_1\|}{\|A_1\|}A_1^*A_1+\frac{\|B_2\|}{\|A_2\|}A_2^*A_2+\frac{\|A_1\|}{\|B_1\|}B_1B_1^*+\frac{\|A_2\|}{\|B_2\|}B_2B_2^*\right\|}{\left\|\frac{\|B_1\|}{\|A_1\|}A_1^*A_1+\frac{\|B_2\|}{\|A_2\|}A_2^*A_2\right\|+\left\|\frac{\|A_1\|}{\|B_1\|}B_1B_1^*+\frac{\|A_2\|}{\|B_2\|}B_2B_2^*\right\|}}\\&\quad\times\sqrt{\left\|\frac{\|B_1\|}{\|A_1\|}A_1^*A_1+\frac{\|B_2\|}{\|A_2\|}A_2^*A_2\right\|\left\|\frac{\|A_1\|}{\|B_1\|}B_1B_1^*+\frac{\|A_2\|}{\|B_2\|}B_2B_2^*\right\|}\\&\le\sqrt{\frac{\left\|\frac{\|B_1\|}{\|A_1\|}A_1^*A_1+\frac{\|B_2\|}{\|A_2\|}A_2^*A_2+\frac{\|A_1\|}{\|B_1\|}B_1B_1^*+\frac{\|A_2\|}{\|B_2\|}B_2B_2^*\right\|}{\left\|\frac{\|B_1\|}{\|A_1\|}A_1^*A_1+\frac{\|B_2\|}{\|A_2\|}A_2^*A_2\right\|+\left\|\frac{\|A_1\|}{\|B_1\|}B_1B_1^*+\frac{\|A_2\|}{\|B_2\|}B_2B_2^*\right\|}}\\&\quad\times\sqrt{\max\left\{\|A_1\|\|B_1\|,\|A_2\|\|B_2\|\right\}+\sqrt{\frac{\|B_1\|\|B_2\|}{\|A_1\|\|A_2\|}}\left\|A_1A_2^*\right\|}\\&\quad\times\sqrt{\max\left\{\|A_1\|\|B_1\|,\|A_2\|\|B_2\|\right\}+\sqrt{\frac{\|A_1\|\|A_2\|}{\|B_1\|\|B_2\|}}\left\|B_1^*B_2\right\|}.\end{aligned}$$

Corollary 2.14. *Let* $S,T\in\mathbb{B}(\mathbb{H})$. *Then for any* $0\le t\le 1$,

$$\begin{aligned}&\omega\left(S+T\right)\\&\le\sqrt{\frac{\left\||S|^{2t}+|T|^{2t}+|S^*|^{2(1-t)}+|T^*|^{2(1-t)}\right\|}{\left\||S|^{2t}+|T|^{2t}\right\|+\left\||S^*|^{2(1-t)}+|T^*|^{2(1-t)}\right\|}}\sqrt{\left\||S|^{2t}+|T|^{2t}\right\|\left\||S^*|^{2(1-t)}+|T^*|^{2(1-t)}\right\|}.\end{aligned}$$

Proof. Let $S=V|S|$ and $T=U|T|$ be the polar decomposition of S and T. By letting $A_1=|S|^t$, $B_1=V|S|^{1-t}$, $A_2=|T|^t$, $B_2=U|T|^{1-t}$, in Corollary 2.12, we reach to the desired result. ∎

Corollary 2.15. *Let $T \in \mathbb{B}(\mathbb{H})$. Then for any $0 \le t \le 1$,*

$$\omega(T) \le \frac{1}{2}\sqrt{\frac{\left\| |T|^{2t} + |T^*|^{2t} + |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}{\left\| |T|^{2t} + |T^*|^{2t} \right\| + \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}} \sqrt{\left\| |T|^{2t} + |T^*|^{2t} \right\| \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}.$$

Proof. Letting $S = T^*$ in Corollary 2.14, we get

$$2\|\Re T\| \le \sqrt{\frac{\left\| |T|^{2t} + |T^*|^{2t} + |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}{\left\| |T|^{2t} + |T^*|^{2t} \right\| + \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}} \sqrt{\left\| |T|^{2t} + |T^*|^{2t} \right\| \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}.$$

Substituting T by $e^{i\theta}T$, in the above inequality, implies that

$$2\left\|\Re\left(e^{i\theta}T\right)\right\| \le \sqrt{\frac{\left\| |T|^{2t} + |T^*|^{2t} + |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}{\left\| |T|^{2t} + |T^*|^{2t} \right\| + \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}} \sqrt{\left\| |T|^{2t} + |T^*|^{2t} \right\| \left\| |T|^{2(1-t)} + |T^*|^{2(1-t)} \right\|}.$$

The result follows by taking the supremum over $\theta \in \mathbb{R}$. □

Remark 2.16. The case $t = \frac{1}{2}$, namely

$$\omega(T) \le \frac{1}{2}\| \, |T| + |T^*| \, \|,$$

has been shown in [27, Ineq. (8)].

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