Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 17
DOI : 10.15672/hujms.xx

Research Article

On transitive permutation groups with bounded movement

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Abstract

Let G be a permutation group on a set Ω. Then for each $g \in G$, we define the movement of g, denoted by move(g), the maximal cardinality $|\Delta^g \backslash \Delta|$ of $\Delta^g \backslash \Delta$ over all subsets Δ of Ω. And the movement of G is defined as the maximum of move(g) over all $g \in G$, denoted by move(G). A permutation group G is said to have bounded movement if it has movement bounded by some positive integer m, that is move$(G) \leq m$. In this paper, we consider the finite transitive permutation groups G with movement move$(G) = m$ for some positive integer $m > 4$, where G is not a 2-group but in which every non-identity element has the movement m or $m - 4$, and there is at least one non-identity element that has the movement $m - 4$. We give a characterization for elements of G in Theorem 1.1. Further, we apply Theorem 1.1 to characterize transitive permutation group G in Theorem 1.2. These results give a partial answer to the open problem posed by the authors in 2024.

Mathematics Subject Classification (2020). 20B05

Keywords. transitive permutation group, movement, action, non-identity element, p-group

1. Introduction

Let G be a permutation group on a set Ω. Then for each $g \in G$, we define the movement of g, denoted by move(g), the maximal cardinality $|\Delta^g \backslash \Delta|$ of $\Delta^g \backslash \Delta$ over all subsets Δ of Ω. And the movement of G is defined as the maximum of move(g) over all $g \in G$, denoted by move(G). A permutation group G is said to have bounded movement if it has movement bounded by some positive integer m, that is move$(G) \leq m$.

The permutation groups with bounded movement have been studied extensively in the past a few decades, see [1–6, 8–10]. It was shown in [9] that if permutation group G has bounded movement m, and if G has no fixed points in Ω, then Ω is finite, and $|\Omega|$ is

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Received: 05.11.2024; Accepted: 23.03.2025

bounded by a function of m. In particular, if G is transitive, then $|\Omega| \le 3m$. In [8], the authors completed the proof of a conjecture of Gardiner and Praeger that the only transitive groups on a set of size $3m$ which have movement m are transitive permutation groups of exponent 3 when m is a power of 3, the symmetric group S_3 in its natural representation on a set of three points, and the alternating groups A_4 and A_5, in their transitive representation on six points. The transitive permutation groups with bounded movement having maximal degree were classified by A.Hassni et al in [6]. In 2005, Alaeiyan and Yoshiara considered the permutation groups G of minimal movement, and showed that if G is not a 2-group and p is the least odd prime diving the order of G, then $|\Omega| \le 4m - p$ or $n = 4m - p + 2$. Moreover, the groups G attaining the maximum bound were classified, see [4]. Recently, the transitive permutation groups G with bounded movement m, such that G is not a 2-group but in which every non-identity element has movement m, m or $m-1$, and m or $m-2$ are classified in [1–3], respectively.

In 2024, we characterized all transitive permutation groups G with movement $\text{move}(G) = m$ for some positive integer m, where G is not a 2-group but in which every non-identity element has the movement m or $m-3$, and there is at least one non-identity element that has the movement $m-3$ in [7]. In the same paper, we posed an open problem.

Open problem. Characterize the finite transitive permutation groups G with movement $\text{move}(G) = m$, where G is not a 2-group but in which there is at least one non-identity element that the movement is less than m.

In this paper, we give a characterization of transitive permutation groups G with movement $\text{move}(G) = m$, where G is not a 2-group but in which every non-identity element has the movement m or $m-4$, and there is at least one non-identity element that has the movement $m-4$. This gives a partial answer to the open problem above. First, we give a characterization for elements in G.

Theorem 1.1. Let G be a transitive permutation group on a set Ω with no fixed point in Ω, and let $\text{move}(G) = m$ for some positive integer $m > 4$. Suppose that every non-identity element in G has the movement m or $m-4$. Let $1 \ne g \in G$ and $g = c_1c_2\cdots c_t$ as a product of disjoint cycles of lengths $l_1, l_2, \cdots, l_t$. Then one of the following holds:

(1) $g := g^*_{2^a} = c_1c_2\cdots c_t$ with $l_1 = l_2 = \cdots = l_t = 2^a$ for $a \ge 1$, and $\text{move}(g) = t2^{a-1}$; $g := g^*_p = c_1c_2\cdots c_t$ with $l_1 = l_2 = \cdots = l_t = p$ an odd prime, and $\text{move}(g) = t\frac{p-1}{2}$;
(2) $t = 1$, $g := g_{8p} = c_1$ with p an odd prime, $l_1 = 8p$, and $\text{move}(g) = 4p$;
(3) $t = 2$, $g := g_{5,25} = c_1c_2$ with $l_1 = 5$ and $l_2 = 25$, and $\text{move}(g) = 14$;
(4) $t = 2$, $g := g_{25,25} = c_1c_2$ with $l_1 = l_2 = 25$, and $\text{move}(g) = 24$;
(5) $t = 2$, $g := g_{4,4p} = c_1c_2$ with p an odd prime, $l_1 = 4$ and $l_2 = 4p$, and $\text{move}(g) = 2 + 2p$;
(6) $t = 2$, $g := g_{4p,4p} = c_1c_2$ with p an odd prime, $l_1 = l_2 = 4p$, and $\text{move}(g) = 4p$;
(7) $t = 3$, $g := g_{5,5,40} = c_1c_2c_3$ with $l_1 = l_2 = 5$ and $l_3 = 40$, and $\text{move}(g) = 24$;
(8) $t = 3$, $g := g_{5,5,8} = c_1c_2c_3$ with $l_1 = l_2 = 5$ and $l_3 = 8$, and $\text{move}(g) = 8$;
(9) $t = 4$, $g := g_{4,4,5,5} = c_1c_2c_3c_4$ with $l_1 = l_2 = 4$ and $l_3 = l_4 = 5$, and $\text{move}(g) = 8$;
(10) $t = 4$, $g := g_{3,3,3,9} = c_1c_2c_3c_4$ with $l_1 = l_2 = l_3 = 3$ and $l_4 = 9$, and $\text{move}(g) = 7$;
(11) $t = 4$, $g := g_{3,3,9,9} = c_1c_2c_3c_4$ with $l_1 = l_2 = 3$ and $l_3 = l_4 = 9$, and $\text{move}(g) = 10$;
(12) $t = 4$, $g := g_{3,9,9,9} = c_1c_2c_3c_4$ with $l_1 = 3$ and $l_2 = l_3 = l_4 = 9$, and $\text{move}(g) = 13$;
(13) $t = 4$, $g := g_{9,9,9,9} = c_1c_2c_3c_4$ with $l_1 = l_2 = l_3 = l_4 = 9$, and $\text{move}(g) = 16$;
(14) $t = 4$, $g := g_{2,2,2,2p} = c_1c_2c_3c_4$ with p an odd prime, $l_1 = l_2 = l_3 = 2$ and $l_4 = 2p$, and $\text{move}(g) = 3 + p$;
(15) $t = 4$, $g := g_{2,2,2p,2p} = c_1c_2c_3c_4$ with p an odd prime, $l_1 = l_2 = 2$ and $l_3 = l_4 = 2p$, and $\text{move}(g) = 2p + 2$;

(16) $t = 4$, $g := g_{2,2p,2p,2p} = c_1c_2c_3c_4$ with p an odd prime, $l_1 = 2$ and $l_2 = l_3 = l_4 = 2p$, and $\mathrm{move}(g) = 3p + 1$;
(17) $t = 4$, $g := g_{2p,2p,2p,2p} = c_1c_2c_3c_4$ with p an odd prime, $l_1 = l_2 = l_3 = l_4 = 2p$, and $\mathrm{move}(g) = 4p$;
(18) $t = 4$, $g := g_{3,3,15,15} = c_1c_2c_3c_4$ with $l_1 = l_2 = 3$ and $l_3 = l_4 = 15$, and $\mathrm{move}(g) = 16$;
(19) $t = 4$, $g := g_{4,5,5,20} = c_1c_2c_3c_4$ with $l_1 = 4$, $l_2 = l_3 = 5$ and $l_4 = 20$, and $\mathrm{move}(g) = 16$;
(20) $t = 4$, $g := g_{5,5,20,20} = c_1c_2c_3c_4$ with $l_1 = l_2 = 5$ and $l_3 = l_4 = 20$, and $\mathrm{move}(g) = 24$;
(21) $t = 5$, $g := g_{3,3,3,3,8} = c_1c_2c_3c_4c_5$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = 8$, and $\mathrm{move}(g) = 8$;
(22) $t = 5$, $g := g_{3,3,3,5,15} = c_1c_2c_3c_4c_5$ with $l_1 = l_2 = l_3 = 3$, $l_4 = 5$ and $l_5 = 15$, and $\mathrm{move}(g) = 12$;
(23) $t = 5$, $g := g_{3,3,3,3,24} = c_1c_2c_3c_4c_5$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = 24$, and $\mathrm{move}(g) = 16$;
(24) $t = 6$, $g := g_{3,3,3,3,5,5} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = l_6 = 5$,and $\mathrm{move}(g) = 8$;
(25) $t = 6$, $g := g_{3,3,3,3,4,4} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = l_6 = 4$,and $\mathrm{move}(g) = 8$;
(26) $t = 6$, $g := g_{2,2,2,2,5,5} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = l_4 = 2$ and $l_5 = l_6 = 5$,and $\mathrm{move}\,(g) = 8$;
(27) $t = 6$, $g := g_{2,2,2,5,5,10} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = 2$, $l_4 = l_5 = 5$ and $l_6 = 10$, and $\mathrm{move}(g) = 12$;
(28) $t = 6$, $g := g_{2,2,5,5,10,10} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = 2$, $l_3 = l_4 = 5$ and $l_5 = l_6 = 10$, and $\mathrm{move}(g) = 16$;
(29) $t = 6$, $g := g_{2,5,5,10,10,10} = c_1c_2c_3c_4c_5c_6$ with $l_1 = 2$, $l_2 = l_3 = 5$, $l_4 = l_5 = l_6 = 10$, and $\mathrm{move}(g) = 20$;
(30) $t = 6$, $g := g_{5,5,10,10,10,10} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = 5$ and $l_3 = l_4 = l_5 = l_6 = 10$, and $\mathrm{move}(g) = 24$;
(31) $t = 6$, $g := g_{3,3,3,3,4,12} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = l_4 = 3$, $l_5 = 4$ and $l_6 = 12$, and $\mathrm{move}(g) = 12$;
(32) $t = 6$, $g := g_{3,3,3,3,12,12} = c_1c_2c_3c_4c_5c_6$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = l_6 = 12$, and $\mathrm{move}(g) = 16$;
(33) $t = 8$, $g := g_{2,2,2,2,3,3,3,3} = c_1c_2c_3c_4c_5c_6c_7c_8$ with $l_1 = l_2 = l_3 = l_4 = 2$ and $l_5 = l_6 = l_7 = l_8 = 3$, and $\mathrm{move}(g) = 8$;
(34) $t = 8$, $g := g_{2,2,2,3,3,3,3,6} = c_1c_2c_3c_4c_5c_6c_7c_8$ with $l_1 = l_2 = l_3 = 2$, $l_4 = l_5 = l_6 = l_7 = 3$ and $l_8 = 6$, and $\mathrm{move}(g) = 10$;
(35) $t = 8$, $g := g_{2,2,3,3,3,3,6,6} = c_1c_2c_3c_4c_5c_6c_7c_8$ with $l_1 = l_2 = 2$, $l_3 = l_4 = l_5 = l_6 = 3$ and $l_7 = l_8 = 6$, and $\mathrm{move}(g) = 12$;
(36) $t = 8$, $g := g_{2,3,3,3,3,6,6,6} = c_1c_2c_3c_4c_5c_6c_7c_8$ with $l_1 = 2$, $l_2 = l_3 = l_4 = l_5 = 3$ and $l_6 = l_7 = l_8 = 6$, and $\mathrm{move}(g) = 14$;
(37) $t = 8$, $g := g_{3,3,3,3,6,6,6,6} = c_1c_2c_3c_4c_5c_6c_7c_8$ with $l_1 = l_2 = l_3 = l_4 = 3$ and $l_5 = l_6 = l_7 = l_8 = 6$, and $\mathrm{move}(g) = 16$;
(38) $t \geq 2$, $g := g_{8,2^b} = c_1c_2 \cdots c_t$ with $l_1 = l_2 = l_3 \cdots = l_{t-1} = 2^b$ for $b > 3$ and $l_t = 8$, and $\mathrm{move}(g) = (t-1)2^{b-1} + 4$;
(39) $t \geq 3$, $g := g_{4,4,2^b} = c_1c_2 \cdots c_t$ with $l_1 = l_2 = l_3 \cdots = l_{t-2} = 2^b$ for $b > 2$ and $l_{t-1} = l_t = 4$, and $\mathrm{move}(g) = (t-2)2^{b-1} + 4$;
(40) $t \geq 5$, $g := g_{2,2,2,2,2^b} = c_1c_2 \cdots c_t$ with $l_1 = l_2 = l_3 \cdots = l_{t-4} = 2^b$ for $b > 1$ and $l_{t-3} = l_{t-2} = l_{t-1} = l_t = 2$, and $\mathrm{move}(g) = (t-4)2^{b-1} + 4$.

Remark 1.1. For distinct positive integer a an a', both $g^*_{2^a}(t=t_1)$ and $g^*_{2^{a'}}(t=t'_1)$ represent elements with the form in (1). Similarly, we have the following symbols: $g^*_{p_1}(t=t_2)$ and $g^*_{p'_1}(t=t'_2)$ in (1), $g_{8p_2}(p=p_2)$ and $g_{8p'_2}(p=p'_2)$ in (2), $g_{4,4p_3}(p=p_3)$ and $g_{4,4p'_3}(p=p'_3)$ in (5), $g_{4p_4,4p_4}(p=p_4)$ and $g_{4p'_4,4p'_4}(p=p'_4)$ in (6), $g_{2,2,2,2p_5}(p=p_5)$ and $g_{2,2,2,2p'_5}(p=p'_5)$ in (14), $g_{2,2,2p_6,2p_6}(p=p_6)$ and $g_{2,2,2p'_6,2p'_6}(p=p'_6)$ in (15), $g_{2,2p_7,2p_7,2p_7}(p=p_7)$ and $g_{2,2p'_7,2p'_7,2p'_7}(p'=p_7)$ in (16), $g_{2p_8,2p_8,2p_8,2p_8}(p=p_8)$ and $g_{2p'_8,2p'_8,2p'_8,2p'_8}(p=p'_8)$ in (17), $g_{8,2^{b_1}}(t=t_3)$ and $g_{8,2^{b'_1}}(t=t'_3)$ in (38), $g_{4,4,2^{b_2}}(t=t_4)$ and $g_{4,4,2^{b'_2}}(t=t'_4)$ in (39), $g_{2,2,2,2,2^{b_3}}(t=t_5)$ and $g_{2,2,2,2,2^{b'_3}}(t=t'_5)$ in (40), respectively.

For two groups K and P, we use $K{:}P$ to denote a semidirect product of K by P. For a prime p, Z_p denotes a cyclic group of order p. Given a real number r, $\lfloor r \rfloor$ is the greatest integer less than or equal to r.

Next, we apply Theorem 1.1 to characterize transitive permutation group G with $\mathrm{move}(G)=m$ and $|G|\neq 2^e$ for any positive integer e, but in which every non-identity element which has the movement m or $m-4$, and there is at least one non-identity element which has the movement $m-4$. The main result is the following.

Theorem 1.2. Let G be a transitive permutaion group on a set Ω of size n, with no fixed point in Ω, and let $\mathrm{move}(G)=m$ for some positive integer $m>4$ and $|G|\neq 2^e$ for any positive integer e. Suppose that every non-identity element g in G has the movement m or $m-4$, and there is at least one non-identity element which has the movement $m-4$. Let p be the least odd prime dividing $|G|$. Then one of the following holds:

(1) $m=7$, $p\in\{3,7\}$, $14\leq n\leq 20$ and $g\in\{g^*_2(t_1=3), g^*_3(t_2=3), g^*_7(t_2=1), g_{3,3,3,9}, g^*_2(t_1=7)\}$;
(2) $m=8$, $p\in\{3,5,17\}$, $16\leq n\leq 23$ and $g\in\{g^*_2(t_1=4), g^*_{2^2}(t_1{=}2), g^*_{2^3}(t_1=1), g^*_3(t_2=4), g^*_5(t_2{=}2), g_{5,5,8}, g_{4,4,5,5}, g_{3,3,3,3,8}, g_{3,3,3,3,5,5}, g_{3,3,3,3,4,4}, g_{2,2,2,2,3,3,3,3}, g_{2,2,2,2,5,5}, g^*_2(t_1=8), g^*_{2^2}(t_1{=}4), g^*_{2^3}(t_1{=}2), g^*_{2^4}(t_1{=}1), g^*_5(t_2{=}4), g^*_{17}(t_2{=}1), g_{2,2,2,10}, g_{4,12}, g_{2,2,6,6}, g_{4,4,2^3}(t_4{=}3), g_{2,2,2,2,2^2}(t_5=6), g_{2,2,2,2,2^3}(t_5=5)\}$;
(3) $m=10$, $p\in\{3,5,7,11,13\}$, $20\leq n\leq 29$ and $g\in\{g^*_2(t_1=6), g^*_{2^2}(t_1=3), g^*_3(t_2=6), g^*_5(t_2=3), g^*_7(t_2=2), g^*_{13}(t_2=1), g_{2,2,2,6}, g_{2,2,2,2,2^2}(t_5=5), g_{3,3,9,9}, g_{2,2,2,3,3,3,3,6}, g^*_2(t_1=10), g^*_{2^2}(t_1=5), g^*_5(t_2=5), g^*_{11}(t_2=2), g_{2,2,2,14}, g_{2,6,6,6}, g_{2,2,2,2,2^2}(t_5=7)\}$;
(4) $m=11$, $p\in\{3,23\}$, $22\leq n\leq 32$ and $g\in\{g_{3,3,3,9}, g^*_2(t_1=7), g^*_3(t_2=7), g^*_2(t_1=11), g^*_{23}(t_2=1)\}$;
(5) $m=12$, $p\in\{3,5,7,13,17\}$, $24\leq n\leq 35$ and $g\in\{g_{5,5,8}, g_{4,4,5,5}, g_{3,3,3,3,8}, g_{3,3,3,3,5,5}, g_{3,3,3,3,4,4}, g_{2,2,2,2,5,5}, g_{2,2,2,2,3,3,3,3}, g^*_2(t_1=8), g^*_{2^2}(t_1{=}4), g^*_{2^3}(t_1{=}2), g^*_{2^4}(t_1=1), g^*_3(t_2=8), g^*_5(t_2=4), g^*_{17}(t_2=1), g_{4,12}, g_{2,2,2,10}, g_{2,2,6,6}, g_{4,4,2^3}(t_4=3), g_{2,2,2,2,2^2}(t_5{=}6), g_{2,2,2,2,2^3}(t_5=5), g_{3,3,3,5,15}, g_{2,2,2,5,5,10}, g_{3,3,3,3,4,12}, g_{2,2,3,3,3,3,6,6}, g^*_2(t_1=12), g^*_{2^2}(t_1=6), g^*_{2^3}(t_1=3), g^*_5(t_2=6), g^*_7(t_2=4), g^*_{13}(t_2=2), g_{24}, g_{4,20}, g_{12,12}, g_{2,2,10,10}, g_{6,6,6,6}, g_{8,2^4}(t_3{=}2), g_{4,4,2^3}(t_4=4), g_{4,4,2^4}(t_4=3), g_{2,2,2,2,2^2}(t_5{=}8), g_{2,2,2,2,2^3}(t_5{=}6), g_{2,2,2,2,2^4}(t_5=5)\}$;
(6) $m=13$, $p\in\{3,7,19\}$, $26\leq n\leq 38$ and $g\in\{g^*_2(t_1=9), g^*_3(t_2=9), g^*_7(t_2=3), g^*_{19}(t_2=1), g_{3,9,9,9}, g^*_2(t_1=13)\}$;
(7) $m=14$, $p\in\{3,5,7,11,29\}$, $28\leq n\leq 41$ and $g\in\{g_{3,3,9,9}, g_{2,2,2,3,3,3,3,6}, g^*_2(t_1=10), g^*_{2^2}(t_1=5), g^*_3(t_2=10), g^*_5(t_2=5), g^*_{11}(t_2=2), g_{2,2,2,14}, g_{2,6,6,6}, g_{2,2,2,2,2^2}(t_5=7), g_{5,25}, g_{2,3,3,3,3,6,6,6}, g^*_2(t_1=14), g^*_{2^2}(t_1=7), g^*_5(t_2{=}7), g^*_{29}(t_2{=}1), g_{2,2,2,22}, g_{2,2,2,2,2^2}(t_5=9)\}$;
(8) $m=16$, $p\in\{3,5,7,13,17\}$, $32\leq n\leq 47$ and $g\in\{g_{3,3,3,5,15}, g_{2,2,2,5,5,10}, g_{3,3,3,3,4,12}, g_{2,2,3,3,3,3,6,6}, g^*_2(t_1=12), g^*_{2^2}(t_1=6), g^*_{2^3}(t_1=3), g^*_3(t_2=12), g^*_5(t_2=6), g^*_7(t_2{=}4), g^*_{13}(t_2=2), g_{24}, g_{4,20}, g_{2,2,10,10}, g_{8,2^4}(t_3{=}2), g_{4,4,2^3}(t_4=4), g_{4,4,2^4}(t_4=3), g_{2,2,2,2,2^2}(t_5=8), g_{2,2,2,2,2^3}(t_5=6), g_{2,2,2,2,2^4}(t_5=5), g_{9,9,9,9}, g_{3,3,15,15}, g_{4,5,5,20}, g_{2,2,5,5,10,10},$

$g_{3,3,3,3,24}, g_{3,3,3,3,12,12}, g_{3,3,3,3,6,6,6,6}, g_2^*(t_1 = 16), g_{2^2}^*(t_1 = 8), g_{2^3}^*(t_1 = 4), g_{2^4}^*(t_1{=}2), g_{2^5}^*(t_1 = 1), g_5^*(t_2 = 8), g_{17}^*(t_2{=}2), g_{4,28}, g_{2,2,2,26}, g_{2,2,14,14}, g_{2,10,10,10}, g_{4,4,2^3}(t_4{=}5), g_{2,2,2,2,2^2}(t_5 = 10), g_{2,2,2,2,2^3}(t_5 = 7)\}$;

(9) $m = 17$, $p = 3$, $34 \le n \le 50$ and $g \in \{g_{3,9,9,9}, g_2^*(t_1 = 13), g_3^*(t_2 = 13), g_2^*(t_1 = 17)\}$;

(10) $m = 18$, $p \in \{3, 5, 7, 11, 13, 19, 29, 37\}$, $36 \le n \le 53$ and $g{\in}\{g_{5,25}, g_{2,3,3,3,3,6,6,6}, g_2^*(t_1 = 14), g_{2^2}^*(t_1 = 7), g_3^*(t_2 = 14), g_5^*(t_2 = 7), g_{29}^*(t_2 = 1), g_{2,2,2,22}, g_{2,2,2,2,2^2}(t_5 = 9), g_2^*(t_1 = 18), g_{2^2}^*(t_1 = 9), g_5^*(t_2 = 9), g_7^*(t_2 = 6), g_{13}^*(t_2 = 3), g_{19}^*(t_2 = 2), g_{37}^*(t_2 = 1), g_{2,2,2,2,2^2}(t_5 = 11)\}$;

(11) $m = 20$, $p \in \{3, 5, 7, 11, 13, 17, 41\}$, $40 \le n \le 59$ and $g \in \{g_{9,9,9,9}, g_{3,3,15,15}, g_{4,5,5,20}, g_{3,3,3,3,24}, g_{2,2,5,5,10,10}, g_{3,3,3,3,12,12}, g_{3,3,3,3,6,6,6,6}, g_2^*(t_1 = 16), g_{2^2}^*(t_1 = 8), g_{2^3}^*(t_1 = 4), g_{2^4}^*(t_1 = 2), g_{2^5}^*(t_1 = 1), g_3^*(t_2 = 16), g_5^*(t_2{=}8), g_{17}^*(t_2 = 2), g_{4,28}, g_{2,2,2,26}, g_{2,2,14,14}, g_{2,10,10,10}, g_{4,4,2^3}(t_4 = 5), g_{2,2,2,2,2^2}(t_5 = 10), g_{2,2,2,2,2^3}(t_5 = 7), g_{2,5,5,10,10,10}, g_2^*(t_1 = 20), g_{2^2}^*(t_1 = 10), g_{2^3}^*(t_1 = 5), g_5^*(t_2 = 10), g_{11}^*(t_2 = 4), g_{41}^*(t_2 = 1), g_{40}, g_{20,20}, g_{2,2,2,34}, g_{10,10,10,10}, g_{8,2^4}(t_3 = 3), g_{8,2^5}(t_3 = 2), g_{4,4,2^3}(t_4 = 6), g_{4,4,2^4}(t_4 = 4), g_{4,4,2^5}(t_4 = 3), g_{2,2,2,2,2^2}(t_5 = 12), g_{2,2,2,2,2^3}(t_5 = 8), g_{2,2,2,2,2^4}(t_5 = 6), g_{2,2,2,2,2^5}(t_5 = 5)\}$;

(12) $m = 24$, $p{\in}\{3, 5, 7, 11, 13, 17, 41\}$, $48{\le}n{\le}71$ and $g \in \{g_{2,5,5,10,10,10}, g_2^*(t_1{=}20), g_{2^2}^*(t_1{=}10), g_{2^3}^*(t_1 = 5), g_3^*(t_2 = 20), g_5^*(t_2 = 10), g_{11}^*(t_2{=}4), g_{41}^*(t_2{=}1), g_{40}, g_{20,20}, g_{2,2,2,34}, g_{10,10,10,10}, g_{8,2^4}(t_3 = 3), g_{8,2^5}(t_3 = 2), g_{4,4,2^3}(t_4 = 6), g_{4,4,2^4}(t_4 = 4), g_{4,4,2^5}(t_4 = 3), g_{2,2,2,2,2^2}(t_5{=}12), g_{2,2,2,2,2^3}(t_5{=}8), g_{2,2,2,2,2^4}(t_5{=}6), g_{2,2,2,2,2^5}(t_5 = 5), g_{25,25}, g_{5,5,40}, g_{5,5,20,20}, g_{5,5,10,10,10,10}, g_2^*(t_1 = 24), g_{2^2}^*(t_1 = 12), g_{2^3}^*(t_1 = 6), g_{2^4}^*(t_1 = 3), g_5^*(t_2 = 12), g_7^*(t_2 = 8), g_{13}^*(t_2 = 4), g_{17}^*(t_2 = 3), g_{4,44}, g_{2,2,22,22}, g_{4,4,2^3}(t_4 = 7), g_{2,2,2,2,2^2}(t_5 = 14), g_{2,2,2,2,2^3}(t_5 = 9)\}$;

(13) $m = 28$, $p \in \{3, 5, 7, 11, 13, 17, 29\}$, $56 \le n \le 83$ and $g \in \{g_{25,25}, g_{5,5,40}, g_{5,5,20,20}, g_{5,5,10,10,10,10}, g_2^*(t_1 = 24), g_{2^2}^*(t_1 = 12), g_{2^3}^*(t_1 = 6), g_{2^4}^*(t_1 = 3), g_3^*(t_2 = 24), g_5^*(t_2 = 12), g_7^*(t_2 = 8), g_{13}^*(t_2 = 4), g_{17}^*(t_2 = 3), g_{4,44}, g_{2,2,22,22}, g_{4,4,2^3}(t_4{=}7), g_{2,2,2,2,2^2}(t_5{=}14), g_{2,2,2,2,2^3}(t_5 = 9), g_2^*(t_1{=}28), g_{2^2}^*(t_1{=}14), g_{2^3}^*(t_1{=}7), g_5^*(t_2 = 14), g_{29}^*(t_2 = 2), g_{56}, g_{4,52}, g_{28,28}, g_{2,2,26,26}, g_{14,14,14,14}, g_{8,2^4}(t_3 = 4), g_{4,4,2^3}(t_4 = 8), g_{4,4,2^4}(t_4 = 5), g_{2,2,2,2,2^2}(t_5 = 16), g_{2,2,2,2,2^3}(t_5 = 10), g_{2,2,2,2,2^4}(t_5 = 7)\}$;

(14) $m = t_1 2^{a-1}$, $p \in \{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_1', p_2', p_3', p_4', p_5', p_6', p_7', p_8'\}$, $2^a t_1 \le n \le \lfloor 2^a t_1 \frac{p}{p-1} \rfloor - 1$ and $g \in \{g_{2^{a'}}^*(t_1' 2^{a'-1} + 4 = t_1 2^{a-1}), g_{p_1'}^*(t_2' \frac{p_1'-1}{2} + 4 = t_1 2^{a-1}), g_{8p_2'}(4p_2' + 4 = t_1 2^{a-1}), g_{4,4p_3'}(6 + 2p_3' = t_1 2^{a-1}), g_{4p_4',4p_4'}(4p_4' + 4 = t_1 2^{a-1}), g_{2,2,2,2p_5'}(7 + p_5' = t_1 2^{a-1}), g_{2,2,2p_6',2p_6'}(2p_6' + 6 = t_1 2^{a-1}), g_{2,2p_7',2p_7',2p_7'}(3p_7' + 5 = t_1 2^{a-1}), g_{2p_8',2p_8',2p_8',2p_8'}(4p_8' + 4 = t_1 2^{a-1}), g_{8,2^{b_1'}}((t_3' - 1)2^{b_1'-1} + 8 = t_1 2^{a-1}), g_{4,4,2^{b_2'}}((t_4' - 2)2^{b_2'-1} + 8{=}t_1 2^{a-1}), g_{2,2,2,2,2^{b_3'}}((t_5'{-}4)2^{b_3'-1}{+}8{=}t_1 2^{a-1}), g_{2^a}^*, g_{p_1}^*(t_2 \frac{p_1-1}{2} = t_1 2^{a-1}), g_{8p_2}(4p_2 = t_1 2^{a-1}), g_{4,4p_3}(2 + 2p_3 = t_1 2^{a-1}), g_{4p_4,4p_4}(4p_4 = t_1 2^{a-1}), g_{2,2,2,2p_5}(3 + p_5 = t_1 2^{a-1}), g_{2,2,2p_6,2p_6}(2p_6 + 2 = t_1 2^{a-1}), g_{2,2p_7,2p_7,2p_7}(3p_7 + 1 = t_1 2^{a-1}), g_{2p_8,2p_8,2p_8,2p_8}(4p_8 = t_1 2^{a-1}), g_{8,2^{b_1}}((t_3{-}1)2^{b_1-1}{+}4 = t_1 2^{a-1}), g_{4,4,2^{b_2}}((t_4{-}2)2^{b_2-1}{+}4 = t_1 2^{a-1}), g_{2,2,2,2,2^{b_3}}((t_5 - 4)2^{b_3-1} + 4 = t_1 2^{a-1})\}$;

(15) $m = t_2 \frac{p_1-1}{2}$, $p{\in}\{p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_1', p_2', p_3', p_4', p_5', p_6', p_7', p_8'\}$, $t_2(p_1 - 1) \le n \le \lfloor t_2(p_1{-}1)\frac{p}{p-1} \rfloor - 1$ and $g{\in}\{g_{2^{a'}}^*(t_1' 2^{a'-1}{+}4{=}t_2 \frac{p_1-1}{2}), g_{p_1'}^*(t_2' \frac{p_1'-1}{2}{+}4{=}t_2 \frac{p_1-1}{2}), g_{8p_2'}(4p_2' + 4 = t_2 \frac{p_1-1}{2}), g_{4,4p_3'}(6 + 2p_3'{=}t_2 \frac{p_1-1}{2}), g_{4p_4',4p_4'}(4p_4' + 4{=}t_2 \frac{p_1-1}{2}), g_{2,2,2,2p_5'}(7 + p_5' {=}t_2 \frac{p_1-1}{2}), g_{2,2,2p_6',2p_6'}(2p_6' + 6{=}t_2 \frac{p_1-1}{2}), g_{2,2p_7',2p_7',2p_7'}(3p_7' + 5{=}t_2 \frac{p_1-1}{2}), g_{2p_8',2p_8',2p_8',2p_8'}(4p_8' + 4 = t_2 \frac{p_1-1}{2}), g_{8,2^{b_1'}}((t_3' - 1)2^{b_1'-1} + 8 = t_2 \frac{p_1-1}{2}), g_{4,4,2^{b_2'}}((t_4' - 2)2^{b_2'-1} + 8 = t_2 \frac{p_1-1}{2}), g_{2,2,2,2,2^{b_3'}}((t_5' - 4)2^{b_3'-1} + 8 = t_2 \frac{p_1-1}{2}), g_{p_1}^*, g_{8p_2}(4p_2 = t_2 \frac{p_1-1}{2}), g_{4,4p_3}(2 + 2p_3 = t_2 \frac{p_1-1}{2}), g_{4p_4,4p_4}(4p_4 = t_2 \frac{p_1-1}{2}), g_{2,2,2,2p_5}(3 + p_5 = t_2 \frac{p_1-1}{2}), g_{2,2,2p_6,2p_6}(2p_6 + 2 = t_2 \frac{p_1-1}{2}), g_{2,2p_7,2p_7,2p_7}(3p_7{+}1 = t_2 \frac{p_1-1}{2}), g_{2p_8,2p_8,2p_8,2p_8}(4p_8 = t_2 \frac{p_1-1}{2}), g_{8,2^{b_1}}((t_3-$

$1)2^{b_1-1}+4=t_2\frac{p_1-1}{2}), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=t_2\frac{p_1-1}{2}), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=t_2\frac{p_1-1}{2})\}$;

(16) $m=4p_2$, $p\in\{p_2,p_3,p_4,p_5,p_6,p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $8p_2\leq n\leq\lfloor 8p_2\frac{p}{p-1}\rfloor-1$ and $g\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=4p_2), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=4p_2), g_{8p'_2}(4p'_2+4=4p_2), g_{4,4p'_3}(6+2p'_3=4p_2), g_{4p'_4,4p'_4}(4p'_4+4=4p_2), g_{2,2,2,2p'_5}(7+p'_5=4p_2), g_{2,2,2p'_6,2p'_6}(2p'_6+6=4p_2), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=4p_2), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=4p_2), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=4p_2), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=4p_2), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=4p_2), g_{8p_2}, g_{4,4p_3}(2+2p_3=4p_2), g_{4p_4,4p_4}(p_4=p_2), g_{2,2,2,2p_5}(3+p_5=4p_2), g_{2,2,2p_6,2p_6}(2p_6+2=4p_2), g_{2,2p_7,2p_7,2p_7}(3p_7+1=4p_2), g_{2p_8,2p_8,2p_8,2p_8}(p_8=p_2), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=4p_2), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=4p_2), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=4p_2)\}$;

(17) $m=2+2p_3$, $p\in\{p_3,p_4,p_5,p_6,p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $4+4p_3\leq n\leq\lfloor(4+4p_3)\frac{p}{p-1}\rfloor-1$ and $g\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=2+2p_3), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=2+2p_3), g_{8p'_2}(4p'_2+4=2+2p_3), g_{4,4p'_3}(6+2p'_3=2+2p_3), g_{4p'_4,4p'_4}(4p'_4+4=2+2p_3), g_{2,2,2,2p'_5}(7+p'_5=2+2p_3), g_{2,2,2p'_6,2p'_6}(2p'_6+6=2+2p_3), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=2+2p_3), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=2+2p_3), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=2+2p_3), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=2+2p_3), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=2+2p_3), g_{4,4p_3}, g_{4p_4,4p_4}(4p_4=2+2p_3), g_{2,2,2,2p_5}(3+p_5=2+2p_3), g_{2,2,2p_6,2p_6}(2p_6+2=2+2p_3), g_{2,2p_7,2p_7,2p_7}(3p_7+1=2+2p_3), g_{2p_8,2p_8,2p_8,2p_8}(4p_8=2+2p_3), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=2+2p_3), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=2+2p_3), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=2+2p_3)\}$;

(18) $m=4p_4$, $p\in\{p_4,p_5,p_6,p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $8p_4\leq n\leq\lfloor 8p_4\frac{p}{p-1}\rfloor-1$ and $g\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=4p_4), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=4p_4), g_{8p'_2}(4p'_2+4=4p_4), g_{4,4p'_3}(6+2p'_3=4p_4), g_{4p'_4,4p'_4}(4p'_4+4=4p_4), g_{2,2,2,2p'_5}(7+p'_5=4p_4), g_{2,2,2p'_6,2p'_6}(2p'_6+6=4p_4), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=4p_4), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=4p_4), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=4p_4), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=4p_4), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=4p_4), g_{4p_4,4p_4}, g_{2,2,2,2p_5}(3+p_5=4p_4), g_{2,2,2p_6,2p_6}(2p_6+2=4p_4), g_{2,2p_7,2p_7,2p_7}(3p_7+1=4p_4), g_{2p_8,2p_8,2p_8,2p_8}(p_8=p_4), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=4p_4), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=4p_4), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=4p_4)\}$;

(19) $m=3+p_5$, $p\in\{p_5,p_6,p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $6+2p_5\leq n\leq\lfloor(6+2p_5)\frac{p}{p-1}\rfloor-1$ and $g\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=3+p_5), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=3+p_5), g_{8p'_2}(4p'_2+4=3+p_5), g_{4,4p'_3}(6+2p'_3=3+p_5), g_{4p'_4,4p'_4}(4p'_4+4=3+p_5), g_{2,2,2,2p'_5}(7+p'_5=3+p_5), g_{2,2,2p'_6,2p'_6}(2p'_6+6=3+p_5), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=3+p_5), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=3+p_5), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=3+p_5), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=3+p_5), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=3+p_5), g_{2,2,2,2p_5}, g_{2,2,2p_6,2p_6}(2p_6+2=3+p_5), g_{2,2p_7,2p_7,2p_7}(3p_7+1=3+p_5), g_{2p_8,2p_8,2p_8,2p_8}(4p_8=3+p_5), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=3+p_5), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=3+p_5), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=3+p_5)\}$;

(20) $m=2p_6+2$, $p\in\{p_6,p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $4p_6+4\leq n\leq\lfloor(4p_6+4)\frac{p}{p-1}\rfloor-1$ and $g\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=2p_6+2), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=2p_6+2), g_{8p'_2}(4p'_2+4=2p_6+2), g_{4,4p'_3}(6+2p'_3=2p_6+2), g_{4p'_4,4p'_4}(4p'_4+4=2p_6+2), g_{2,2,2,2p'_5}(7+p'_5=2p_6+2), g_{2,2,2p'_6,2p'_6}(2p'_6+6=2p_6+2), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=2p_6+2), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=2p_6+2), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=2p_6+2), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=2p_6+2), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=2p_6+2), g_{2,2,2p_6,2p_6}, g_{2,2p_7,2p_7,2p_7}(3p_7+1=$

$2p_6+2), g_{2p_8,2p_8,2p_8,2p_8}(4p_8=2p_6+2), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=2p_6+2), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=2p_6+2), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=2p_6+2)\}$;

(21) $m=3p_7+1$, $p\in\{p_7,p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $6p_7+2\le n\le\lfloor(6p_7+2)\frac{p}{p-1}\rfloor-1$ and $p\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=3p_7+1), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=3p_7+1), g_{8p'_2}(4p'_2+4=3p_7+1), g_{4,4p'_3}(6+2p'_3=3p_7+1), g_{4p'_4,4p'_4}(4p'_4+4=3p_7+1), g_{2,2,2,2p'_5}(7+p'_5=3p_7+1), g_{2,2,2p'_6,2p'_6}(2p'_6+6=3p_7+1), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=3p_7+1), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=3p_7+1), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=3p_7+1), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=3p_7+1), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=3p_7+1), g_{2,2p_7,2p_7,2p_7}, g_{2p_8,2p_8,2p_8,2p_8}(4p_8=3p_7+1), g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=3p_7+1), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=3p_7+1), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=3p_7+1)\}$;

(22) $m=4p_8$, $p\in\{p_8,p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $8p_8\le n\le\lfloor 8p_8\frac{p}{p-1}\rfloor-1$ and $p\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=4p_8), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=4p_8), g_{8p'_2}(4p'_2+4=4p_8), g_{4,4p'_3}(6+2p'_3=4p_8), g_{4p'_4,4p'_4}(4p'_4+4=4p_8), g_{2,2,2,2p'_5}(7+p'_5=4p_8), g_{2,2,2p'_6,2p'_6}(2p'_6+6=4p_8), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=4p_8), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=4p_8), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=4p_8), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=4p_8), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=4p_8), g_{2p_8,2p_8,2p_8,2p_8}, g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=4p_8), g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=4p_8), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=4p_8)\}$;

(23) $m=(t_3-1)2^{b_1-1}+4$, $p\in\{p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $(t_3-1)2^{b_1}+8\le n\le\lfloor((t_3-1)2^{b_1}+8)\frac{p}{p-1}\rfloor-1$ and $p\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=(t_3-1)2^{b_1-1}+4), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=(t_3-1)2^{b_1-1}+4), g_{8p'_2}(4p'_2+4=(t_3-1)2^{b_1-1}+4), g_{4,4p'_3}(6+2p'_3=(t_3-1)2^{b_1-1}+4), g_{4p'_4,4p'_4}(4p'_4+4=(t_3-1)2^{b_1-1}+4), g_{2,2,2,2p'_5}(7+p'_5=(t_3-1)2^{b_1-1}+4), g_{2,2,2p'_6,2p'_6}(2p'_6+6=(t_3-1)2^{b_1-1}+4), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=(t_3-1)2^{b_1-1}+4), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=(t_3-1)2^{b_1-1}+4), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=(t_3-1)2^{b_1-1}+4), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=(t_3-1)2^{b_1-1}+4), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=(t_3-1)2^{b_1-1}+4), g_{8,2^{b_1}}, g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=(t_3-1)2^{b_1-1}+4)), g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=(t_3-1)2^{b_1-1}+4))\}$;

(24) $m=(t_4-2)2^{b_2-1}+4$, $p\in\{p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $(t_4-2)2^{b_2}+8\le n\le\lfloor((t_4-2)2^{b_2}+8)\frac{p}{p-1}\rfloor-1$ and $p\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=(t_4-2)2^{b_2-1}+4), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=(t_4-2)2^{b_2-1}+4), g_{8p'_2}(4p'_2+4=(t_4-2)2^{b_2-1}+4), g_{4,4p'_3}(6+2p'_3=(t_4-2)2^{b_2-1}+4), g_{4p'_4,4p'_4}(4p'_4+4=(t_4-2)2^{b_2-1}+4), g_{2,2,2,2p'_5}(7+p'_5=(t_4-2)2^{b_2-1}+4), g_{2,2,2p'_6,2p'_6}(2p'_6+6=(t_4-2)2^{b_2-1}+4), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=(t_4-2)2^{b_2-1}+4), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=(t_4-2)2^{b_2-1}+4), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=(t_4-2)2^{b_2-1}+4), g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8=(t_4-2)2^{b_2-1}+4), g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8=(t_4-2)2^{b_2-1}+4), g_{4,4,2^{b_2}}, g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=(t_3-1)2^{b_1-1}+4))\}$;

(25) $m=(t_5-4)2^{b_3-1}+4$, $p\in\{p'_1,p'_2,p'_3,p'_4,p'_5,p'_6,p'_7,p'_8\}$, $(t_5-4)2^{b_3}+8\le n\le\lfloor((t_5-4)2^{b_3}+8)\frac{p}{p-1}\rfloor-1$ and $p\in\{g^*_{2^{a'}}(t'_1 2^{a'-1}+4=(t_5-4)2^{b_3-1}+4), g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4=(t_5-4)2^{b_3-1}+4), g_{8p'_2}(4p'_2+4=(t_5-4)2^{b_3-1}+4), g_{4,4p'_3}(6+2p'_3=(t_5-4)2^{b_3-1}+4), g_{4p'_4,4p'_4}(4p'_4+4=(t_5-4)2^{b_3-1}+4), g_{2,2,2,2p'_5}(7+p'_5=(t_5-4)2^{b_3-1}+4), g_{2,2,2p'_6,2p'_6}(2p'_6+6=(t_5-4)2^{b_3-1}+4), g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5=(t_5-4)2^{b_3-1}+4), g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4=(t_5-4)2^{b_3-1}+4), g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8=(t_5-4)2^{b_3-1}+$

4), $g_{4,4,2^{b_2'}}((t_4' - 2)2^{b_2'-1} + 8 = (t_5 - 4)2^{b_3-1} + 4), g_{2,2,2,2,2^{b_3'}}((t_5' - 4)2^{b_3'-1} + 8 = (t_5 - 4)2^{b_3-1} + 4), g_{2,2,2,2,2^{b_3}}\}$;

(26) $m = 2^{s-1}(p-1)$, $n = 2^s p$ with $1 < 2^s < p$ and $p \geq 5$, and $G = K : P$ with K a 2-group and $P = Z_p$ is fixed point free on Ω; K has p-orbit of length 2^s, and each element of K moves at most $2^s(p-1)$ point of Ω;

(27) G is a 3-group of exponent 3.

2. Proof of Theorem 1.1

Let G be a permutation group on a set Ω, and let $1 \neq g \in G$ and $g = c_1c_2\cdots c_t$ as a product of disjoint cycles of lengths $l_1, l_2, \cdots, l_t$, where $c_i = (a_{i1}\ a_{i2} \ldots a_{il_i})$ for $1 \leq i \leq t$. Let

$$\Delta(g) = \{a_{12}, a_{14}, \cdots, a_{1k_1}, a_{22}, a_{24}, \cdots, a_{2k_2}, \cdots, a_{t2}, a_{t4}, \cdots, a_{tk_t}\},$$

where $k_i = l_i$ if l_i is even, and $k_i = l_i - 1$ if l_i is odd. Then $|\Delta(g)^g \setminus \Delta(g)| = |\Delta(g)| = \sum_{i=1}^{t} \lfloor \frac{l_i}{2} \rfloor$.

The next lemma gives an upper bound for $|\Delta^g \setminus \Delta|$ for an arbitrary subest Δ of Ω, see ([6, Lemma 2.1]).

Lemma 2.1. Let G be a permutation group on a set Ω and suppose that $\Delta \subseteq \Omega$. Then for each $g \in G$, $|\Delta^g \setminus \Delta| \leq \sum_{i=1}^{t} \lfloor \frac{l_i}{2} \rfloor$, with equality if $\Delta = \Delta(g)$, where l_i is the length of the i^{th} cycle of g, and t is the number of nontrivial cycles of g in its disjoint cycle representation.

The following result is crucial to the proof of Theorem 1.1.

Lemma 2.2. Let G be a permutation group on a set Ω. Let g be a cycle of length pk for some odd prime p and positive integer $k > 1$. Then $\mathrm{move}(g) - \mathrm{move}(g^k) = \lfloor \frac{k}{2} \rfloor$.

Proof. Since p is an odd prime, we see that g^k is k cycles of length p. If k is odd, then $k = 2t + 1$ for some positve integer t. It follows that

$$\mathrm{move}(g) = \lfloor \frac{kp}{2} \rfloor = \lfloor \frac{(2t+1)p}{2} \rfloor = tp + \frac{p-1}{2},$$

and

$$\mathrm{move}(g^k) = k\lfloor \frac{p}{2} \rfloor = (2t+1)\frac{p-1}{2} = \mathrm{move}(g) - t.$$

Thus $\mathrm{move}(g) - \mathrm{move}(g^k) = \frac{k-1}{2}$.

If k is even, then $k = 2t$ for some positive integer t, and so

$$\mathrm{move}(g) = \lfloor \frac{kp}{2} \rfloor = \lfloor \frac{(2t)p}{2} \rfloor = tp,$$

and

$$\mathrm{move}(g^k) = k\lfloor \frac{p}{2} \rfloor = (2t)\frac{p-1}{2} = \mathrm{move}(g) - t.$$

Therefore $\mathrm{move}(g) - \mathrm{move}(g^k) = \frac{k}{2}$. ∎

Let G be a permutation group on a set Ω and $\mathrm{move}(G) = m$, in which every non-identity element has the movement m or $m - 4$. Then we can characterize the structures of elements in G.

Proof of Theorem 1.1. Let $1 \neq g \in G$, and $g = c_1c_2\cdots c_t$ as a product of disjoint cycles of lengths $l_1, l_2, \cdots, l_t$. Let $h = g^{l_i}$ for $1 \leq i \leq t$. Then by Lemma 2.1, $\mathrm{move}(g) = \sum_{i=1}^{t} \lfloor l_i/2 \rfloor$ and $\mathrm{move}(h) \leq \sum_{j \neq i} \lfloor l_j/2 \rfloor < \sum_{i=1}^{t} \lfloor l_i/2 \rfloor = \mathrm{move}(g)$.

Case 1. $\mathrm{move}(g) = m - 4$.

In this case, $h = g^{l_i} = 1$ for $1 \le i \le t$ as $\mathrm{move}(h) < \mathrm{move}(g)$. It follows that $l := l_1 = l_2 = \cdots = l_t$. Assume that $l \neq 2^n$ for any positive integer n. Then $l = pk$ for some odd prime p and positive integer k. Note that $\mathrm{move}(g) = \mathrm{move}(g^k)$. Then by Lemma 2.2 we have

$$t\lfloor \frac{l}{2} \rfloor = tk\lfloor \frac{p}{2} \rfloor.$$

It follows that $k = 1$ and thus $l = p$. Therefore (1) holds.

Case 2. $\mathrm{move}(g) = m$.

In this case, $\mathrm{move}(h) = \mathrm{move}(g^{l_i}) = m - 4$ or $h = g^{l_i} = 1$, for some $1 \le i \le t$, since $\mathrm{move}(h) < \mathrm{move}(g)$.

Subcase 2.1. $\mathrm{move}(h) = \mathrm{move}(g^{l_i}) = m - 4$.

Let $h = c_1^{'} c_2^{'} \cdots c_t^{'}$. Then $o(c_1^{'}) = o(c_2^{'}) = \cdots = o(c_t^{'}) = \frac{l_1}{(l_1,l_i)} = \frac{l_2}{(l_2,l_i)} = \cdots = \frac{l_{i-1}}{(l_{i-1},l_i)} = \frac{l_{i+1}}{(l_{i+1},l_i)} = \cdots = \frac{l_t}{(l_t,l_i)}$ by Case 1. It follows that either $\{c_j\} \subseteq \{c_1^{'}, c_2^{'}, \cdots, c_t^{'}\}$ or $\{c_j\} \cap \{c_1^{'}, c_2^{'}, \cdots, c_t^{'}\} = \phi$ for each $1 \le j \le t$. Without loss of generality, we assume that $\{c_1^{'}, c_2^{'}, \cdots, c_t^{'}\} = \{c_1, c_2, \cdots, c_n\}$, where $n < t$ and $n + 1 \le i \le t$. Then

$$\mathrm{move}(g) - \mathrm{move}(h) = \sum_{j=n+1}^{t} \lfloor \frac{l_j}{2} \rfloor.$$

It follows that $1 \le t - n \le 4$.

If $t = n + 1$, then $l_t = l_i = 8$ or 9. It follows that g is either a product of four cycles of length 3 and one cycle of length 8, or two cycles of length 5 and one cycle of length 8. Thus (8) and (21) hold.

If $t = n+2$, then either $l_i = 6$ or 7, and $l_j = 2$ or 3, where $n+1 \le i \neq j \le n+2$, or $l_i = 4$ or 5, and $l_j = 4$ or 5 where $n + 1 \le i \neq j \le n + 2$. For the former, it is straightforward to verify that no such g exists. For the letter, then g is either a product of four cycles of length 3 and two cycles of length 4, four cycles of length 3 and two cycles of length 5, four cycles of length 2 and two cycles of length 5, or two cycles of length 4 and two cycles of length 5. Thus (9), (24), (25) and (26) hold.

If $t = n+3$, then $l_i = 4$ or 5, $l_j = 2$ or 3, and $l_k = 2$ or 3 where $n+1 \le i \neq j \neq k \le n+3$. It is straightforward to verify that no such g exists.

If $t = n + 4$, then $l_i = 2$ or 3, $l_j = 2$ or 3, $l_k = 2$ or 3, and $l_z = 2$ or 3 where $n + 1 \le i \neq j \neq k \neq z \le n + 4$. It follows that g is a product of four cycles of length 2 and four cycles of length 3. Thus (33) holds.

Subcase 2.2. $h = g^{l_i} = 1$ for some $1 \le i \le t$. We may assume that $i = 1$.

In this case, $l_j | l_1$ for all $1 \le j \le t$. First we suppose that l_1 is not a power of 2. Then $l_1 = pk$ with an odd prime p and positive integer $k \ge 1$. If $k = 1$, then g is a product of t cycles of length p. Thus we may assume that $k > 1$. Suppose taht $p|k$. Then $1 \neq g^k$ and $\mathrm{move}(g) - \mathrm{move}(g^k) = 0$ or 4. Hence $\mathrm{move}(c_1) - \mathrm{move}(c_1^k) = \lfloor \frac{k}{2} \rfloor \le 4$.

If $\lfloor \frac{k}{2} \rfloor = 4$, then $k = 8$ or 9. It follows that $p = 3$. By simple calculation, there is no g satisfying the assumption.

If $\lfloor \frac{k}{2} \rfloor = 3$, then $k = 6$ or 7. It follows that $p = 3$ or 7. By simple calculation, there is no g satisfying the assumption.

If $\lfloor \frac{k}{2} \rfloor = 2$, then $k = 4$ or 5. It follows that $p = 5$. So g is a product of one cycle of length 5 and one cycle of length 25, or g is two cycles of 25. Thus (3) and (4) hold.

If $\lfloor \frac{k}{2} \rfloor = 1$, then $k = 2$ or 3. It follows that $p = 3$. So g is a product of three cycles of length 3 and one cycle of length 9, two cycles of length 3 and two cycles of length 9, one cycle of length 3 and three cycles of length 9, or four cycles of length 9. Thus (10), (11), (12) and (13) hold.

Next we suppose that $p \nmid k$, and g has a cycles of length k, b cycles of length p, and d cycles of length dividing k. Then g has $t - a - b - d$ cycles of length pk. Since $\mathrm{move}(c_1) - \mathrm{move}(c_1^k) = \lfloor \frac{k}{2} \rfloor$, we have $\lfloor \frac{k}{2} \rfloor = 1, 2, 3$ or 4.

If $\lfloor \frac{k}{2} \rfloor = 1$, then $k = 2$ or 3 and $d = 0$. It follows that $t - a - b = 1, 2, 3$ or 4. Let $k = 2$. If $t - a - b = 1$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 1 = 4$, and so we have $a = 3$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is either a product of three cycles of length 2, four cycles of length 3 and one cycle of length 6, or a product of three cycles of length 2, two cycles of length 5 and one cycle of length 10. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of three cycles of length 2 and one cycle of length $2p$. Thus (14), (27) and (34) hold.

If $t - a - b = 2$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 2 = 4$, and so we have $a = 2$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is either a product of two cycles of length 2, four cycles of length 3 and two cycles of length 6, or a product of two cycles of length 2, two cycles of length 5 and two cycles of length 10. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of two cycles of length 2 and two cycles of length $2p$. Thus (15), (28) and (35) hold.

If $t - a - b = 3$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 3 = 4$, and so we have $a = 1$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is either a product of one cycle of length 2, four cycles of length 3 and three cycles of length 6, or a product of one cycle of length 2, two cycles of length 5 and three cycles of length 10. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of one cycle of length 2 and three cycles of length $2p$. Thus (16), (29) and (36) hold.

If $t - a - b = 4$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 4 = 4$, and so we have $a = 0$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then we see that $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is either a product of four cycles of length 3 and four cycles of length 6, or a product of two cycles of length 5 and four cycles of length 10. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of four cycles of length $2p$. Thus (17), (30) and (37) hold.

Let $k = 3$. If $t - a - b = 1$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 1 = 4$, and so we have $a = 3$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we see that $p = 5$ and $b = 1$. Thus g is a product of three cycles of length 3, one cycle of length 5 and one cycle of length 15. Thus (22) holds.

If $t - a - b = 2$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 2 = 4$, and so we have $a = 2$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we see that $p = 5$ and $b = 0$. Thus g is a product of two cycles of length 3 and two cycles of length 15. Thus (18) holds.

If $t - a - b = 3$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 3 = 4$, and so we have $a = 1$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we see that $p = 3$ and $b = 1$, a contradiction.

If $t - a - b = 4$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = a + 4 = 4$, and so we have $a = 0$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we see that $p = 3$ and $b = 1$, a contradiction.

If $\lfloor \frac{k}{2} \rfloor = 2$, then $k = 4$ or 5, and $t - a - b - d \leq 2$. Let $k = 4$. If $t - a - b - d = 1$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = 2a + d + 2 = 4$, and so $a = 1$ and $d = 0$, or $a = 0$ and $d = 2$. Assume that $a = 1$ and $d = 0$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is a product of one cycle of 4, four cycles of length of 3 and one cycle of length of 12, or a product of one cycle of length 4, two cycles of length 5 and one cycle of length 20. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of one cycle of length 4 and one cycle of length $4p$. Thus (5), (19) and (31) hold.

Assume that $a = 0$ and $d = 2$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is a product of four cycles of length 3, two cycles of length 2 and one cycle of length 12, or a product of two cycles of length 5, two cycles of length 2 and one cycle of length 20. But $\mathrm{move}(g) - \mathrm{move}(g^2) = 2$, a contradiction. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of two cycles of length 2 and one cycle of length $4p$. But $\mathrm{move}(g) - \mathrm{move}(g^2) = 2$, a contradiction.

If $t - a - b - d = 2$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = 2a + d + 4 = 4$, and so $a = d = 0$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$, or $p = 5$ and $b = 2$. Thus g is either a product of four cycles of length 3 and two cycles of length 12, or a product of two cycles of length 5 and two cycles of length 20. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a product of two cycles of length $4p$. Thus(6), (20) and (32) hold.

Let $k = 5$, then $d = 0$. If $t - a - b = 1$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = 2a + 2 = 4$, and so we have $a = 1$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we see that $p = 3$ and $b = 3$, and so g is a product of one cycle of length 5, three cycles of length 3 and one cycle of length 15. Thus (22) holds.

If $t - a - b = 2$, then $\mathrm{move}(g) - \mathrm{move}(g^k) = 2a + 4 = 4$, and so we have $a = 0$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we conclude that $p = 3$ and $b = 2$, and so g is a product of two cycles of length 3 and two cycles of length 15. Thus (18) holds.

If $\lfloor \frac{k}{2} \rfloor = 3$, then $k = 6$ or 7, and $t-a-b-d = 1$. Let $k = 6$. Then $\mathrm{move}(g)-\mathrm{move}(g^k) = 4 \geq 3a + 3$, and so $a = 0$ and $d = 1$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 5$ and $b = 2$. It follows that g is either a product of two cycles of length 5, one cycle of length 2 and one cycle of length 30, or a product of two cycles of length 5, one cycle of length 3 and one cycle of length 30. But $\mathrm{move}(g)-\mathrm{move}(g^2) = 1$ or 2, a contradiction. If $\mathrm{move}(g)-\mathrm{move}(g^p) = 0$, then $b = 0$. It follows that g is either a product of one cycle of length 2 and one cycle of length $6p$, or a product of one cycle of length 3 and one cycle of length $6p$. But $\mathrm{move}(g) - \mathrm{move}(g^2) = 1$ or 2, a contradiction. Let $k = 7$. Then $d = 0$. It follows that $\mathrm{move}(g) - \mathrm{move}(g^k) = 3a + 3 = 4$, and so $a = \frac{1}{3}$, a contradiction.

If $\lfloor \frac{k}{2} \rfloor = 4$, then $k = 8$ or 9, and $t-a-b-d = 1$. Let $k = 8$. Then $\mathrm{move}(g)-\mathrm{move}(g^k) = 4 \geq 4a + 3$, and so $a = d = 0$. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, then $p = 3$ and $b = 4$ or $p = 5$ and $b = 2$. Thus g is either a product of four cycles of length 3 and one cycle of 24, or a product of two cycles of length 5 and one cycle of length 40. If $\mathrm{move}(g) - \mathrm{move}(g^p) = 0$, then $b = 0$, and so g is a cycle of length $8p$. Thus (2), (7) and (23) hold.

Let $k = 9$. Then $a = d = 0$. Since $\mathrm{move}(g) - \mathrm{move}(g^p) = 4$, we have $p = 5$ and $b = 1$. Thus g is a product of one cycle of length 5 and one cycle of length 45. But $\mathrm{move}(g) - \mathrm{move}(g^{15}) = 9$, a contradiction.

Now we suppose that $l_1 = 2^b$ for some positive integer b. Then $l_i = 2^{b_i}$ with $b_i \leq b$ for $2 \leq i \leq t$. If $b_i = b$ for $2 \leq i \leq t$, then g is a product of t cycles of length 2^b. Thus (1) hold. If $b_i < b$ for some i, then $g^{2^{b_i}} \neq 1$ and $1 \leq b_i \leq 3$. It follows that g is a product of $(t-4)$-cycles of length a power of 2^b and four cycles of length 2 for $t \geq 5$, $(t-2)$-cycles of length a power of 2^b and two cycles of length 4 for $t \geq 3$, or $(t-1)$-cycles of length a power of 2^b and one cycle of length 8 for $t \geq 2$. Thus (38), (39) and (40) hold. ∎

3. Proof of Theorem 1.2

Let G be a transitive permutation group on a set Ω with bounded movement m. Suppose that G is not a 2-group. Then the upper bound of $|\Omega|$ is given in [9, Lemma 2.2].

Lemma 3.1. Let G be a permutation group on Ω which has no fixed points on Ω. Suppose that G is not a 2-group and $\text{move}(G) = m$ with a positive integer m. Aussme that p is the least odd prime dividing $|G|$. Then $|\Omega| \leq \lfloor \frac{2mp}{p-1} \rfloor$.

Now we can give a characterization for all transitive permutation groups G satisfying the hypotheses of Theorem 1.2.

Proof of Theorem 1.2. Suppose that p is the least odd prime dividing $|G|$. Then by Lemma 3.1, $n := |\Omega| \leq \lfloor \frac{2mp}{p-1} \rfloor$. Suppose that $n = \lfloor \frac{2mp}{p-1} \rfloor$. If $p = 3$, then $n = 3m$. By [8, Theorem], G is a 3-group of exponenet 3, or G is one of S_3, A_4 or A_5 of degree 3, 6 and 6, respectively. Note that $\text{move}(S_3) = 1$ and $\text{move}(A_4) = \text{move}(A_5) = 2$. Thus G is a 3-group of exponent 3. If $p \geq 5$, then by [3, Theorem 1.1] and [6, Theorem 1.2], $n = 2^s p$, $m = 2^{s-1}(p-1)$, $1 < 2^s < p$, and $G = K : P$ with K a 2-group and $P = Z_p$ is fixed point free on Ω; K has p-orbit of length 2^s, and each element of K moves at most $2^s(p-1)$ points of Ω.

Next we suppose that $n < \lfloor \frac{2mp}{p-1} \rfloor$. Let $1 \neq g \in G$. Then by Theorem 1.1, $g \in \{g^*_{2^a}, g^*_{p_1}, g_{8p_2}, g_{5,25}, g_{25,25}, g_{4,4p_3}, g_{4p_4,4p_4}, g_{5,5,40}, g_{5,5,8}, g_{4,4,5,5}, g_{3,3,3,9}, g_{3,3,9,9}, g_{3,9,9,9}, g_{9,9,9,9}, g_{2,2,2,2p_5}, g_{2,2,2p_6,2p_6}, g_{2,2p_7,2p_7,2p_7}, g_{2p_8,2p_8,2p_8,2p_8}, g_{3,3,15,15}, g_{4,5,5,20}, g_{5,5,20,20}, g_{3,3,3,3,8}, g_{3,3,3,5,15}, g_{3,3,3,3,24}, g_{3,3,3,3,5,5}, g_{3,3,3,3,4,4}, g_{2,2,2,2,5,5}, g_{2,2,2,5,5,10}, g_{2,2,5,5,10,10}, g_{2,5,5,10,10,10}, g_{5,5,10,10,10,10}, g_{3,3,3,3,4,12}, g_{3,3,3,3,12,12}, g_{2,2,2,2,3,3,3,3}, g_{2,2,2,3,3,3,3,6}, g_{2,2,3,3,3,3,6,6}, g_{2,3,3,3,3,6,6,6}, g_{3,3,3,3,6,6,6,6}, g_{8,2^{b_1}}, g_{4,4,2^{b_2}}, g_{2,2,2,2,2^{b_3}}\}$. It follows that $m \in \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2 \frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2, 3p_7+1, 4p_8, (t_3-1)2^{b_1-1}+4, (t_4-2)2^{b_2-1}+4, (t_5-4)2^{b_3-1}+4\}$.

If $m = 7$, then at least one of $g^*_2(t_1 = 3)$, $g^*_3(t_2 = 3)$ and $g^*_7(t_2 = 1)$ belongs to G, and at least one of $g_{3,3,3,9}$, $g^*_2(t_1 = 7)$ and $g^*_3(t_2 = 7)$ belongs to G. By Lemma 3.1, we have $14 \leq n \leq 20$. Thus we can exclude $g^*_3(t_2 = 7)$.

If $m = 8$, then at least one of $g^*_2(t_1 = 4)$, $g^*_{2^2}(t_1 = 2)$, $g^*_{2^3}(t_1 = 1)$, $g^*_3(t_2 = 4)$ and $g^*_5(t_2 = 2)$ belongs to G, and at least one of $g_{5,5,8}$, $g_{4,4,5,5}$, $g_{3,3,3,3,8}$, $g_{3,3,3,3,5,5}$, $g_{3,3,3,3,4,4}$, $g_{2,2,2,2,5,5}$, $g_{2,2,2,2,3,3,3,3}$, $g^*_2(t_1 = 8)$, $g^*_{2^2}(t_1 = 4)$, $g^*_{2^3}(t_1 = 2)$, $g^*_{2^4}(t_1 = 1)$, $g^*_3(t_2 = 8)$, $g^*_5(t_2 = 4)$, $g^*_{17}(t_2 = 1)$, $g_{4,12}$, $g_{2,2,2,10}$, $g_{2,2,6,6}$, $g_{4,4,2^3}(t_4 = 3)$, $g_{2,2,2,2,2^2}(t_5 = 6)$ and $g_{2,2,2,2,2^3}(t_5 = 5)$ belongs to G. By Lemma 3.1, we have $16 \leq n \leq 23$. Thus we can exclude $g^*_3(t_2 = 8)$.

If $m = 10$, then at least one of $g^*_2(t_1 = 6)$, $g^*_{2^2}(t_1 = 3)$, $g^*_3(t_2 = 6)$, $g^*_5(t_2 = 3)$, $g^*_7(t_2 = 2)$, $g^*_{13}(t_2 = 1)$, $g_{2,2,2,6}$ and $g_{2,2,2,2,2^2}(t_5 = 5)$ belongs to G, and at least one of $g_{3,3,9,9}$, $g_{2,2,2,3,3,3,3,6}$, $g^*_2(t_1 = 10)$, $g^*_{2^2}(t_1 = 5)$, $g^*_3(t_2 = 10)$, $g^*_5(t_2 = 5)$, $g^*_{11}(t_2 = 2)$, $g_{2,2,2,14}$, $g_{2,6,6,6}$, and $g_{2,2,2,2,2^2}(t_5 = 7)$ belongs to G. By Lemma 3.1, we have $20 \leq n \leq 29$. Thus we can exclude $g^*_3(t_2 = 10)$.

If $m = 12$, then at least one of $g_{5,5,8}$, $g_{4,4,5,5}$, $g_{3,3,3,3,8}$, $g_{3,3,3,3,5,5}$, $g_{3,3,3,3,4,4}$, $g_{2,2,2,2,5,5}$, $g_{2,2,2,2,3,3,3,3}$, $g^*_2(t_1 = 8)$, $g^*_{2^2}(t_1 = 4)$, $g^*_{2^3}(t_1 = 2)$, $g^*_{2^4}(t_1 = 1)$, $g^*_3(t_2 = 8)$, $g^*_5(t_2 = 4)$, $g^*_{17}(t_2 = 1)$, $g_{4,12}$, $g_{2,2,2,10}$, $g_{2,2,6,6}$, $g_{4,4,2^3}(t_4 = 3)$, $g_{2,2,2,2,2^2}(t_5 = 6)$ and $g_{2,2,2,2,2^3}(t_5 = 5)$ belongs to G, and at least one of $g_{3,3,3,5,15}$, $g_{2,2,2,5,5,10}$, $g_{3,3,3,3,4,12}$, $g_{2,2,3,3,3,3,6,6}$, $g^*_2(t_1 = 12)$, $g^*_{2^2}(t_1 = 6)$, $g^*_{2^3}(t_1 = 3)$, $g^*_3(t_2 = 12)$, $g^*_5(t_2 = 6)$, $g^*_7(t_2 = 4)$, $g^*_{13}(t_2 = 2)$, g_{24}, $g_{4,20}$, $g_{12,12}$, $g_{2,2,10,10}$, $g_{6,6,6,6}$, $g_{8,2^4}(t_3 = 2)$, $g_{4,4,2^3}(t_4 = 4)$, $g_{4,4,2^4}(t_4 = 3)$, $g_{2,2,2,2,2^2}(t_5 = 8)$, $g_{2,2,2,2,2^3}(t_5 = 6)$ and $g_{2,2,2,2,2^4}(t_5 = 5)$ belongs to G. By Lemma 3.1, we have $24 \leq n \leq 35$. Thus we can exclude $g^*_3(t_2 = 12)$.

If $m = 13$, then at least one of $g^*_2(t_1 = 9)$, $g^*_3(t_2 = 9)$, $g^*_7(t_2 = 3)$ and $g^*_{19}(t_2 = 1)$ belongs to G, and at least one of $g_{3,9,9,9}$, $g^*_2(t_1 = 13)$ and $g^*_3(t_2 = 13)$ belongs to G. By Lemma 3.1, we have $26 \leq n \leq 38$. Thus we can exclude $g^*_3(t_2 = 13)$.

If $m = 14$, then at least one of $g_{3,3,9,9}$, $g_{2,2,2,3,3,3,3,6}$, $g^*_2(t_1 = 10)$, $g^*_{2^2}(t_1 = 5)$, $g^*_3(t_2 = 10)$, $g^*_5(t_2 = 5)$, $g^*_{11}(t_2 = 2)$, $g_{2,2,2,14}$, $g_{2,6,6,6}$, and $g_{2,2,2,2,2^2}(t_5 = 7)$ belongs to G, and at least one of $g_{5,25}$, $g_{2,3,3,3,3,6,6,6}$, $g_2^*(t_1=14)$, $g_{2^2}^*(t_1=7)$, $g_3^*(t_2=14)$, $g_5^*(t_2=7)$, $g_{29}^*(t_2=1)$, $g_{2,2,2,22}$ and $g_{2,2,2,2,2^2}(t_5=9)$ belongs to G. By Lemma 3.1, we have $28\le n\le 41$. Thus we can exclude $g_3^*(t_2=14)$.

If $m=16$, then at least one of $g_{3,3,3,5,15}$, $g_{2,2,2,5,5,10}$, $g_{3,3,3,3,4,12}$, $g_{2,2,3,3,3,3,6,6}$, $g_2^*(t_1=12)$, $g_{2^2}^*(t_1=6)$, $g_{2^3}^*(t_1=3)$, $g_3^*(t_2=12)$, $g_5^*(t_2=6)$, $g_7^*(t_2=4)$, $g_{13}^*(t_2=2)$, g_{24}, $g_{4,20}$, $g_{12,12}$, $g_{2,2,10,10}$, $g_{6,6,6,6}$, $g_{8,2^4}(t_3=2)$, $g_{4,4,2^3}(t_4=4)$, $g_{4,4,2^4}(t_4=3)$, $g_{2,2,2,2,2^2}(t_5=8)$, $g_{2,2,2,2,2^3}(t_5=6)$ and $g_{2,2,2,2,2^4}(t_5=5)$ belongs to G, and at least one of $g_{9,9,9,9}$, $g_{3,3,15,15}$, $g_{4,5,5,20}$, $g_{3,3,3,3,24}$, $g_{2,2,5,5,10,10}$, $g_{3,3,3,3,12,12}$, $g_{3,3,3,3,6,6,6,6}$, $g_2^*(t_1=16)$, $g_{2^2}^*(t_1=8)$, $g_{2^3}^*(t_1=4)$, $g_{2^4}^*(t_1=2)$, $g_{2^5}^*(t_1=1)$, $g_3^*(t_2=16)$, $g_5^*(t_2=8)$, $g_{17}^*(t_2=2)$, $g_{4,28}$, $g_{2,2,2,26}$, $g_{2,2,14,14}$, $g_{2,10,10,10}$, $g_{4,4,2^3}(t_4=5)$, $g_{2,2,2,2,2^2}(t_5=10)$ and $g_{2,2,2,2,2^3}(t_5=7)$ belongs to G. By Lemma 3.1, we have $32\le n\le 47$. Thus we can exclude $g_3^*(t_2=16)$.

If $m=20$, then at least one of $g_{9,9,9,9}$, $g_{3,3,15,15}$, $g_{4,5,5,20}$, $g_{3,3,3,3,24}$, $g_{2,2,5,5,10,10}$, $g_{3,3,3,3,12,12}$, $g_{3,3,3,3,6,6,6,6}$, $g_2^*(t_1=16)$, $g_{2^2}^*(t_1=8)$, $g_{2^3}^*(t_1=4)$, $g_{2^4}^*(t_1=2)$, $g_{2^5}^*(t_1=1)$, $g_3^*(t_2=16)$, $g_5^*(t_2=8)$, $g_{17}^*(t_2=2)$, $g_{4,28}$, $g_{2,2,2,26}$, $g_{2,2,14,14}$, $g_{2,10,10,10}$, $g_{4,4,2^3}(t_4=5)$, $g_{2,2,2,2,2^2}(t_5=10)$ and $g_{2,2,2,2,2^3}(t_5=7)$ belongs to G, and at least one of $g_{2,5,5,10,10,10}$, $g_2^*(t_1=20)$, $g_{2^2}^*(t_1=10)$, $g_{2^3}^*(t_1=5)$, $g_3^*(t_2=20)$, $g_5^*(t_2=10)$, $g_{11}^*(t_2=4)$, $g_{41}^*(t_2=1)$, g_{40}, $g_{20,20}$, $g_{2,2,2,34}$, $g_{10,10,10,10}$, $g_{8,2^4}(t_3=3)$, $g_{8,2^5}(t_3=2)$, $g_{4,4,2^3}(t_4=6)$, $g_{4,4,2^4}(t_4=4)$, $g_{4,4,2^5}(t_4=3)$, $g_{2,2,2,2,2^2}(t_5=12)$, $g_{2,2,2,2,2^3}(t_5=8)$, $g_{2,2,2,2,2^4}(t_5=6)$ and $g_{2,2,2,2,2^5}(t_5=5)$ belongs to G. By Lemma 3.1, we have $40\le n\le 59$. Thus we can exclude $g_3^*(t_2=20)$.

If $m=24$, then at least one of $g_{2,5,5,10,10,10}$, $g_2^*(t_1=20)$, $g_{2^2}^*(t_1=10)$, $g_{2^3}^*(t_1=5)$, $g_3^*(t_2=20)$, $g_5^*(t_2=10)$, $g_{11}^*(t_2=4)$, $g_{41}^*(t_2=1)$, g_{40}, $g_{20,20}$, $g_{2,2,2,34}$, $g_{10,10,10,10}$, $g_{8,2^4}(t_3=3)$, $g_{8,2^5}(t_3=2)$, $g_{4,4,2^3}(t_4=6)$, $g_{4,4,2^4}(t_4=4)$, $g_{4,4,2^5}(t_4=3)$, $g_{2,2,2,2,2^2}(t_5=12)$, $g_{2,2,2,2,2^3}(t_5=8)$, $g_{2,2,2,2,2^4}(t_5=6)$ and $g_{2,2,2,2,2^5}(t_5=5)$ belongs to G, and at least one of $g_{25,25}$, $g_{5,5,40}$, $g_{5,5,20,20}$, $g_{5,5,10,10,10,10}$, $g_2^*(t_1=24)$, $g_{2^2}^*(t_1=12)$, $g_{2^3}^*(t_1=6)$, $g_{2^4}^*(t_1=3)$, $g_3^*(t_2=24)$, $g_5^*(t_2=12)$, $g_7^*(t_2=8)$, $g_{13}^*(t_2=4)$, $g_{17}^*(t_2=3)$, $g_{4,44}$, $g_{2,2,22,22}$, $g_{4,4,2^3}(t_4=7)$, $g_{2,2,2,2,2^2}(t_5=14)$ and $g_{2,2,2,2,2^3}(t_5=9)$ belongs to G. By Lemma 3.1, we have $48\le n\le 71$. Thus we can exclude $g_3^*(t_2=24)$.

Assume that $m=t_1 2^{a-1}$ and $m\notin\{7,8,10,12,13,14,16,20,24\}$. If $m-4=7$, then $m=11$. It follows that at least one of $g_{3,3,3,9}$, $g_2^*(t_1=7)$ and $g_3^*(t_2=7)$ belongs to G, and at least one of $g_2^*(t_1=11)$, $g_3^*(t_2=11)$ and $g_{23}^*(t_2=1)$ belongs to G. By Lemma 3.1, we have $22\le n\le 32$. Thus we can exclude $g_3^*(t_2=11)$.

If $m-4=13$, then $m=17$. It follows that at least one of $g_{3,9,9,9}$, $g_2^*(t_1=13)$ and $g_3^*(t_2=13)$ belongs to G, and at least one of $g_2^*(t_1=17)$ and $g_3^*(t_2=17)$ belongs to G. By Lemma 3.1, we have $34\le n\le 50$. Thus we can exclude $g_3^*(t_2=17)$.

If $m-4=14$, then $m=18$. It follows that at least one of $g_{5,25}$, $g_{2,3,3,3,3,6,6,6}$, $g_2^*(t_1=14)$, $g_{2^2}^*(t_1=7)$, $g_3^*(t_2=14)$, $g_5^*(t_2=7)$, $g_{29}^*(t_2=1)$, $g_{2,2,2,22}$ and $g_{2,2,2,2,2^2}(t_5=9)$ belongs to G, and at least one of $g_2^*(t_1=18)$, $g_{2^2}^*(t_1=9)$, $g_3^*(t_2=18)$, $g_5^*(t_2=9)$, $g_7^*(t_2=6)$, $g_{13}^*(t_2=3)$, $g_{19}^*(t_2=2)$, $g_{37}^*(t_2=1)$ and $g_{2,2,2,2,2^2}(t_5=11)$ belongs to G. By Lemma 3.1, we have $36\le n\le 53$. Thus we can exclude $g_3^*(t_2=18)$.

If $m-4=24$, then $m=28$. It follows that at least one of $g_{25,25}$, $g_{5,5,40}$, $g_{5,5,20,20}$, $g_{5,5,10,10,10,10}$, $g_2^*(t_1=24)$, $g_{2^2}^*(t_1=12)$, $g_{2^3}^*(t_1=6)$, $g_{2^4}^*(t_1=3)$, $g_3^*(t_2=24)$, $g_5^*(t_2=12)$, $g_7^*(t_2=8)$, $g_{13}^*(t_2=4)$, $g_{17}^*(t_2=3)$, $g_{4,44}$, $g_{2,2,22,22}$, $g_{4,4,2^3}(t_4=7)$, $g_{2,2,2,2,2^2}(t_5=14)$ and $g_{2,2,2,2,2^3}(t_5=9)$ belongs to G, and at least one of $g_2^*(t_1=28)$, $g_{2^2}^*(t_1=14)$, $g_{2^3}^*(t_1=7)$, $g_3^*(t_2=28)$, $g_5^*(t_2=14)$, $g_{29}^*(t_2=2)$, g_{56}, $g_{4,52}$, $g_{28,28}$, $g_{2,2,26,26}$, $g_{14,14,14,14}$, $g_{8,2^4}(t_3=4)$, $g_{4,4,2^3}(t_4=8)$, $g_{4,4,2^4}(t_4=5)$, $g_{2,2,2,2,2^2}(t_5=16)$, $g_{2,2,2,2,2^3}(t_5=10)$ and $g_{2,2,2,2,2^4}(t_5=7)$ belongs to G. By Lemma 3.1, we have $56\le n\le 83$. Thus we can exclude $g_3^*(t_2=28)$.

Now we assume that $m-4\notin\{7,8,10,12,13,14,16,20,24\}$. Then at least one of $g_{2^{a'}}^*$ $(t_1'2^{a'-1}+4=t_1 2^{a-1})$, $g_{p_1'}^*(t_2\frac{p_1'-1}{2}+4=t_1 2^{a-1})$, $g_{8p_2'}(4p_2'+4=t_1 2^{a-1})$, $g_{4,4p_3'}(6+2p_3'=$

$t_12^{a-1})$, $g_{4p_4',4p_4'}(4p_4'+4=t_12^{a-1})$, $g_{2,2,2,2p_5'}(7+p_5'=t_12^{a-1})$, $g_{2,2,2p_6',2p_6'}(2p_6'+6=t_12^{a-1})$, $g_{2,2p_7',2p_7',2p_7'}(3p_7'+5=t_12^{a-1})$, $g_{2p_8',2p_8',2p_8',2p_8'}(4p_8'+4=t_12^{a-1})$, $g_{8,2^{b_1'}}((t_3'-1)2^{b_1'-1}+8=t_12^{a-1})$, $g_{4,4,2^{b_2'}}((t_4'-2)2^{b_2'-1}+8=t_12^{a-1})$ and $g_{2,2,2,2,2^{b_3'}}((t_5'-4)2^{b_3'-1}+8=t_12^{a-1})$ belongs to G, and at least one of $g^*_{2^a}$, $g^*_{p_1}(t_2\frac{p_1-1}{2}=t_12^{a-1})$, $g_{8p_2}(4p_2=t_12^{a-1})$, $g_{4,4p_3}(2+2p_3=t_12^{a-1})$, $g_{4p_4,4p_4}(4p_4=t_12^{a-1})$, $g_{2,2,2,2p_5}(3+p_5=t_12^{a-1})$, $g_{2,2,2p_6,2p_6}(2p_6+2=t_12^{a-1})$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1=t_12^{a-1})$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8=t_12^{a-1})$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=t_12^{a-1})$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=t_12^{a-1})$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=t_12^{a-1})$ belongs to G. By Lemma 3.1, we have $2^at_1\le n\le\lfloor 2^at_1\frac{p}{p-1}\rfloor-1$, where $p\in\{p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_1',p_2',p_3',p_4',p_5',p_6',p_7',p_8'\}$.

Assume that $m=t_2\frac{p_1-1}{2}$. We may let $m\notin\{7,8,10,12,13,14,16,20,24,t_12^{a-1}\}$ and $m-4\notin\{7,8,10,12,13,14,16,20,24\}$. Then at least one of $g^*_{2^{a'}}(t_1'2^{a'-1}+4=t_2\frac{p_1-1}{2})$, $g^*_{p_1'}(t_2'\frac{p_1'-1}{2}+4=t_2\frac{p_1-1}{2})$, $g_{8p_2'}(4p_2'+4=t_2\frac{p_1-1}{2})$, $g_{4,4p_3'}(6+2p_3'=t_2\frac{p_1-1}{2})$, $g_{4p_4',4p_4'}(4p_4'+4=t_2\frac{p_1-1}{2})$, $g_{2,2,2,2p_5'}(7+p_5'=t_2\frac{p_1-1}{2})$, $g_{2,2,2p_6',2p_6'}(2p_6'+6=t_2\frac{p_1-1}{2})$, $g_{2,2p_7',2p_7',2p_7'}(3p_7'+5=t_2\frac{p_1-1}{2})$, $g_{2p_8',2p_8',2p_8',2p_8'}(4p_8'+4=t_2\frac{p_1-1}{2})$, $g_{8,2^{b_1'}}((t_3'-1)2^{b_1'-1}+8=t_2\frac{p_1-1}{2})$, $g_{4,4,2^{b_2'}}((t_4'-2)2^{b_2'-1}+8=t_2\frac{p_1-1}{2})$ and $g_{2,2,2,2,2^{b_3'}}((t_5'-4)2^{b_3'-1}+8=t_2\frac{p_1-1}{2})$ belongs to G, and at least one of $g^*_{p_1}$, $g_{8p_2}(4p_2=t_2\frac{p_1-1}{2})$, $g_{4,4p_3}(2+2p_3=t_2\frac{p_1-1}{2})$, $g_{4p_4,4p_4}(4p_4=t_2\frac{p_1-1}{2})$, $g_{2,2,2,2p_5}(3+p_5=t_2\frac{p_1-1}{2})$, $g_{2,2,2p_6,2p_6}(2p_6+2=t_2\frac{p_1-1}{2})$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1=t_2\frac{p_1-1}{2})$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8=t_2\frac{p_1-1}{2})$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=t_2\frac{p_1-1}{2})$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=t_2\frac{p_1-1}{2})$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=t_2\frac{p_1-1}{2})$ belongs to G. By Lemma 3.1, we have $t_2(p_1-1)\le n\le\lfloor t_2(p_1-1)\frac{p}{p-1}\rfloor-1$, where $p\in\{p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_1',p_2',p_3',p_4',p_5',p_6',p_7',p_8'\}$.

Assume that $m=4p_2$. We may let $m\notin\{7,8,10,12,13,14,16,20,24,t_12^{a-1},t_2\frac{p_1-1}{2}\}$, and $m-4\notin\{7,8,10,12,13,14,16,20,24\}$. Then at least one of $g^*_{2^{a'}}(t_1'2^{a'-1}+4=4p_2)$, $g^*_{p_1'}(t_2'\frac{p_1'-1}{2}+4=4p_2)$, $g_{8p_2'}(4p_2'+4=4p_2)$, $g_{4,4p_3'}(6+2p_3'=4p_2)$, $g_{4p_4',4p_4'}(4p_4'+4=4p_2)$, $g_{2,2,2,2p_5'}(7+p_5'=4p_2)$, $g_{2,2,2p_6',2p_6'}(2p_6'+6=4p_2)$, $g_{2,2p_7',2p_7',2p_7'}(3p_7'+5=4p_2)$, $g_{2p_8',2p_8',2p_8',2p_8'}(4p_8'+4=4p_2)$, $g_{8,2^{b_1'}}((t_3'-1)2^{b_1'-1}+8=4p_2)$, $g_{4,4,2^{b_2'}}((t_4'-2)2^{b_2'-1}+8=4p_2)$ and $g_{2,2,2,2,2^{b_3'}}((t_5'-4)2^{b_3'-1}+8=4p_2)$ belongs to G, and at least one of g_{8p_2}, $g_{4,4p_3}(2+2p_3=4p_2)$, $g_{4p_4,4p_4}(p_4=p_2)$, $g_{2,2,2,2p_5}(3+p_5=4p_2)$, $g_{2,2,2p_6,2p_6}(2p_6+2=4p_2)$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1=4p_2)$, $g_{2p_8,2p_8,2p_8,2p_8}(p_8=p_2)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=4p_2)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=4p_2)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=4p_2)$ belongs to G. By Lemma 3.1, we have $8p_2\le n\le\lfloor 8p_2\frac{p}{p-1}\rfloor-1$, where $p\in\{p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_1',p_2',p_3',p_4',p_5',p_6',p_7',p_8'\}$.

Assume that $m=2+2p_3$. We may let $m\notin\{7,8,10,12,13,14,16,20,24,t_12^{a-1},t_2\frac{p_1-1}{2},4p_2\}$ and $m-4\notin\{7,8,10,12,13,14,16,20,24\}$. Then at least one of $g^*_{2^{a'}}(t_1'2^{a'-1}+4=2+2p_3)$, $g^*_{p_1'}(t_2'\frac{p_1'-1}{2}+4=2+2p_3)$, $g_{8p_2'}(4p_2'+4=2+2p_3)$, $g_{4,4p_3'}(6+2p_3'=2+2p_3)$, $g_{4p_4',4p_4'}(4p_4'+4=2+2p_3)$, $g_{2,2,2,2p_5'}(7+p_5'=2+2p_3)$, $g_{2,2,2p_6',2p_6'}(2p_6'+6=2+2p_3)$, $g_{2,2p_7',2p_7',2p_7'}(3p_7'+5=2+2p_3)$, $g_{2p_8',2p_8',2p_8',2p_8'}(4p_8'+4=2+2p_3)$, $g_{8,2^{b_1'}}((t_3'-1)2^{b_1'-1}+8=2+2p_3)$, $g_{4,4,2^{b_2'}}((t_4'-2)2^{b_2'-1}+8=2+2p_3)$ and $g_{2,2,2,2,2^{b_3'}}((t_5'-4)2^{b_3'-1}+8=2+2p_3)$ belongs to G, and at least one of $g_{4,4p_3}$, $g_{4p_4,4p_4}(4p_4=2+2p_3)$, $g_{2,2,2,2p_5}(3+p_5=2+2p_3)$, $g_{2,2,2p_6,2p_6}(2p_6+2=2+2p_3)$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1=2+2p_3)$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8=2+2p_3)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4=2+2p_3)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4=2+2p_3)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4=2+2p_3)$ belongs to G. By Lemma 3.1, we have $4+4p_3\le n\le\lfloor(4+4p_3)\frac{p}{p-1}\rfloor-1$, where $p\in\{p_3,p_4,p_5,p_6,p_7,p_8,p_1',p_2',p_3',p_4',p_5',p_6',p_7',p_8'\}$.

Assume that $m = 4p_4$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = 4p_4)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = 4p_4)$, $g_{8p'_2}(4p'_2+4 = 4p_4)$, $g_{4,4p'_3}(6+2p'_3 = 4p_4)$, $g_{4p'_4,4p'_4}(4p'_4+4 = 4p_4)$, $g_{2,2,2,2p'_5}(7+p'_5 = 4p_4)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = 4p_4)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = 4p_4)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = 4p_4)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = 4p_4)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = 4p_4)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = 4p_4)$ belongs to G, and at least one of $g_{4p_4,4p_4}$, $g_{2,2,2,2p_5}(3+p_5 = 4p_4)$, $g_{2,2,2p_6,2p_6}(2p_6+2 = 4p_4)$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1 = 4p_4)$, $g_{2p_8,2p_8,2p_8,2p_8}(p_8 = p_4)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4 = 4p_4)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = 4p_4)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = 4p_4)$ belongs to G. By Lemma 3.1, we have $8p_4 \le n \le \lfloor 8p_4\frac{p}{p-1}\rfloor - 1$, where $p \in \{p_4, p_5, p_6, p_7, p_8, p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = 3+p_5$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = 3+p_5)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = 3+p_5)$, $g_{8p'_2}(4p'_2+4 = 3+p_5)$, $g_{4,4p'_3}(6+2p'_3 = 3+p_5)$, $g_{4p'_4,4p'_4}(4p'_4+4 = 3+p_5)$, $g_{2,2,2,2p'_5}(7+p'_5 = 3+p_5)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = 3+p_5)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = 3+p_5)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = 3+p_5)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = 3+p_5)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = 3+p_5)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = 3+p_5)$ belongs to G, and at least one of $g_{2,2,2,2p_5}$, $g_{2,2,2p_6,2p_6}(2p_6+2 = 3+p_5)$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1 = 3+p_5)$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8 = 3+p_5)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4 = 3+p_5)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = 3+p_5)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = 3+p_5)$ belongs to G. By Lemma 3.1, we have $6+2p_5 \le n \le \lfloor(6+2p_5)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p_5, p_6, p_7, p_8, p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = 2p_6+2$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = 2p_6+2)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = 2p_6+2)$, $g_{8p'_2}(4p'_2+4 = 2p_6+2)$, $g_{4,4p'_3}(6+2p'_3 = 2p_6+2)$, $g_{4p'_4,4p'_4}(4p'_4+4 = 2p_6+2)$, $g_{2,2,2,2p'_5}(7+p'_5 = 2p_6+2)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = 2p_6+2)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = 2p_6+2)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = 2p_6+2)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = 2p_6+2)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = 2p_6+2)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = 2p_6+2)$ belongs to G, and at least one of $g_{2,2,2p_6,2p_6}$, $g_{2,2p_7,2p_7,2p_7}(3p_7+1 = 2p_6+2)$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8 = 2p_6+2)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4 = 2p_6+2)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = 2p_6+2)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = 2p_6+2)$ belongs to G. By Lemma 3.1, we have $4p_6+4 \le n \le \lfloor(4p_6+4)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p_6, p_7, p_8, p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = 3p_7+1$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = 3p_7+1)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = 3p_7+1)$, $g_{8p'_2}(4p'_2+4 = 3p_7+1)$, $g_{4,4p'_3}(6+2p'_3 = 3p_7+1)$, $g_{4p'_4,4p'_4}(4p'_4+4 = 3p_7+1)$, $g_{2,2,2,2p'_5}(7+p'_5 = 3p_7+1)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = 3p_7+1)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = 3p_7+1)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = 3p_7+1)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = 3p_7+1)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = 3p_7+1)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = 3p_7+1)$ belongs to G, and at least one of $g_{2,2p_7,2p_7,2p_7}$, $g_{2p_8,2p_8,2p_8,2p_8}(4p_8 = 3p_7+1)$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4 = 3p_7+1)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = 3p_7+1)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = 3p_7+1)$ belongs to G. By Lemma 3.1, we have $6p_7+2 \le n \le \lfloor(6p_7+2)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p_7, p_8, p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = 4p_8$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2, 3p_7+1\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = 4p_8)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = 4p_8)$, $g_{8p'_2}(4p'_2+4 = 4p_8)$, $g_{4,4p'_3}(6+2p'_3 = 4p_8)$, $g_{4p'_4,4p'_4}(4p'_4+4 = 4p_8)$, $g_{2,2,2,2p'_5}(7+p'_5 = 4p_8)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = 4p_8)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = 4p_8)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = 4p_8)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = 4p_8)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = 4p_8)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = 4p_8)$ belongs to G, and at least one of $g_{2p_8,2p_8,2p_8,2p_8}$, $g_{8,2^{b_1}}((t_3-1)2^{b_1-1}+4 = 4p_8)$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = 4p_8)$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = 4p_8)$ belongs to G. By Lemma 3.1, we have $8p_8 \le n \le \lfloor 8p_8\frac{p}{p-1}\rfloor - 1$, where $p \in \{p_8, p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = (t_3-1)2^{b_1-1}+4$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2, 3p_7+1, 4p_8\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = (t_3-1)2^{b_1-1}+4)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = (t_3-1)2^{b_1-1}+4)$, $g_{8p'_2}(4p'_2+4 = (t_3-1)2^{b_1-1}+4)$, $g_{4,4p'_3}(6+2p'_3 = (t_3-1)2^{b_1-1}+4)$, $g_{4p'_4,4p'_4}(4p'_4+4 = (t_3-1)2^{b_1-1}+4)$, $g_{2,2,2,2p'_5}(7+p'_5 = (t_3-1)2^{b_1-1}+4)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = (t_3-1)2^{b_1-1}+4)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = (t_3-1)2^{b_1-1}+4)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = (t_3-1)2^{b_1-1}+4)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = (t_3-1)2^{b_1-1}+4)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = (t_3-1)2^{b_1-1}+4)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = (t_3-1)2^{b_1-1}+4)$ belongs to G, and at least one of $g_{8,2^{b_1}}$, $g_{4,4,2^{b_2}}((t_4-2)2^{b_2-1}+4 = (t_3-1)2^{b_1-1}+4))$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = (t_3-1)2^{b_1-1}+4))$ belongs to G. By Lemma 3.1, we have $(t_3-1)2^{b_1}+8 \le n \le \lfloor((t_3-1)2^{b_1}+8)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = (t_4-2)2^{b_2-1}+4$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2, 3p_7+1, 4p_8, (t_3-1)2^{b_1-1}+4\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = (t_4-2)2^{b_2-1}+4)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = (t_4-2)2^{b_2-1}+4)$, $g_{8p'_2}(4p'_2+4 = (t_4-2)2^{b_2-1}+4)$, $g_{4,4p'_3}(6+2p'_3 = (t_4-2)2^{b_2-1}+4)$, $g_{4p'_4,4p'_4}(4p'_4+4 = (t_4-2)2^{b_2-1}+4)$, $g_{2,2,2,2p'_5}(7+p'_5 = (t_4-2)2^{b_2-1}+4)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = (t_4-2)2^{b_2-1}+4)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = (t_4-2)2^{b_2-1}+4)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = (t_4-2)2^{b_2-1}+4)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = (t_4-2)2^{b_2-1}+4)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = (t_4-2)2^{b_2-1}+4)$, and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = (t_4-2)2^{b_2-1}+4)$ belongs to G, and at least one of $g_{4,4,2^{b_2}}$ and $g_{2,2,2,2,2^{b_3}}((t_5-4)2^{b_3-1}+4 = (t_3-1)2^{b_1-1}+4))$ belongs to G. By Lemma 3.1, we have $(t_4-2)2^{b_2}+8 \le n \le \lfloor((t_4-2)2^{b_2}+8)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$.

Assume that $m = (t_5-4)2^{b_3-1}+4$. We may let $m \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24, t_1 2^{a-1}, t_2\frac{p_1-1}{2}, 4p_2, 2+2p_3, 4p_4, 3+p_5, 2p_6+2, 3p_7+1, 4p_8, (t_3-1)2^{b_1-1}+4, (t_4-2)2^{b_2-1}+4\}$ and $m-4 \notin \{7, 8, 10, 12, 13, 14, 16, 20, 24\}$. Then at least one $g^*_{2^{a'}}(t'_1 2^{a'-1}+4 = (t_5-4)2^{b_3-1}+4)$, $g^*_{p'_1}(t'_2\frac{p'_1-1}{2}+4 = (t_5-4)2^{b_3-1}+4)$, $g_{8p'_2}(4p'_2+4 = (t_5-4)2^{b_3-1}+4)$, $g_{4,4p'_3}(6+2p'_3 = (t_5-4)2^{b_3-1}+4)$, $g_{4p'_4,4p'_4}(4p'_4+4 = (t_5-4)2^{b_3-1}+4)$, $g_{2,2,2,2p'_5}(7+p'_5 = (t_5-4)2^{b_3-1}+4)$, $g_{2,2,2p'_6,2p'_6}(2p'_6+6 = (t_5-4)2^{b_3-1}+4)$, $g_{2,2p'_7,2p'_7,2p'_7}(3p'_7+5 = (t_5-4)2^{b_3-1}+4)$, $g_{2p'_8,2p'_8,2p'_8,2p'_8}(4p'_8+4 = (t_5-4)2^{b_3-1}+4)$, $g_{8,2^{b'_1}}((t'_3-1)2^{b'_1-1}+8 = (t_5-4)2^{b_3-1}+4)$, $g_{4,4,2^{b'_2}}((t'_4-2)2^{b'_2-1}+8 = (t_5-4)2^{b_3-1}+4)$ and $g_{2,2,2,2,2^{b'_3}}((t'_5-4)2^{b'_3-1}+8 = (t_5-4)2^{b_3-1}+4)$ belongs to G, and $g_{2,2,2,2,2^{b_3}}$ belongs to G. By Lemma 3.1, we have $(t_5-4)2^{b_3}+8 \le n \le \lfloor((t_5-4)2^{b_3}+8)\frac{p}{p-1}\rfloor - 1$, where $p \in \{p'_1, p'_2, p'_3, p'_4, p'_5, p'_6, p'_7, p'_8\}$. ∎

In the future, we want to explore the constructions of these groups in Theorem 1.2.

Acknowledgments

This work was partially supported by National Natural Science Foundation of China (12301026), China Scholarship Council (202208360148), and Jiangxi Provincial Natural Science Foundation (20232BAB211006).

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