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A Picture Fuzzy Based Approach Using Proximal Relator Spaces

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Abstract. Picture fuzzy sets are extensions of the fuzzy sets and the intuitonistic fuzzy sets which consist of membership, a neutral membership and a non-membership degrees. In this paper, we define the picture fuzzy proximity relations which are extension of picture fuzzy sets and fuzzy proximity relations, and give some examples. Also, we give the definition of picture fuzzy spatial and descriptive Lodato proximity space.

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1. Introduction

Proximity spaces were defined by Efremoviĉ [5]. The set U, together with the proximity relation that M is near(proximal) N for subsets M and N of any set U, was known as proximity space. Readers can easily access many resources about proximity spaces. Naimpally and Warrack wrote a wonderful book about proximity space and presented it to their readers [11]. There is a more visual form of proximity called descriptive proximity, which is used in a number of applications. A general descriptive proximity is an Efremoviĉ proximity.

Peters defined $(U, \mathcal{R}_\delta)$ relator space for a type of proximity spaces relations $\mathcal{R}_\delta$ on U [17]. To understand easily, he used two proximity relations like the Efremoviĉ proximity and the descriptive proximity δ_Φ in defining $\mathcal{R}_{\delta_\Phi}$ [14,16,18].

A non-empty $\mathcal{R}$ of binary relations on a non-empty set U is known as *a relator* on U, $(U, \mathcal{R})$ is known as *a relator space* which is a generalized uniform space lacking all the conditions of uniform space except the reflexivity of the corresponding relations [20]. By using proximity relations $\mathcal{R}_\delta$ on U, Peters presented a proximal relator space $(U, \mathcal{R}_\delta)$ [18]. To be clear, it can be considered only three proximity relations, namely the Lodato proximity [8,9] (denoted by δ), the descriptive Lodato proximity δ [18] (denoted by δ_Φ), an extension of the Lodato proximity, the descriptive Lodato fuzzy proximity [18] (denoted by τ_{δ_Φ}), and also a fuzzy proximal relator [12] (denoted by $\mathcal{R}_{\tau_{\delta_\Phi}}$ is defined by $\mathcal{R}_{\tau_{\delta_\Phi}} = \{\delta, \delta_\Phi, \tau_{\delta_\Phi}\}$).

Zadeh introduced the fuzzy sets (FS) which are characterized by a membership function which associates with each point a real number in the interval $[0,1]$ [25]. We can see fuzzy set theory various areas such as decision making, health science and so on. However, there may be instabilities in real life, such as hesitation or uncertain decisions get involved, and the fuzzy sets are not available to process some information with fuzziness and uncertainty. Therefore, to handle such problems, Atanassov generalized fuzzy sets to intuitionistic fuzzy sets (IFS) [1] that can deal with some information with fuzziness and uncertainty in many fields.

Researchers still have problems in some situations which is non-identifiable by intuitionistic fuzzy sets such as voting problems. In the voting problems, voters have four options: vote for, vote against, abstain and refuse. To solve

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such problems like voting, picture fuzzy sets (PFSs) have been introduced by Cuong and Kreinovich [3]. The picture fuzzy set (PFS) is a direct generalization of FS and IFS. Fundamentally, we see PFSs mostly real-life situtations such as person thoughts including mostly answers kinds: yes, neutral, no, and refuse. Besides, it is also used to solve practical dilemmas as image processing, medical diagnosing and so on.

Coung studied a lot of operation as distance measure of PFSs, convex combination of PFSs, interval-valued PFSs, PF relations, picture fuzzy soft set (PFSS), and also he applied PFSs to a simple decision making problems [4]. By investigating PFS, researchers defined a new parameter which is the neutral function. It solves the complex problems in a better way. Using PFSs is vital to modeling a number of real-life decision-making problems alligning with similarity measure and distance measures. Many researchers have been studied the problems by using the PFSs environment, such as Li et al. [7], Kumar et al. [6], Luo and Li [10], Singh and Ganie [19], Verma et al [24].

Öztürk et al presented fuzzy proximal relator spaces [12], L-fuzzy proximity [21] and complex fuzzy proximity [22]. Afterward, Tekin defined spherical and Pythagorean fuzzy proximity relations by using fuzzy relations via relator spaces [23]. Besides, Öztürk carried fuzzy proximities to intuitionistic fuzzy environments [13].

The picture fuzzy proximity relations are an effective tool to determine proximity of different groups that have similar or different. In the existing fuzzy proximity relations, it is composed of just a membership function [12,21,22]. However, we defined a picture fuzzy proximity relation which combines a membership function, a neutral membership function and a non-membership function. When we compare proposed and existing fuzzy proximity relation, it shows that the picture fuzzy proximity relation is an important tool to evaluate the degree of proximity between different objects by three sides which are membership function, a neutral membership function and a non-membership function.

When we would like to make more understandable our paper with an example, PFPR could be used for classification some objects like computers. They possess similar or different properties like ram capacity, memory, operating systems and so on. We can choose 3 groups of computers like 3 sets, but they must have intersection. Thus, we can find out the proximity of them by determining a membership function, a neutral membership function and a non-membership function.

Furthermore, we investigate a picture fuzzy spatial proximity relation and a picture fuzzy spatial Lodato proximity relation based on the proposed picture fuzzy proximity relation. We use picture fuzzy proximity relation to figure out practical problems that have different objects spatially far but descriptively near to each other. It is essential to find the proximity and difference between objects. The picture fuzzy proximity relations are useful tools to find out the level of connection and difference between three or more sets. Thus, our main aim in this paper is to create a theoretical infrastructure for subsequent studies.

2. Preliminaries

In this section, it will be given briefly the basic definition, operation of picture fuzzy set and proximal relator spaces.

Definition 2.1. In the following section, some information is generally reproduced verbatim from [23] because the same information is used in the both of articles.

Let U be a nonempty set. A basic proximity δ is a relation on the power set of a nonempty set U, the collection of all subsets of U, which satisfies the following axioms for all subsets M,N,Q of U:

(B_0) $M \,\delta\, N \rightarrow M \neq \emptyset, N \neq \emptyset$

(B_1) $M \,\delta\, N$ implies $N \,\delta\, M$.

(B_2) $M \cap N \neq \emptyset$ implies $M \,\delta\, N$.

(B_3) $(M \cup N)\ \delta\ Q \Longleftrightarrow M \,\delta\, Q$ or $N \,\delta\, Q$.

Further, δ is called Lodato proximity if it satisfies (B_0)-(B_3) and the following axioms:

(B_4) ($M\delta N$and $n\delta Q$ for each $n \in N$) $\rightarrow M\delta Q$.

Furthermore, δ is called separated proximity if it satisfies (B_0)-(B_4) and the following axioms:

(B_5) $n\delta m \rightarrow n = m$.

If δ is a basic proximity and holds the condition (B_6) below, then it is known as an Efremovič proximity on U.

(B_6) If, for any $M, N \subset U$, $M \not\delta N$, there exists $Q, P \subset U$, $Q \cup P = U$ such that $M \not\delta N$ and $N \not\delta P$.

The pair (U, δ) is called a basic (Lodato, Efremovič) proximity space. When we write $M\ \delta N$, we read M is near to N, while when we write $M \not\delta N$, we read M is far from N.

2.1. **Spatially Near.** [15] Let U be a nonempty set of points and $\mathcal{P}(U)$ be the power set of U. The closure of a subset $M \in \mathcal{P}(U)$ defined by $clM = u \in U : u\delta M$. The closure of a subset M is the set of all points u in U that are near M.

Let δ be a spatial nearness (proximity) relation on a nonempty U. $M\delta N$ means M is spatially near N, if $M \cap N \neq \emptyset$, in the other words, the intersection of M and N is not empty (clM and clN have at least on point in common). The spatial proximity (nearness) relation δ is defined as for $M, N \in \mathcal{P}(U)$.

$$\delta = \{(M, N) \in \mathcal{P}(U) \times \mathcal{P}(U) : clM \cap clN \neq \emptyset\}.$$

2.2. **Descriptively Near.** [15] Let U be a nonempty set endowed with a descriptive proximity relation δ_Φ, $x \in U$ and $M, N \in \mathcal{P}(U)$. Also, $\Phi = \{\phi_1, \phi_2, \cdots, \phi_k\}$ is a set of probe functions such that $\phi_i : U \to \mathbb{R}$ represent features of each x, where $\phi_i(x)$ equals a feature value of x. Let $\Phi(x)$ denote a feature vector for the object x, i.e, a vector of feature values that describe x, where

$$\Phi(x) = \{\phi_1(x), \phi_2(x), \cdots, \phi_k(x)\}.$$

A feature vector gives a description of an object. $\mathcal{Q}(M)$ and $\mathcal{Q}(N)$ indicate sets of descriptions of points in M, N, respectively.

$$\mathcal{Q}(M) = \{\Phi(a) : a \in M\}.$$

M is descriptively near N (denoted by $M\ \delta_\Phi N$). The descriptive proximity of M and N is defined by

$$M\delta_\Phi N \Leftrightarrow \mathcal{Q}(M) \cap \mathcal{Q}(N) \neq \emptyset.$$

Similarly, $M\ \not\delta_\Phi N$ reads M is descriptively far from N. The descriptive remoteness of M and N given by

$$M\ \not\delta_\Phi N \Leftrightarrow \mathcal{Q}(M) \cap \mathcal{Q}(N) = \emptyset.$$

The pair (U, δ_Φ) is called a descriptive proximity space [2].

Definition 2.2 ([15]). Let U be a nonempty set. A descriptive EF-proximity δ_Φ is defined on $\mathcal{P}(U)$ which satisfies the following axioms for all subsets M, N, P of $\mathcal{P}(U)$:

(D_0) $M\ \delta_\Phi\ N$ implies $M \neq \emptyset$, $N \neq \emptyset$

(D_1) $M\ \delta_\Phi\ N$ implies $N\ \delta_\Phi\ M$ (descriptive symmetry).

(D_2) $M \cap N \neq \emptyset$ implies $M\ \delta_\Phi\ N$.

(D_3) $(M \cup N)\ \delta_\Phi\ P \Longleftrightarrow M\ \delta_\Phi\ P$ or $N\ \delta_\Phi\ P$.

(D_4) If, for any $M, N \subset U$, $M\ \not\delta_\Phi\ N$, there exists $P, Q \subset U, P \cup Q = U$ such that $M\ \not\delta_\Phi\ P$ and $N\ \not\delta_\Phi\ Q$. The structure (U, δ_Φ) is celled a descriptive EF-proximity space.

Definition 2.3 ([12]). Let $(U, \mathcal{R})$ be a proximal relator space,

$$\begin{array}{lll}\tau_\mathcal{R}: & \mathcal{P}(U) \times \mathcal{P}(U) & \longrightarrow [0,1] \\ & (M, N) & \longmapsto \tau_\mathcal{R}(M, N)\end{array}$$

be a fuzzy relation and $M, N \subset U$; then

$$\mathcal{R} = \{((M, N), \tau_\mathcal{R}(M, N)) | (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$$

is defined as a fuzzy proximity relation $M, N, P \in \mathcal{P}(U)$, if it holds the following conditions:

1) $\tau_\mathcal{R}(M, \emptyset) = 0$.

2) $\tau_\mathcal{R}(M, N) = \tau_\mathcal{R}(N, M)$.

3) $\tau_\mathcal{R}(M, N) \neq 0$ implies $M\ \mathcal{R}\ N$.

4) $\tau_\mathcal{R}(M, (N \cup P)) \neq 0 \Rightarrow \tau_\mathcal{R}(M, N) \neq 0$ and $M\ \mathcal{R}\ N$ or $\tau_\mathcal{R}(M, P) \neq 0$ and $M\ \mathcal{R}\ P$.

Definition 2.4 ([3]). Let U be nonempty set, a Picture fuzzy set (PFS) is presented as:

$$M = \{(u, f_M(u), g_M(u), h_M(u),) \mid \forall u \in U\}.$$

$f_M : U \to [0, 1]$ represents the grade of membership, $h_M : U \to [0, 1]$ represents the grade of uncertainty, and $g_M : U \to [0, 1]$ represents the grade of non-membership of the element $u \in U$ to the set M, if $0 \le f_M(u)+g_M(u)+h_M(u) \le 1$ and $1 - (f_M(u) + g_M(u) + h_M(u))$ is called refusal grade of u in M.

3. Picture Fuzzy Proximal Relator Spaces

Definition 3.1. Let $(U, \mathcal{R})$ be a proximal relator space, $\mathcal{P}(U)$ be power set of U. A picture fuzzy proximity relation on $\mathcal{P}(U)$ is given a set below.

$$\mathcal{R}_{\mathcal{P}} = \{((M, N), f_{\mathcal{R}}(M, N), g_{\mathcal{R}}(M, N), h_{\mathcal{R}}(M, N)) \mid (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$$

If it satisfies the following axioms for all $M, N, Q \in \mathcal{P}(U)$:

$PF1$) If $M = \emptyset$ or $N = \emptyset$, then $f_{\mathcal{R}}(M, N) = g_{\mathcal{R}}(M, N) = h_{\mathcal{R}}(M, N) = 0$.

$PF2$) $f_{\mathcal{R}}(M, N) = f_{\mathcal{R}}(N, M)$ and $g_{\mathcal{R}}(M, N) = g_{\mathcal{R}}(N, M)$ and $h_{\mathcal{R}}(M, N) = h_{\mathcal{R}}(N, M)$.

$PF3$) $f_{\mathcal{R}}(M, N) \neq 0$ or $h_{\mathcal{R}}(M, N) \neq 0$ or $h_{\mathcal{R}}(M, N) \neq 0$ implies $M\mathcal{R}_{\mathcal{P}}N$.

$PF4$) $f_{\mathcal{R}}(M, N \cup Q) \neq 0$ and $g_{\mathcal{R}}(M, N \cup Q) \neq 0$ and $h_{\mathcal{R}}(M, N \cup Q) \neq 0$ imply $f_{\mathcal{R}}(M, N) \neq 0$, $g_{\mathcal{R}}(M, N) \neq 0$ and $h_{\mathcal{R}}(M, N) \neq 0$; $M\mathcal{R}_{\mathcal{P}}N$ or $f_{\mathcal{R}}(M, Q) \neq 0$, $g_{\mathcal{R}}(M, Q) \neq 0$ and $h_{\mathcal{R}}(M, Q) \neq 0$; $M\mathcal{R}_{\mathcal{P}}Q$,
where the values of $f_{\mathcal{R}}(M, N)$, $g_{\mathcal{R}}(M, N)$ and $h_{\mathcal{R}}(M, N)$ are form $[0, 1]$ and $0 \leq f_{\mathcal{R}}(M, N) + g_{\mathcal{R}}(M, N) + h_{\mathcal{R}}(M, N) \leq 1$ for all $(M, N) \in \mathcal{P}(U)$.

Furthermore, the set of all picture fuzzy proximity relations on $\mathcal{P}(U) \times \mathcal{P}(U)$ is represented by $PFR(\mathcal{P}(U))$. The triplet $(f_{\mathcal{R}}, g_{\mathcal{R}}, h_{\mathcal{R}})$ is known as picture fuzzy proximity number (PFPN). In this case, for all $(M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)$, $f_{\mathcal{R}}(M, N)$ symbolizes picture fuzzy proximity membership degree, $g_{\mathcal{R}}(M, N)$ symbolizes picture fuzzy proximity abstinence degree, and $h_{\mathcal{R}}(M, N)$ symbolizes picture fuzzy proximity non-membership degree. It is also presented with the $t \times t$ relational matrix:

$$\mathcal{R}_{\mathcal{P}} = \begin{array}{c} \\ M_1 \\ \vdots \\ M_t \end{array} \begin{array}{c} \begin{array}{ccc} N_1 & \cdots & N_t \end{array} \\ \begin{bmatrix} (f_{\mathcal{R}}(M_1, N_1), g_{\mathcal{R}}(M_1, N_1), h_{\mathcal{R}}(M_1, N_1)) & \cdots & (f_{\mathcal{R}}(M_1, N_t), g_{\mathcal{R}}(M_1, N_t), h_{\mathcal{R}}(M_1, N_t)) \\ \vdots & \ddots & \vdots \\ (f_{\mathcal{R}}(M_t, N_1), g_{\mathcal{R}}(M_t, N_1), h_{\mathcal{R}}(M_t, N_1)) & \cdots & (f_{\mathcal{R}}(M_t, N_t), g_{\mathcal{R}}(M_t, N_t), h_{\mathcal{R}}(M_t, N_t)) \end{bmatrix} \end{array}$$

Moreover, $f_{\mathcal{R}}(M, N) + g_{\mathcal{R}}(M, N) + h_{\mathcal{R}}(M, N)$ is called picture fuzzy proximity measure. PFP measure gives us the level of two sets how proximal(near) each other for different properties. Then, we give the definition of the complementary relation of $\mathcal{R}_{\mathcal{P}}$ as

$$\mathcal{R}_{\mathcal{P}}^{C} = \{((M, N), h_{\mathcal{R}}(M, N), g_{\mathcal{R}}(M, N), f_{\mathcal{R}}(M, N)) \mid (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}.$$

Besides, $\not{\mathcal{R}}_{\mathcal{P}}(M, N) = 1 - (f_{\mathcal{R}}(M, N) + g_{\mathcal{R}}(M, N) + h_{\mathcal{R}}(M, N))$ is called picture fuzzy proximity far measure.

Example 3.2. Let $U = \{m, n, o, p, r, s, t, u, w, y, z\}$. Also, $M = \{m, n, p, r, s, q\}$, $N = \{o, n, p, r, s, w\}$, $P = \{o, n, p, r, s, q\}$ and $Q = \{m, n, p, r, s, y\}$ are subsets of U.

$$\begin{aligned} f_{\mathcal{R}} : \quad & \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow [0, 1] \\ & (M, N) \longmapsto f_{\mathcal{R}}(M, N) = \tfrac{|M \cap N|}{|M| + |N|}, \end{aligned}$$

$$\begin{aligned} g_{\mathcal{R}} : \quad & \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow [0, 1] \\ & (M, N) \longmapsto g_{\mathcal{R}}(M, N) = \tfrac{|N \setminus M|}{|M|} \end{aligned}$$

and

$$\begin{aligned} h_{\mathcal{R}} : \quad & \mathcal{P}(U) \times \mathcal{P}(U) \longrightarrow [0, 1] \\ & (M, N) \longmapsto h_{\mathcal{R}}(M, N) = \tfrac{|M \setminus N|}{|M \cup N|}. \end{aligned}$$

(U, δ) forms a basic proximity space with δ. Since all sets satisfy the basic proximity conditions such that $M \ \delta \ N :\Leftrightarrow M \cap N \neq \emptyset$. Similarly, other sets can be seen. Then, we find $f_{\mathcal{R}}, g_{\mathcal{R}}, h_{\mathcal{R}}$ which are shown below.

$$
\begin{aligned}
f_{\mathcal{R}}(M,M) &= \tfrac{|M\cap M|}{|M|+|M|} = \tfrac{|M|}{2|M|} = \tfrac{1}{2} = 0.5\\
f_{\mathcal{R}}(M,N) &= \tfrac{|M\cap N|}{|M|+|N|} = \tfrac{4}{12} = 0.333\\
f_{\mathcal{R}}(M,P) &= \tfrac{|M\cap P|}{|M|+|P|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(M,Q) &= \tfrac{|M\cap Q|}{|M|+|Q|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(N,M) &= \tfrac{|N\cap M|}{|N|+|M|} = \tfrac{4}{12} = 0.333\\
f_{\mathcal{R}}(N,N) &= \tfrac{|N\cap N|}{|N|+|N|} = \tfrac{|N|}{2|N|} = \tfrac{1}{2} = 0.5\\
f_{\mathcal{R}}(N,P) &= \tfrac{|N\cap P|}{|N|+|P|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(N,Q) &= \tfrac{|N\cap Q|}{|N|+|Q|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(P,M) &= \tfrac{|P\cap M|}{|P|+|M|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(P,N) &= \tfrac{|P\cap N|}{|P|+|N|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(P,P) &= \tfrac{|P\cap P|}{|P|+|P|} = \tfrac{|P|}{2|P|} = \tfrac{1}{2} = 0.5\\
f_{\mathcal{R}}(P,Q) &= \tfrac{|P\cap Q|}{|P|+|Q|} = \tfrac{4}{12} = 0.333\\
f_{\mathcal{R}}(Q,M) &= \tfrac{|Q\cap M|}{|Q|+|M|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(Q,N) &= \tfrac{|Q\cap N|}{|Q|+|N|} = \tfrac{5}{12} = 0.416\\
f_{\mathcal{R}}(Q,P) &= \tfrac{|Q\cap P|}{|Q|+|P|} = \tfrac{4}{12} = 0.416\\
f_{\mathcal{R}}(Q,Q) &= \tfrac{|Q\cap Q|}{|Q|+|Q|} = \tfrac{|Q|}{2|Q|} = \tfrac{1}{2} = 0.5,
\end{aligned}
$$

$$
\begin{aligned}
g_{\mathcal{R}}(M,M) &= \tfrac{|M\setminus M|}{|M|} = \tfrac{|\emptyset|}{|M|} = \tfrac{0}{6} = 0\\
g_{\mathcal{R}}(M,N) &= \tfrac{|M\setminus N|}{|M|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(M,P) &= \tfrac{|P\setminus M|}{|M|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(M,Q) &= \tfrac{|Q\setminus M|}{|M|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(N,M) &= \tfrac{|M\setminus N|}{|N|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(N,N) &= \tfrac{|N\setminus N|}{|N|} = \tfrac{|\emptyset|}{|N|} = \tfrac{0}{6} = 0\\
g_{\mathcal{R}}(N,P) &= \tfrac{|P\setminus N|}{|N|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(N,Q) &= \tfrac{|Q\setminus N|}{|N|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(P,M) &= \tfrac{|M\setminus P|}{|P|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(P,B) &= \tfrac{|B\setminus C|}{|P|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(P,P) &= \tfrac{|P\setminus P|}{|P|} = \tfrac{|\emptyset|}{|P|} = \tfrac{0}{6} = 0\\
g_{\mathcal{R}}(P,Q) &= \tfrac{|Q\setminus P|}{|P|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(Q,M) &= \tfrac{|M\setminus Q|}{|Q|} = \tfrac{1}{6} = 0.166\\
g_{\mathcal{R}}(Q,N) &= \tfrac{|B\setminus D|}{|Q|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(Q,P) &= \tfrac{|P\setminus Q|}{|Q|} = \tfrac{2}{6} = 0.333\\
g_{\mathcal{R}}(Q,Q) &= \tfrac{|Q\setminus Q|}{|Q|} = \tfrac{|\emptyset|}{|Q|} = \tfrac{0}{6} = 0
\end{aligned}
$$

and

$$\begin{aligned}
h_{\mathcal{R}}(M,M) &= \tfrac{|M\setminus M|}{|M\cup M|} = \tfrac{|\emptyset|}{|M|} = \tfrac{0}{6} = 0\\
h_{\mathcal{R}}(M,N) &= \tfrac{|M\setminus N|}{|M\cup N|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(M,P) &= \tfrac{|M\setminus P|}{|M\cup Q|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(M,Q) &= \tfrac{|M\setminus Q|}{|M\cup Q|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(N,M) &= \tfrac{|N\setminus M|}{|N\cup M|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(N,N) &= \tfrac{|N\setminus N|}{|N\cup N|} = \tfrac{|\emptyset|}{|N|} = \tfrac{0}{6} = 0\\
h_{\mathcal{R}}(N,P) &= \tfrac{|N\setminus P|}{|N\cup P|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(N,Q) &= \tfrac{|N\setminus Q|}{|N\cup Q|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(P,M) &= \tfrac{|P\setminus M|}{|P\cup M|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(P,N) &= \tfrac{|P\setminus N|}{|P\cup N|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(P,P) &= \tfrac{|P\setminus P|}{|P\cup P|} = \tfrac{|\emptyset|}{|P|} = \tfrac{0}{6} = 0\\
h_{\mathcal{R}}(P,Q) &= \tfrac{|P\setminus Q|}{|P\cup Q|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(Q,M) &= \tfrac{|Q\setminus M|}{|Q\cup M|} = \tfrac{1}{7} = 0.142\\
h_{\mathcal{R}}(Q,N) &= \tfrac{|Q\setminus N|}{|Q\cup N|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(Q,P) &= \tfrac{|Q\setminus P|}{|Q\cup P|} = \tfrac{2}{8} = 0.25\\
h_{\mathcal{R}}(Q,Q) &= \tfrac{|Q\setminus Q|}{|Q\cup Q|} = \tfrac{|\emptyset|}{|Q|} = \tfrac{0}{6} = 0,
\end{aligned}$$

where they satisfy the condition $0 \le f_{\mathcal{R}}(M,N) + g_{\mathcal{R}}(M,N) + h_{\mathcal{R}}(M,N) \le 1$.

$$\begin{aligned}
&f_{\mathcal{R}}(M,M) + g_{\mathcal{R}}(M,M) + h_{\mathcal{R}}(M,M) = 0.5 + 0 + 0 = 0.5 \le 1,\\
&f_{\mathcal{R}}(M,N) + g_{\mathcal{R}}(M,N) + h_{\mathcal{R}}(M,N) = 0.333 + 0.333 + 0.25 = 0.916 \le 1,\\
&f_{\mathcal{R}}(M,P) + g_{\mathcal{R}}(M,P) + h_{\mathcal{R}}(M,P) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(M,Q) + g_{\mathcal{R}}(M,Q) + h_{\mathcal{R}}(M,Q) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(N,M) + g_{\mathcal{R}}(N,M) + h_{\mathcal{R}}(N,M) = 0.333 + 0.333 + 0.25 = 0.916 \le 1,\\
&f_{\mathcal{R}}(N,N) + g_{\mathcal{R}}(N,N) + h_{\mathcal{R}}(N,N) = 0.5 + 0 + 0 = 0.5 \le 1,\\
&f_{\mathcal{R}}(N,P) + g_{\mathcal{R}}(N,P) + h_{\mathcal{R}}(N,P) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(N,Q) + g_{\mathcal{R}}(N,Q) + h_{\mathcal{R}}(N,Q) = 0.416 + 0.333 + 0.25 = 1 \le 1,\\
&f_{\mathcal{R}}(P,M) + g_{\mathcal{R}}(P,M) + h_{\mathcal{R}}(P,M) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(P,N) + g_{\mathcal{R}}(P,N) + h_{\mathcal{R}}(P,N) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(P,P) + g_{\mathcal{R}}(P,P) + h_{\mathcal{R}}(P,P) = 0.5 + 0 + 0 = 0.5 \le 1,\\
&f_{\mathcal{R}}(P,Q) + g_{\mathcal{R}}(P,Q) + h_{\mathcal{R}}(P,Q) = 0.333 + 0.333 + 0.25 = 0.916 \le 1,\\
&f_{\mathcal{R}}(Q,M) + g_{\mathcal{R}}(Q,M) + h_{\mathcal{R}}(Q,M) = 0.416 + 0.166 + 0.142 = 0.726 \le 1,\\
&f_{\mathcal{R}}(Q,N) + g_{\mathcal{R}}(Q,N) + h_{\mathcal{R}}(Q,N) = 0.416 + 0.333 + 0.25 = 1 \le 1,\\
&f_{\mathcal{R}}(Q,P) + g_{\mathcal{R}}(Q,P) + h_{\mathcal{R}}(Q,P) = 0.333 + 0.333 + 0.25 = 0.916 \le 1,\\
&f_{\mathcal{R}}(Q,Q) + g_{\mathcal{R}}(Q,Q) + h_{\mathcal{R}}(Q,Q) = 0.5 + 0 + 0 = 0.5 \le 1.
\end{aligned}$$

From here, $\mathcal{R}_{\mathcal{P}}$ satisfies the axioms $(PF1) - (PF4)$. Thus, we say $\mathcal{R}_{\mathcal{P}}$ is a picture fuzzy proximity relation. It is shown that

M is $(0.333, 0.333, 0.25)$ picture fuzzy proximal to N ($M\, \mathcal{R}_{(0.333,0.333,0.25)} N$),
M is $(0.416, 0.166, 0.142)$ picture fuzzy proximal to P ($M\, \mathcal{R}_{(0.416,0.166,0.142)} P$),
M is $(0.416, 0.166, 0.142)$ picture fuzzy proximal to Q ($M\, \mathcal{R}_{(0.416,0.166,0.142)} Q$),
N is $(0.416, 0.166, 0.142)$ picture fuzzy proximal to P ($N\, \mathcal{R}_{(0.416,0.166,0.142)} P$),
N is $(0.416, 0.333, 0.25)$ picture fuzzy proximal to Q ($N\, \mathcal{R}_{(0.416,0.333,0.25)} Q$),
P is $(0.333, 0.333, 0.25)$ picture fuzzy proximal to Q ($P\, \mathcal{R}_{(0.333,0.333,0.25)} Q$).

Picture fuzzy far measures are

M is 0.084 picture fuzzy far to N ($M \not\mathcal{R}_{0.084} N$),
M is 0.274 picture fuzzy far to P ($M \not\mathcal{R}_{0.274} P$),
M is 0.274 picture fuzzy far to Q ($M \not\mathcal{R}_{0.274} Q$),
N is 0.274 picture fuzzy far to P ($N \not\mathcal{R}_{0.274} P$),
N is 0 picture fuzzy far to Q ($N \not\mathcal{R}_{0} Q$),
P is 0.084 picture fuzzy far to Q ($P \not\mathcal{R}_{0.084} Q$).

The relational matrix is given below:

$$\mathcal{R}_{\mathcal{P}} = \begin{array}{c} \\ M \\ N \\ P \\ Q \end{array} \begin{array}{c} \begin{array}{cccc} M & N & P & Q \end{array} \\ \left[\begin{array}{cccc} (0.5,0,0) & (0.333,0.333,0.25) & (0.416,0.166,0.142) & (0.416,0.166,0.142) \\ (0.333,0.333,0.25) & (0.5,0,0) & (0.416,0.166,0.142) & (0.416,0.333,0.25) \\ (0.416,0.166,0.142) & (0.416,0.166,0.142) & (0.5,0,0) & (0.333,0.333,0.25) \\ (0.416,0.166,0.142) & (0.416,0.333,0.25) & (0.333,0.333,0.25) & (0.5,0,0) \end{array}\right] \end{array}.$$

Definition 3.3. Let $U \neq \emptyset$ and $\mathcal{R}_{\mathcal{P}}$ be a picture fuzzy proximity relation on $\mathcal{P}(U)$. In this case, $(U, \mathcal{R}_{\mathcal{P}})$ is called a picture fuzzy proximal space.

Definition 3.4. Suppose that $(U, \mathcal{R})$ be a proximal relator space. If $\mathcal{R}_{\mathcal{P}}$ is a picture fuzzy proximity relation on $\mathcal{P}(U)$, $(U, \mathcal{R}, \mathcal{R}_{\mathcal{P}})$ is called a picture fuzzy proximal relator space.

When we take in consideration Example 3.2, $(U, \mathcal{R}_{\mathcal{P}})$ is a picture fuzzy proximal space. Because of the fact that (U, δ) is a proximal relator space with basic proximity, and so $(U, \delta, \delta_{\mathcal{P}})$ is a picture fuzzy proximal relator space.

Definition 3.5. Suppose that (U, δ) be a proximity space and $\delta_{\mathcal{P}}$ be a picture fuzzy proximity relation. If $\delta_{\mathcal{P}}$ satisfies the following conditions $(PF\delta 1)$-$(PF\delta 4)$, it is known to be a picture fuzzy spatial proximity relation.

$PFS1$) $f_\delta(M, \emptyset) = 0$ or $h_\delta(M, \emptyset) = 0$; $g_\delta(M, \emptyset) = 0$.

$PFS2$) $f_\delta(M, N) = f_\delta(N, M)$ and $g_\delta(M, N) = g_\delta(N, M)$ and $h_\delta(M, N) = h_\delta(N, M)$.

$PFS3$) $f_\delta(M, N) \neq 0$ or $g_\delta(M, N) \neq 0$ or $h_\delta(M, N) \neq 0$ imply $M \delta_{\mathcal{P}} N$.

$PFS4$) $f_\delta(M, N \cup P) \neq 0$ and $g_\delta(M, N \cup P) \neq 0$ and $h_\delta(M, N \cup P) \neq 0$ imply $f_\delta(M, N) \neq 0$, $g_\delta(M, N) \neq 0$ and $h_\delta(M, N) \neq 0$; $M \delta_{\mathcal{P}} N$ or $f_\delta(M, P) \neq 0$, $g_\delta(M, P) \neq 0$ and $h_\delta(M, P) \neq 0$; $M \delta_{\mathcal{P}} P$.

Furthermore, $(U, \delta, \delta_{\mathcal{P}})$ is called picture fuzzy spatial proximity space.

Definition 3.6. Suppose that (U, δ) be a proximity space and $\delta_{\mathcal{P}}$ be a picture fuzzy proximity relation. If $\delta_{\mathcal{P}}$ satisfies the conditions $(PFS1)$-$(PFS4)$ and the following axiom $(PFS5)$, it is known to be as a picture fuzzy spatial Lodato proximity relation. Furthermore, $(U, \delta, \delta_{\mathcal{P}})$ is known to be as a picture fuzzy spatial Lodato proximity space.

$PFS5$) $f_\delta(M, N) \neq 0$ and $f_\delta(n, P) \neq 0$; $g_\delta(M, N) \neq 0$ and $g_\delta(n, P) \neq 0$; $h_\delta(M, N) \neq 0$ and $h_\delta(n, P) \neq 0$ for all for all $n \in N$ implies $f_\delta(M, P) \neq 0$, $g_\delta(M, P) \neq 0$, $h_\delta(M, P) \neq 0$ and $M \delta_{\mathcal{P}} P$.

Definition 3.7. Assume that (U, δ_Φ) be a descriptive proximity space and $\delta_{\Phi\mathcal{P}}$ be a picture fuzzy proximity relation. $(U, \delta_\Phi, \delta_{\Phi\mathcal{P}})$ is known to be picture fuzzy descriptive Lodato proximity space, if $\delta_{\Phi\mathcal{P}}$ holds the following conditions:

$PFD1$) $f_{\delta_\Phi}(M, \emptyset) = 0$ or $h_{\delta_\Phi}(M, \emptyset) = 0$; $g_{\delta_\Phi}(M, \emptyset) = 0$.

$PFD2$) $f_{\delta_\Phi}(M, N) = f_{\delta_\Phi}(N, M)$ and $g_{\delta_\Phi}(M, N) = g_{\delta_\Phi}(N, M)$ and $h_{\delta_\Phi}(M, N) = h_{\delta_\Phi}(N, M)$.

$PFD3$) $f_{\delta_\Phi}(M, N) \neq 0$ or $g_{\delta_\Phi}(M, N) \neq 0$ or $h_{\delta_\Phi}(M, N) \neq 0$ imply $M \delta_{\Phi\mathcal{P}} N$.

$PFD4$) $f_{\delta_\Phi}(M, N \cup P) \neq 0$ and $g_{\delta_\Phi}(M, N \cup P) \neq 0$ and $h_{\delta_\Phi}(M, N \cup P) \neq 0$ imply $f_{\delta_\Phi}(M, N) \neq 0$, $g_{\delta_\Phi}(M, N) \neq 0$ and $h_{\delta_\Phi}(M, N) \neq 0$; $M \delta_{\Phi\mathcal{P}} N$ or $f_{\delta_\Phi}(M, P) \neq 0$, $g_{\delta_\Phi}(M, Q) \neq 0$ and $h_{\delta_\Phi}(M, P) \neq 0$; $M \delta_{\Phi\mathcal{P}} P$.

$PFD5$) $f_{\delta_\Phi}(M, N) \neq 0$ and $f_{\delta_\Phi}(n, P) \neq 0$; $g_{\delta_\Phi}(M, N) \neq 0$ and $g_{\delta_\Phi}(n, P) \neq 0$; $h_{\delta_\Phi}(M, N) \neq 0$ and $h_{\delta_\Phi}(n, P) \neq 0$ for all $n \in N$ implies $f_{\delta_\Phi}(M, P) \neq 0$, $g_{\delta_\Phi}(M, P) \neq 0$, $h_{\delta_\Phi}(M, P) \neq 0$ and $M \delta_{\Phi\mathcal{P}} P$.

Definition 3.8. For a picture fuzzy proximity relation $\mathcal{R}_{\mathcal{P}}$ on $\mathcal{P}(U) \times \mathcal{P}(U)$, it is given that $\mathcal{R}_{\mathcal{P}}^{-1}$ on $\mathcal{P}(U)$ is inverse relation of $\mathcal{R}_{\mathcal{P}}$ and defined as

$$\mathcal{R}_{\mathcal{P}}^{-1} = \{((N, M), f_{\mathcal{R}}(N, M), g_{\mathcal{R}}(N, M), h_{\mathcal{R}}(N, M)) \mid (M, N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}.$$

$\mathcal{R}_{\mathcal{P}}^{-1}$ is the transpose of picture fuzzy relational matrix, that is, $f_{\mathcal{R}^{-1}}(N, M) = f_{\mathcal{R}}(M, N)$, $g_{\mathcal{R}^{-1}}(N, M) = g_{\mathcal{R}}(M, N)$ and $h_{\mathcal{R}^{-1}}(N, M) = h_{\mathcal{R}}(M, N)$.

Theorem 3.9. *Let $\mathcal{R}_{\mathcal{P}}$ be a picture fuzzy proximity relation. Then, $\mathcal{R}_{\mathcal{P}}^{-1}$ is also a picture fuzzy proximity relation.*

Proof. The proof is clear from the definition of $\mathcal{R}_{\mathcal{P}}^{-1}$. □

Example 3.10. By using Example 3.2, we can easily find $\mathcal{R}_{\mathcal{P}}^{-1}$

$$\mathcal{R}_{\mathcal{P}}^{-1}=\begin{bmatrix} (0.5,0,0) & (0.333,0.333,0.25) & (0.416,0.166,0.142) & (0.416,0.166,0.142) \\ (0.333,0.333,0.25) & (0.5,0,0) & (0.416,0.166,0.142) & (0.416,0.333,0.25) \\ (0.416,0.166,0.142) & (0.416,0.166,0.142) & (0.5,0,0) & (0.333,0.333,0.25) \\ (0.416,0.166,0.142) & (0.416,0.333,0.25) & (0.333,0.333,0.25) & (0.5,0,0) \end{bmatrix}.$$

Hence, $\mathcal{R}_{\mathcal{P}}^{-1}$ satisfies the axioms $(\mathcal{PFR}1)-(\mathcal{PFR}4)$ and so $\mathcal{R}_{\mathcal{P}}^{-1}$ is a picture fuzzy proximity relation. Also, for $\mathcal{R}=\{\delta\}$, $\left(X,\delta,\delta_{\mathcal{P}}^{-1}\right)$ is a picture fuzzy proximal relator space.

Theorem 3.11. *Let $\mathcal{R}_{\mathcal{P}_1}$, $\mathcal{R}_{\mathcal{P}_2}$ and $\mathcal{R}_{\mathcal{P}}$ be picture fuzzy proximity relations. The following properties hold:*
1) $\mathcal{R}_{\mathcal{P}_1}\leq\mathcal{R}_{\mathcal{P}_2}$ *implies* $\mathcal{R}_{\mathcal{P}_1}^{-1}\leq\mathcal{R}_{\mathcal{P}_2}^{-1}$.
2) $\left(\mathcal{R}_{\mathcal{P}}^{-1}\right)^{-1}=\mathcal{R}_{\mathcal{P}}$.
3) $\left(\mathcal{R}_{\mathcal{P}_1}\wedge\mathcal{R}_{\mathcal{P}_2}\right)^{-1}=\mathcal{R}_{\mathcal{P}_1}^{-1}\wedge\mathcal{R}_{\mathcal{P}_2}^{-1}$.
4) $\left(\mathcal{R}_{\mathcal{P}_1}\vee\mathcal{R}_{\mathcal{P}_2}\right)^{-1}=\mathcal{R}_{\mathcal{P}_1}^{-1}\vee\mathcal{R}_{\mathcal{P}_2}^{-1}$.

Proof. Let $\mathcal{R}_{\mathcal{P}_1}$, $\mathcal{R}_{\mathcal{P}_2}$ and $\mathcal{R}_{\mathcal{P}}$ be picture fuzzy proximity relations.
1) If $\mathcal{R}_{\mathcal{P}_1}\leq\mathcal{R}_{\mathcal{P}_2}$, then

$$f_{\mathcal{R}_{\mathcal{P}_1}^{-1}}(N,M)=f_{\mathcal{R}_{\mathcal{P}_1}}(M,N)\leq f_{\mathcal{R}_{\mathcal{P}_2}}(M,N)=f_{\mathcal{R}_{\mathcal{P}_2}^{-1}}(N,M)$$

for all $(M,N)\in\mathcal{P}(U)\times\mathcal{P}(V)$. Similarly,

$$g_{\mathcal{R}_{\mathcal{P}_1}^{-1}}(N,M)=g_{\mathcal{R}_{\mathcal{P}_1}}(M,N)\leq g_{\mathcal{R}_{\mathcal{P}_2}}(M,N)=g_{\mathcal{R}_{\mathcal{P}_2}^{-1}}(N,M)$$

and

$$h_{\mathcal{R}_{\mathcal{P}_1}^{-1}}(N,M)=h_{\mathcal{R}_{\mathcal{P}_1}}(M,N)\geq h_{\mathcal{R}_{\mathcal{P}_2}}(N,M)=h_{\mathcal{R}_{\mathcal{P}_2}^{-1}}(N,M).$$

Hence, we have that $\mathcal{R}_{\mathcal{P}_1}^{-1}\leq\mathcal{R}_{\mathcal{P}_2}^{-1}$.
2)

$$f_{\left(\mathcal{R}_{\mathcal{P}}^{-1}\right)^{-1}}(M,N)=f_{\mathcal{R}_{\mathcal{P}}^{-1}}(N,M)=f_{\mathcal{R}_{\mathcal{P}}}(M,N)$$

for all $(M,N)\in\mathcal{P}(U)\times\mathcal{P}(U)$. In a similar way,

$$g_{\left(\mathcal{R}_{\mathcal{P}}^{-1}\right)^{-1}}(M,N)=g_{\mathcal{R}_{\mathcal{P}}^{-1}}(N,M)=g_{\mathcal{R}_{\mathcal{P}}}(M,N)$$

and

$$h_{\left(\mathcal{R}_{\mathcal{P}}^{-1}\right)^{-1}}(M,N)=h_{\mathcal{R}_{\mathcal{P}}^{-1}}(N,M)=h_{\mathcal{R}_{\mathcal{P}}}(M,N).$$

Thus, $\left(\mathcal{R}_{\mathcal{P}}^{-1}\right)^{-1}=\mathcal{R}_{\mathcal{P}}$.
3)

$$\begin{aligned} f_{\left(\mathcal{R}_{1_{\mathcal{P}}}\wedge\mathcal{R}_{\mathcal{P}_2}\right)^{-1}}(N,M) &= f_{\left(\mathcal{R}_{\mathcal{P}_1}\wedge\mathcal{R}_{\mathcal{P}_2}\right)}(M,N) &&= f_{\mathcal{R}_{\mathcal{P}_1}}(M,N)\wedge f_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \\ &= f_{\mathcal{R}_{\mathcal{P}_1}{}^{-1}}(N,M)\wedge f_{\mathcal{R}_{\mathcal{P}_2}{}^{-1}}(N,M) &&= f_{\mathcal{R}_{\mathcal{P}_1}^{-1}\wedge\mathcal{R}_{\mathcal{P}_2}^{-1}}(N,M). \end{aligned}$$

Likewise,

$$\begin{aligned} g_{\left(\mathcal{R}_{1_{\mathcal{P}}}\wedge\mathcal{R}_{\mathcal{P}_2}\right)^{-1}}(N,M) &= g_{\left(\mathcal{R}_{\mathcal{P}_1}\wedge\mathcal{R}_{\mathcal{P}_2}\right)}(M,N) &&= g_{\mathcal{R}_{\mathcal{P}_1}}(M,N)\wedge g_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \\ &= g_{\mathcal{R}_{\mathcal{P}_1}{}^{-1}}(N,M)\wedge g_{\mathcal{R}_{\mathcal{P}_2}{}^{-1}}(N,M) &&= g_{\mathcal{R}_{\mathcal{P}_1}^{-1}\wedge\mathcal{R}_{\mathcal{P}_2}^{-1}}(N,M). \end{aligned}$$

The proof for

$$h_{\left(\mathcal{R}_{\mathcal{P}_1}\wedge\mathcal{R}_{\mathcal{P}_2}\right)^{-1}}(N,M)=h_{\mathcal{R}_{\mathcal{P}_1}{}^{-1}\wedge\mathcal{R}_{\mathcal{P}_2}{}^{-1}}(N,M)$$

is done in a similar way.
4) It can be made similar to the proof of (3).

□

Definition 3.12. Let $\mathcal{R}_{\mathcal{P}_1}, \mathcal{R}_{\mathcal{P}_2} \in PFR(\mathcal{P}(U))$. Then,

1) $\mathcal{R}_{\mathcal{P}_1} \leq \mathcal{R}_{\mathcal{P}_2}$ provided that $f_{\mathcal{R}_1}(M,N) \leq f_{\mathcal{R}_2}(M,N)$, $g_{\mathcal{R}_1}(M,N) \leq g_{\mathcal{R}_2}(M,N)$ and $h_{\mathcal{R}_1}(M,N) \geq h_{\mathcal{R}_2}(M,N)$.

2) $\mathcal{R}_{\mathcal{P}_1} \wedge \mathcal{R}_{\mathcal{P}_2} = \{((M,N), f_{\mathcal{R}_1}(M,N) \wedge f_{\mathcal{R}_2}(M,N), g_{\mathcal{R}_1}(M,N) \wedge g_{\mathcal{R}_2}(M,N), h_{\mathcal{R}_1}(M,N) \vee h_{\mathcal{R}_2}(M,N)) \mid (M,N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$.

3) $\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2} = \{((M,N), f_{\mathcal{R}_1}(M,N) \vee f_{\mathcal{R}_2}(M,N), g_{\mathcal{R}_1}(M,N) \wedge g_{\mathcal{R}_2}(M,N), h_{\mathcal{R}_1}(M,N) \wedge h_{\mathcal{R}_2}(M,N)) \mid (M,N) \in \mathcal{P}(U) \times \mathcal{P}(U)\}$.

Theorem 3.13. *Let $\mathcal{R}_{\mathcal{P}}$, $\mathcal{R}_{\mathcal{P}_1}$ and $\mathcal{R}_{\mathcal{P}_2}$ be picture fuzzy proximity relations. The following properties hold:*

1) $\mathcal{R}_{\mathcal{P}} \wedge (\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2}) = (\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}) \vee (\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2})$.

2) $\mathcal{R}_{\mathcal{P}} \vee (\mathcal{R}_{\mathcal{P}_1} \wedge \mathcal{R}_{\mathcal{P}_2}) = (\mathcal{R}_{\mathcal{P}} \vee \mathcal{R}_{\mathcal{P}_1}) \wedge (\mathcal{R}_{\mathcal{P}} \vee \mathcal{R}_{\mathcal{P}_2})$.

3) $\mathcal{R}_{\mathcal{P}_1} \wedge \mathcal{R}_{\mathcal{P}_2} \leq \mathcal{R}_{\mathcal{P}_1}$.

4) $\mathcal{R}_{\mathcal{P}_1} \wedge \mathcal{R}_{\mathcal{P}_2} \leq \mathcal{R}_{\mathcal{P}_2}$.

5) $\mathcal{R}_{\mathcal{P}_1} \leq \mathcal{R}_{\mathcal{P}}$ *and* $\mathcal{R}_{\mathcal{P}_2} \leq \mathcal{R}_{\mathcal{P}}$ *implies that* $\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2} \leq \mathcal{R}_{\mathcal{P}}$.

6) $\mathcal{R}_{\mathcal{P}} \leq \mathcal{R}_{\mathcal{P}_1}$ *and* $\mathcal{R}_{\mathcal{P}} \leq \mathcal{R}_{\mathcal{P}_2}$ *implies that* $\mathcal{R}_{\mathcal{P}} \leq \mathcal{R}_{\mathcal{P}_1} \wedge \mathcal{R}_{\mathcal{P}_2}$.

Proof. Let $\mathcal{R}_{\mathcal{P}}$, $\mathcal{R}_{\mathcal{P}_1}$ and $\mathcal{R}_{\mathcal{P}_2}$ be picture fuzzy proximity relations.

1)We use that the operators $\wedge$ and $\vee$ satisfy the distributive law when they are applied to elements of $[0,1]$.

$$\begin{aligned} f_{\mathcal{R}_{\mathcal{P}} \wedge (\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2})}(M,N) &= f_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge \left\{ f_{\mathcal{R}_{\mathcal{P}_1}}(M,N) \vee f_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \right\} \\ &= \left\{ f_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge f_{\mathcal{R}_{\mathcal{P}_1}}(M,N) \right\} \vee \left\{ f_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge f_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \right\} \\ &= f_{\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}}(M,N) \vee f_{\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2}}(M,N) \\ &= f_{(\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}) \vee (\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2})}(M,N). \end{aligned}$$

Similarly,

$$\begin{aligned} g_{\mathcal{R}_{\mathcal{P}} \wedge (\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2})}(M,N) &= g_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge \left\{ g_{\mathcal{R}_{\mathcal{P}_1}}(M,N) \vee g_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \right\} \\ &= \left\{ g_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge g_{\mathcal{R}_{\mathcal{P}_1}}(M,N) \right\} \vee \left\{ g_{\mathcal{R}_{\mathcal{P}}}(M,N) \wedge g_{\mathcal{R}_{\mathcal{P}_2}}(M,N) \right\} \\ &= g_{\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}}(M,N) \vee g_{\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2}}(M,N) \\ &= g_{(\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}) \vee (\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2})}(M,N). \end{aligned}$$

The proof is similar to the previous one, in case of

$$h_{\mathcal{R}_{\mathcal{P}} \wedge (\mathcal{R}_{\mathcal{P}_1} \vee \mathcal{R}_{\mathcal{P}_2})}(M,N) = h_{(\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_1}) \vee (\mathcal{R}_{\mathcal{P}} \wedge \mathcal{R}_{\mathcal{P}_2})}(M,N).$$

The rest of results can be proved in a similar way to the previous one. □

Definition 3.14. Let $\mathcal{R}_{\mathcal{P}_1} \in PFR(\mathcal{P}(U) \times \mathcal{P}(V))$ and $\mathcal{R}_{\mathcal{P}_2} \in PFR(\mathcal{P}(V) \times \mathcal{P}(W))$ be two picture fuzzy proximity relations such that

$$\mathcal{R}_{\mathcal{P}_1} = \{((M,N), f_{\mathcal{R}_1}(M,N), g_{\mathcal{R}_1}(M,N), h_{\mathcal{R}_1}(M,N)) \mid (M,N) \in \mathcal{P}(U) \times \mathcal{P}(V)\}$$

and

$$\mathcal{R}_{\mathcal{P}_2} = \{((N,P), f_{\mathcal{R}_2}(N,P), g_{\mathcal{R}_2}(N,P), h_{\mathcal{R}_2}(N,P)) \mid (N,P) \in \mathcal{P}(V) \times \mathcal{P}(W)\}.$$

Max–min composed relation is defined as $\mathcal{R}_{\mathcal{P}_1} \circ \mathcal{R}_{\mathcal{P}_2}$
$=\{((M,P), f_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P), g_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P), h_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P)) \mid (M,P) \in \mathcal{P}(U) \times \mathcal{P}(W)\}$ where

$$\begin{aligned} f_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P) &= \bigvee_N \{(f_{\mathcal{R}_1}(M,N) \wedge f_{\mathcal{R}_2}(N,P))\} \\ g_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P) &= \bigwedge_N \{(g_{\mathcal{R}_1}(M,N) \wedge g_{\mathcal{R}_2}(N,P))\} \\ h_{\mathcal{R}_1 \circ \mathcal{R}_2}(M,P) &= \bigwedge_N \{(h_{\mathcal{R}_1}(M,N) \vee h_{\mathcal{R}_2}(M,P))\}. \end{aligned}$$

Example 3.15. Let $\mathcal{R}_{\mathcal{P}}$ be an picture fuzzy proximity relation given as Example 3.2. The max–min composition of $\mathcal{R}_{\mathcal{P}}$.

$$\mathcal{R}_{\mathcal{P}}^2 = \begin{bmatrix} (0.5,0,0) & (0.416,0,0.142) & (0.416,0,0.142) & (0.416,0,0.142) \\ (0.416,0,0.142) & (0.5,0,0) & (0.416,0,0.142) & (0.416,0,0.25) \\ (0.416,0,0.142) & (0.416,0,0.142) & (0.5,0,0) & (0.416,0,0.142) \\ (0.416,0,0.142) & (0.416,0,0.25) & (0.416,0,0.142) & (0.5,0,0) \end{bmatrix}.$$

$\mathcal{R}^2_{\mathcal{P}}$ satisfies the axioms $(PF1)-(PF4)$, and so $\mathcal{R}^2_{\mathcal{P}}$ is a picture fuzzy proximity relation by Definition 3.1. Therefore, $(U^2, \mathcal{R}, \mathcal{R}^2_{\mathcal{P}})$ is a picture fuzzy proximity relator space.

Theorem 3.16. *Let $PFR(\mathcal{P}(U))$ be the family of all picture fuzzy proximity relations. The (max–min, min–max) composition of picture fuzzy proximity relation on $\mathcal{P}(U)$ is associative.*

$$\mathcal{R}_{P_1} \circ (\mathcal{R}_{P_2} \circ \mathcal{R}_{P_3}) = (\mathcal{R}_{P_1} \circ \mathcal{R}_{P_2}) \circ \mathcal{R}_{P_3}.$$

Proof. Let

$$\mathcal{R}_{P_1} = \{((M,N), f_{\mathcal{R}_1}(M,N), g_{\mathcal{R}_1}(M,N), h_{\mathcal{R}_1}(M,N)) \mid M,N \in \mathcal{P}(U)\}$$

and

$$\mathcal{R}_{P_2} \circ \mathcal{R}_{P_3} = \{((N,Q), \max\{\min\{f_{\mathcal{R}_2}(N,P), f_{\mathcal{R}_3}(P,Q)\}\}, \min\{\min\{g_{\mathcal{R}_2}(N,P), g_{\mathcal{R}_3}(N,P)\}\}, \min\{\max\{h_{\mathcal{R}_2}(N,P), h_{\mathcal{R}_3}(P,Q)\}\}) \mid N,P,Q \in \mathcal{P}(U)\}.$$

From here, we find the composition

$\mathcal{R}_{P_1} \circ (\mathcal{R}_{P_2} \circ \mathcal{R}_{P_3}) = \{((M,N), f_{\mathcal{R}_1}(M,N), g_{\mathcal{R}_1}(M,N), h_{\mathcal{R}_1}(M,N)) \mid M,N \in \mathcal{P}(U)\}$
$\circ (\{((N,Q), \max\{\min\{f_{\mathcal{R}_2}(N,P), f_{\mathcal{R}_3}(P,Q)\}\}, \min\{\min\{g_{\mathcal{R}_2}(N,P), g_{\mathcal{R}_3}(N,P)\}\},$
$\min\{\max\{h_{\mathcal{R}_2}(N,P), h_{\mathcal{R}_3}(P,Q)\}\}) \mid N,P,Q \in \mathcal{P}(U)\})$
$= \{((M,Q), \max\{\min\{f_{\mathcal{R}_1}(M,N), \max\{\min\{f_{\mathcal{R}_2}(N,P), f_{\mathcal{R}_3}(P,Q)\}\}\}\}, \min\{\min\{g_{\mathcal{R}_1}(M,N), \min\{\max\{g_{\mathcal{R}_2}(N,P), g_{\mathcal{R}_3}(P,Q)\}\}\}\},$
$\min\{\max\{h_{\mathcal{R}_1}(M,N), \min\{\max\{h_{\mathcal{R}_2}(N,P), h_{\mathcal{R}_3}(P,Q)\}\}\}\}) \mid M,N,P,Q \in \mathcal{P}(U)\}$

and likewise, we obtain

$(\mathcal{R}_{P_1} \circ \mathcal{R}_{P_2}) \circ \mathcal{R}_{P_3} = (\{((M,P), \max\{\min\{f_{\mathcal{R}_1}(M,N), f_{\mathcal{R}_2}(N,P)\}\}, \min\{\min\{g_{\mathcal{R}_1}(M,N), g_{\mathcal{R}_2}(N,P)\}\},$
$\min\{\max\{h_{\mathcal{R}_1}(M,N), h_{\mathcal{R}_2}(N,P)\}\}\}\}) \mid M,N,P \in \mathcal{P}(U)\}) \circ \{((P,Q), f_{\mathcal{R}_3}(P,Q), g_{\mathcal{R}_3}(P,Q), h_{\mathcal{R}_3}(P,Q)) \mid P,Q \in \mathcal{P}(U)\}$
$= \{((M,Q), \max\{\min\{\max\{\min\{f_{\mathcal{R}_1}(M,N), f_{\mathcal{R}_2}(N,P)\}\}, f_{\mathcal{R}_3}(P,Q)\}\}, \min\{\min\{\min\{\min\{g_{\mathcal{R}_1}(M,N), g_{\mathcal{R}_2}(N,P)\}\}, g_{\mathcal{R}_3}(P,Q)\}\},$
$\min\{\max\{\min\{\max\{h_{\mathcal{R}_1}(M,N), h_{\mathcal{R}_2}(N,P)\}\}, h_{\mathcal{R}_3}(P,Q)\}\}) \mid M,N,P,Q \in \mathcal{P}(U)\}.$

By using the feature of the operator max and min, we reach $\mathcal{R}_{P_1} \circ (\mathcal{R}_{P_2} \circ \mathcal{R}_{P_3}) = (\mathcal{R}_{P_1} \circ \mathcal{R}_{P_2}) \circ \mathcal{R}_{P_3}$. □

4. Conclusion

The picture fuzzy set is a recently developed tool to deal with uncertainty which is a direct extension of intuitionistic fuzzy set. To study a new concept of picture fuzzy sets by using proximal relator spaces, it is defined some axioms that have to be fulfilled by fuzzy proximity relations. After investigating some results concerning picture fuzzy proximity relations, our study has focused on the relationship between Lodato proximity and picture fuzzy proximity. Therefore, our main aim is to create a theoretical infrastructure for subsequent studies and to pioneer subsequent applied studies. We are expecting that these structures will guide developing other extensions of fuzzy set. By utilizing these fuzzy sets effectively for some applications, it will be give great advantages.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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