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An efficient method for solving Schlömilch-type integral equations

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Abstract — Schlömilch integral equations have many applications in terrestrial physics and serve as useful tools for various ionospheric problems. Recently, researchers have investigated Schlömilch-type integral equations. Unlike Schlömilch integral equations, there are only a few works in the literature that discuss the classification and solution methods for Schlömilch -type equations. In this study, we mainly focus on introducing an efficient method based on a modified homotopy approach for solving certain Schlömilch-type equations. To demonstrate the efficiency and simplicity of the proposed algorithm, we also present some extensions that enable the solution of important application-related problems.

Keywords: *Fredholm integral equations, homotopy, perturbation theory, Rayleigh equation*

Subject Classification (2020): 45B05, 65H20

1. Introduction

The Schlömilch's integral equation plays an important role in the quasi-transverse approximations. It is considered to be one of the important equations of mathematical physics. Many researchers from various fields have been studying the equation and its solution [1–3]. There is a sufficient number of works that have been published recently on the theoretical and computational aspect of the equation [4–7]. The standard Schlömilch's integral equation admits the following form:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\sin\theta)\, d\theta, \quad -\pi \le x \le \pi$$

where ψ is known and ϕ is the desired function. A solution for this equation has been shown to be of the following form:

$$\phi(x) = \psi(0) + x\int_0^{\pi/2} \psi'(x\sin\theta)\, d\theta$$

where the derivative is taken with respect to the argument $\eta = x\sin\theta$.

There are some other types of Schlömilch's integral equations, such as generalized Schlömilch's integral equations and nonlinear Schlömilch's integral equations. However, we will not consider them here. Instead, we refer the interested readers to [8–11].

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In this study, we aim to study Schlömilch-type integral equations. The standard linear Schlömilch-type integral equation is defined as

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi \tag{1.1}$$

where ψ is known and ϕ is the desired function [9]. Similar to what has been done in standard Schlömilch's integral equation theory, we define and examine two other forms of this equation. These are the *generalized Schlömilch-type integral equation* and the *nonlinear Schlömilch-type integral equation*. They are listed in the following, respectively:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos^r\theta)\,d\theta, \quad -\pi \le x \le \pi \quad \text{and} \quad r \ge 1 \tag{1.2}$$

and

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} N(\phi(x\cos\theta))\,d\theta, \quad -\pi \le x \le \pi \tag{1.3}$$

where $N(\phi(x\cos\theta))$ is a nonlinear function of $\phi(x\cos\theta)$.

In addition, we also consider the equation of the form:

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi \tag{1.4}$$

This is an important equation in fluid dynamics. We see that it is related to the Rayleigh equation, which is considered to be one of the most important equations in fluid dynamics [12, 13].

The rest of the paper is organized as follows: Section 2 reviews the homotopy perturbation method and provides its basic properties. Section 3 considers a modification to homotopy that works for Schlömilch's integral equations. This section shows that it also works for the Schlömilch-type integral equations. In addition, the section introduces and solves a type of equation that produces important results that lead to an easy solution of the Rayleigh equation. Moreover, it applies a method based on a modification of homotopy and obtains an algorithm for the solution. Section 4 considers a special case of (1.4) to get an equation that leads to the Rayleigh equation. The final section discusses the conclusion of this study.

2. Homotopy Perturbation Method

He [14] proposed the homotopy perturbation method in 1999. Since then, it has been used for many different kinds of linear and nonlinear problems. The method is a combination of homotopy in topology and perturbation method. This method is useful in the sense that there is no need to use small parameters or any kind of linearization. The solution algorithm is very simple, and a few iterations lead to highly accurate solutions [15–17]. The homotopy perturbation method has been successfully applied to integral equations. We present recent studies showcasing various applications of homotopy methods, along with other numerical approaches to integral equations, for comparison [18–20].

We explain the method by showing its application to (1.1). Let

$$L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,dt - \psi(x) = 0$$

with solution $\phi(x)$. The most important part is to construct a suitable homotopy with an embedding parameter $p \in [0, 1]$, which will also be used as an expanding parameter.

$$\mathcal{H}(\phi, p) = (1-p)F(\phi) + pL(\phi)$$

where $F(\phi)$ is a functional operator with known solution, say ϕ_0. Setting the homotopy equation to zero and observing the equation as the parameter goes from zero to 1, one can easily interpret this as the trivial problem deforms to the original problem. Another important aspect of the equation is when considering the parameter as an expanding parameter as

$$\phi = \phi_0 + p\phi_1 + p^2\phi_2 + p^3\phi_3 + \cdots$$

As $p \to 1$, the series becomes an approximate solution of (1.1).

3. Modified Homotopy Perturbation Method

The main purpose and main idea for a modification to a homotopy are to accelerate the rate of convergence of the obtained series as a result of the application of the homotopy perturbation method. Thus, if a modification involves introducing a new term to a homotopy, it is expected that this new term will contribute to the rate of convergence of the obtained series [21–23]. Before delving into a modification to the homotopy, we state the following theorem.

Theorem 3.1. ψ is a polynomial function if and only if the solution of (1.1) is a polynomial function of the same degree.

The proof of this theorem is similar to the proof of a similar theorem provided in [5] for standard Schlömilch's integral equation.

Assuming ψ is a polynomial function, we define a homotopy

$$\mathcal{H}(\phi, p) = (1-p)F(\phi) + pL(\phi) + p(1-p)\sum_{k=0}^{n} q_k x^k \tag{3.1}$$

where F is a functional operator with a known solution, L is a linear operator, and q_k's are constants to be determined. We want to make a note that this homotopy works well for the Schlömilch's integral equations. We claim that it also works for the Schlömilch-type integral equations. We assume ψ is a polynomial function throughout the paper unless otherwise stated.

3.1. Schlömilch-type Integral Equation

In the subsection, consider (1.1) with ψ being a polynomial function of degree n. We start with setting the homotopy equation (3.1) equal to zero with

$$F(\phi) = \phi \text{ and } L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\, dt - \psi(x)$$

We then replace ϕ with that $\phi = \phi_0 + p\phi_1 + p^2\phi_2 + \cdots$. Equating the like terms with respect to orders of p along with setting that $\phi_2 = 0$, we end up with the following formula for q_k's.

$$q_k = a_k - \sqrt{\pi}\frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k+1}{2})} a_k, \quad k = 0, 1, 2, \cdots, n$$

where $\psi(x) = \sum_{k=0}^{n} a_k x^k$. This setup produces the solution, namely

$$\phi(x) = \sum_{k=0}^{n} (a_k - q_k) x^k \tag{3.2}$$

Example 3.2. [9, 11] Consider the following Schlömilch-type integral equation

$$1 + 2x = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\, d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = 1$ and $a_1 = 2$. These values provide $q_0 = 0$ and $q_1 = 2 - \pi$. Using (3.2), we get the solution, which is $\phi(x) = 1 + \pi x$. It is easy to verify that this is a solution. Besides, a comparison with the other methods shows how efficient and simple it is to use this algorithm.

Example 3.3. Consider

$$x - \frac{x^3}{3} = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos\theta)\,d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = a_2 = 0$, $a_1 = 1$, and $a_3 = -\frac{1}{3}$. Corresponding q_k values are $q_0 = q_2 = 0$, $q_1 = 1 - \frac{\pi}{2}$, and $q_3 = \frac{\pi}{4} - \frac{1}{3}$. We obtain that $\phi(x) = \frac{\pi}{2}x - \frac{\pi}{4}x^3$. By a simple substitution, one can easily verify that this is a solution.

3.2. Generalized Schlömilch-type Integral Equation

In the subsection, we consider (1.2). Since the procedure is similar to what we did in the preceding case, We will not go into details; rather, we will provide only the results. Following the similar steps done in the preceding case, we have

$$q_k = a_k - \sqrt{\pi}\frac{\Gamma(\frac{rk}{2}+1)}{\Gamma(\frac{rk+1}{2})}a_k, \quad k = 0, 1, 2, \cdots, n$$

where $\psi(x) = \sum_{k=0}^{n} a_k x^k$. The solution is

$$\phi(x) = \sum_{k=0}^{n}(a_k - q_k)x^k$$

Example 3.4. Consider

$$x + 3x^2 = \frac{2}{\pi}\int_0^{\pi/2} \phi(x\cos^2\theta)\,d\theta, \quad -\pi \le x \le \pi$$

We have $a_0 = 0$, $a_1 = 1$, and $a_2 = 3$. Corresponding q_k values are $q_0 = 0$, $q_1 = -1$, and $q_2 = -5$. We obtain that $\phi(x) = 2x + 8x^2$, which is the exact solution.

3.3. Nonlinear Schlömilch-type Integral Equation

In the subsection, we consider (1.3). Again, we will not go into details; rather, we will provide only the results by applying to an example. We require the nonlinear function N to be invertible.

Example 3.5. [9] Consider

$$\frac{4}{3\pi}x^3 = \frac{2}{\pi}\int_0^{\pi/2} \phi^3(x\cos(\theta))\,d\theta, \quad -\pi \le x \le \pi \tag{3.3}$$

We first set $N(\phi(x\cos(\theta))) = \phi^3(x\cos(\theta)) = \Omega(x\cos(\theta))$ to transform (3.3) to the standard form. In other words, (3.3) becomes

$$\frac{4}{3\pi}x^3 = \frac{2}{\pi}\int_0^{\pi/2} \Omega(x\cos(\theta))\,d\theta, \quad -\pi \le x \le \pi$$

This takes us to (1.1). Following the procedure provided for (1.1), we get $\Omega(x) = x^3$. Since $\phi^3 = \Omega$, we get $\phi(x) = x$, which is a solution to (3.3). This is exactly the same solution obtained in [9] using the Regularization-Adomian method.

Equation of the form: $\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\,d\theta$

We now consider (1.4)

$$\psi(x) = \frac{2}{\pi} \int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\, d\theta, \quad -\pi \le x \le \pi$$

We apply the homotopy

$$\mathcal{H}(\phi, p) = (1-p)F(\phi) + pL(\phi) + p(1-p)\sum_{k=0}^{n} q_k x^k$$

with

$$F(\phi) = \phi \text{ and } L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\, d\theta - \psi(x)$$

Following the steps explained above, we first set $\mathcal{H}(\phi, p) = 0$. This amounts

$$\mathcal{H}(f, p) = (1-p)F(\phi) + pL(\phi) + p(1-p)\sum_{k=0}^{n} q_k x^k = 0 \tag{3.4}$$

where $F(\phi) = \phi$ and $L(\phi) = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi(x\cos\theta)\, d\theta - \psi(x)$.
Using p as an expanding parameter, we have $\phi = \phi_0 + p\phi_1 + p^2\phi_2 + \cdots$. Substituting this into (3.4) and equating the like terms, we have

$$\begin{aligned}
\phi_0 &= 0 \\
\phi_1 &= \sum_{k=0}^{n} (a_k - q_k)x^k \\
\phi_2 &= \phi_1 - \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi_1(x\cos\theta)\, d\theta + \sum_{k=0}^{n} q_k x^k \\
&= \sum_{k=0}^{n} (a_k - q_k)(1 - c_k)x^k \\
\phi_{i+1} &= \phi_i - \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\phi_i(x\cos\theta)\, d\theta
\end{aligned}$$

where $c_k = \frac{2}{\pi}\int_0^{\pi/2} f(\cos(\theta))\cos^k(\theta)\, d\theta$. Setting that $\phi_2 = 0$, we first get the q_k values and these are

$$q_k = \frac{a_k(c_k - 1)}{c_k}, \quad k = 1, 2, \cdots, n$$

where $c_k \neq 0$. Once the q_k values are known, then the solution is provided by

$$\begin{aligned}
\phi(x) &= \sum_{k=0}^{n} (a_k - q_k)x^k \\
&= \sum_{k=0}^{n} \frac{a_k}{c_k} x^k
\end{aligned} \tag{3.5}$$

4. An Application Problem

Consider the following equation

$$\psi(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(\theta)\phi(x\cos(\theta))\, d\theta \tag{4.1}$$

This will lead to one of the most important equations in fluid dynamics. In particular, for $\psi(x) = \frac{x}{2} - \frac{x^3}{8}$, (4.1) provides the Rayleigh equation [12, 13].

We apply the algorithm that we described in the previous section. To be more precise, we assume that $f(\cos(\theta)) = \cos(\theta)$ and apply the algorithm to (4.1). It follows that the formula for $c_k = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} f(\cos(\theta))\cos^k(\theta)\,d\theta$ becomes

$$c_k = \frac{2}{\pi}\int_0^{\pi/2} \cos^{k+1}(\theta)\,d\theta$$

This implies that

$$c_0 = \frac{2}{\pi}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{4}{3\pi}, \quad \text{and} \quad c_3 = \frac{3}{8}$$

The solution is then

$$\phi(x) = x - \frac{x^3}{3}$$

which is obtained from (3.5). We want to point out that this is exactly the same solution of the Rayleigh equation obtained in [11] with two different methods, namely, the Homotopy analysis method and the Variational iteration method.

5. Conclusion

This study contains a set of results for solutions of the Schlömilch-type integral equations. We showed that a modification to homotopy that works for the standard Schlömilch's integral equations works well for Schlömilch-type integral equations as well. In addition, we also considered an important case that led to solving some important application problems. As a future direction, we think the algorithm introduced here can be extended and applied to solve some other important application problems. A modification to homotopy can also be redefined or improved to get better algorithms.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] H. Unz, *Schlömilch's integral equation*, Journal of Atmospheric and Terrestrial Physics 25 (2) (1963) 101–102.

[2] P. J. D. Gething, R. G. Maliphant, *Unz's application of Schlömilch's integral equation to oblique incidence observations*, Journal of Atmospheric and Terrestrial Physics 29 (5) (1967) 599–600.

[3] S. De, B. Sarkar, M. Mal, M. De, B. Ghosh, S. Adhikari, *On Schlömilch's integral equation for the ionospheric plasma*, Japanese Journal of Applied Physics 33 (1-7A) (1994) 4154–4156.

[4] L. Bougoffa, M. Al-Haqbani, R. C. Rach, *A convenient technique for solving integral equations of the first kind by the Adomian decomposition method*, Kybernetes 41(1/2) (2012) 145–156.

[5] A. Altürk, *On the solutions of Schlömilch's integral equations*, Celal Bayar University Journal of Science 13 (3) (2017) 671–676.

[6] A. Altürk, H. Arabacıoğlu, *A new modification to homotopy perturbation method for solving Schlömilch's integral equation*, International Journal of Advances in Applied Mathematics and Mechanics 5 (1) (2017) 40–48.

[7] A. Altürk, *A simple and efficient approach based on Laguerre polynomials for solving Schlömilch's integral equation*, Journal of Inequalities and Special Functions 14 (1) (2023) 37–50.

[8] A. M. Wazwaz, Linear and nonlinear integral equations: methods and applications, Heidelberg University Publishing, Berlin, 2011.

[9] A. M. Wazwaz, *Solving Schlömilch's integral equation by the Regularization-Adomian method*, Romanian Journal of Physics 60 (1-2) (2015) 56–71.

[10] P. Kourosh, M. Delkhosh, *Solving the nonlinear Schlömilch's integral equation arising in ionospheric problems*, Afrika Matematika 28 (3) (2017) 459–480.

[11] M. A. Al-Jawary, G. H. Radhi, J. Ravnik, *Two efficient methods for solving Schlömilch's integral equation*, International Journal of Intelligent Computing and Cybernetics 10 (3) (2017) 287–309.

[12] P. J. Ponzo, N. Wax, *Existence and stability of periodic solutions of* $\ddot{y} - \mu F(\dot{y}) + y = 0$, Journal of Mathematical Analysis and Applications 38 (3) (1972) 793–804.

[13] A. D. D. Craik, Wave interactions and fluid flows, Cambridge University Press, Cambridge, 1986.

[14] J. H. He, *Homotopy perturbation technique*, Computer Methods in Applied Mechanics and Engineering 178 (3-4) (1999) 257–262.

[15] S. J. Liao, *An approximate solution technique not depending on small parameters: A special example*, International Journal of Non-Linear Mechanics 30 (3) (1995) 371–380.

[16] J. H. He, *A coupling method of a homotopy technique and a perturbation technique for non-linear problems*, International Journal of Non-Linear Mechanics 35 (1) (2000) 37–43.

[17] J. H. He, *Homotopy perturbation method: A new nonlinear analytical technique*, Applied Mathematics and Computation 135 (1) (2003) 73–79.

[18] J. H. He, *Recent development of the homotopy perturbation method*, Topological Methods in Nonlinear Analysis Journal of the Juliusz Schauder Center 31 (2008) 205–209.

[19] M. A. Noor, *Iterative methods for nonlinear equations using homotopy perturbation technique*, Applied Mathematics & Information Sciences 4 (2) (2010) 227–235.

[20] L. Yuzhen, *Numerical methods for integral equations*, Doctoral Dissertation Syracuse University (2023) New York.

[21] A. Golbabai, B. Keremati, *Modified homotopy perturbation method for solving Fredholm integral equations*, Chaos, Solitons & Fractals 37 (2008) 1528–1537.

[22] J. S. Nadjafi, M. Tamamgar, *Modified homotopy perturbation method for solving integral equations*, International Journal of Modern Physics B 24 (24) (2010) 4741–4746.

[23] M. Sotoodeh, M. A. F. Araghi, *A new modified homotopy perturbation method for solving linear second-order Fredholm integro-differential equations*, International Journal of Mathematical Modelling & Computations 2 (4) (2012) 299–308.