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Simplicial Lie-Rinehart Algebras with Related Structures

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Abstract

In this paper, simplicial Lie-Rinehart algebras and Lie-Rinehart cat^1-algebras will be defined. With the help of these definitions, the relations between Lie-Rinehart crossed modules, cat^1 -algebras and simplicial Lie-Rinehart algebras will be explained.

Anahtar Kelimeler: Lie-Rinehart algebra, simplicial object, crossed module

İlgili Yapılarla Simplisel Lie-Rinehart Cebirler

Öz

Bu çalışmada, simplisel Lie-Rinehart cebirler ve Lie-Rinehart cat^1 -cebirler tanımlanacaktır. Bu tanımlamalar yardımıyla, Lie-Rinehart çaprazlanmış modüller, cat^1-cebirler ve simplisel Lie-Rinehart cebirler arasındaki ilişki açıklanacaktır.

Keywords: Lie-Rinehart cebir, simplisel obje, çaprazlanmış modül

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Introduction

Lie-Rinehart algebras represent a significant structure that examines the interactions between Lie algebras and differential geometry. These structures are defined as general constructions related to both Lie algebras and modules over commutative algebras and offer a rich and profound field of research in topics related to differential algebraic structures. Particularly, the study of modules over Lie-Rinehart algebras is crucial for understanding the geometric and algebraic properties of these structures. In this context, the crossed modules of Lie-Rinehart algebras emerge as a generalization of similar concepts in differential geometry and homological algebra. These were firstly introduced by Herz in (Herz, 1953). Lie-Rinehart algebras have a very close relationship to Lie algebroids and are their section spaces. For more information, see (Huebschmann, 1990; Mackenzie, 1987).

Lie-Rinehart crossed modules represent a significant concept that merges algebraic and differential structures. These modules enable the formulation of a Lie algebra as a module while exploring the interactions of these structures across different mathematical contexts. These structures are critically important, particularly in the representation theory of Lie algebras and the study of geometric structures. The applications of these modules are extensive in fields such as differential geometry, algebraic geometry, and theoretical physics. For instance, Lie-Rinehart modules enable a better understanding of the relationships between differential forms and vector fields, thus providing a foundation for a deeper exploration of symmetries and structures. In conclusion, Lie-Rinehart crossed modules reveal connections between mathematical theories and structures, making significant contributions to both theoretical and applied research. In this context, a thorough investigation of these modules can offer new perspectives in various disciplines of modern mathematics. On the other hand, crossed modules for Lie-Rinehart algebras were defined in (Casas et al., 2004) to provide extensive information about the cohomology of Lie-Rinehart algebras. Afterwards, the authors have contributed the subject with the studied (Casas, 2011; Casas et al., 2005).

Our main purpose in this article is to define simplicial Lie-Rinehart algebras and Lie-Rinehart cat^1-algebras, and give natural equivalences between the category of Lie Rinehart crossed modules, Lie Rinehart cat^1-algebras and one dimensional simplicial Lie-Rinehart algebras. In addition, fort he Lie algebra and categorical aspect with see (Arvasi & Akça, 2002).

Preliminaries

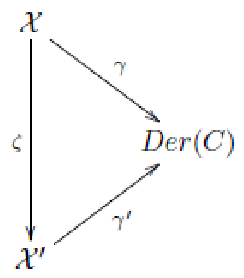
In this section, the fundamental concepts and properties that will be used in the article will be given. For the detailed information about the definitions and theorems that we restate see (Casas et al., 2004; Herz, 1953). Throughout this article, r will be taken as a field, C as a commutative algebra over r, and $Der(C)$ as the set of r-derivations of C.

Definition 1. *Let $\mathcal{X}$ be a Lie r-algebra and an C-module and $\gamma: \mathcal{X} \to Der(C)$ is an C-module and a Lie r-algebra homomorphism. So, $(\mathcal{X},\gamma)$ is called a Lie-Rinehart C-algebra over C (or shortly called L-R algebra) and denoted by $\mathcal{X}$, if*

$$[x, cx'] = c[x, x'] + x(c)x'$$

for all $x, x' \in \mathcal{X}$, $c \in C$ where $x(c) = \gamma(x)(c)$.

Definition 2. *Let $\mathcal{X}$ and $\mathcal{X}'$ be L-R algebras. Let $\zeta: \mathcal{X} \to \mathcal{X}'$ be a Lie algebra homomorphism and an C-module homomorphism. If the diagram*



is commutative then ζ *is called a L-R algebra homomorphism.*

So, we have the category of L-R algebras and we denote this category by $\mathfrak{LR}(C)$.

Example 3. *If* $\gamma = 0$ *for a Lie-Rinehart algebra* $\mathcal{X}$ *then* $\mathcal{X}$ *is a Lie* C*-algebra. Also,* $Der(C)$ *is a L-R algebra.*

Example 4. *If* $\mathcal{X}$ *is a L-R algebra, then* $\mathcal{X} \rtimes C$ *with Lie bracket*

$$[(x,c),(x',c')] = \big([x,x'], x(c') - x'(c)\big)$$

and map

$$\tilde{\gamma}: \mathcal{X} \rtimes C \to Der(C),\ \tilde{\gamma}(x,\mathrm{c}) = \gamma(x)$$

is a L-R algebra, where γ *is* $\gamma: \mathcal{X} \to Der(C)$*.*

Definition 5. *Let* $\mathcal{X}$ *be a L-R algebra . A L-R subalgebra* $\mathcal{N}$ *of* $\mathcal{X}$ *consists of a Lie* r*-subalgebra* $\mathcal{N}$ *which is an* C*-module and* C *acts on* $\mathcal{N}$ *via the composition*

$$\mathcal{N} \hookrightarrow \mathcal{X} \xrightarrow{\gamma} Der(C).$$

It is trivial that a L-R subalgebra $\mathcal{N}$ *of* $\mathcal{X}$ *is an ideal if* $\mathcal{N}$ *is an ideal of* $\mathcal{X}$ *as Lie* r *-algebra with the following composition*

$$\mathcal{N} \hookrightarrow \mathcal{X} \xrightarrow{\gamma} Der(C).$$

Definition 6. *Let* $\mathcal{X}$ *be a L-R algebra and* Y *be a Lie* C *-algebra. The action of* $\mathcal{X}$ *on* Y *is a* r *-linear map*

$$\begin{array}{ccc} \mathcal{X} \times Y & \to & Y \\ (x,y) & \mapsto & {}^{x}y \end{array}$$

satisfies the following axioms

1. $^{[x,x']}y = {}^{x}\left({}^{x'}y\right) - {}^{x'}({}^{x}y)$
2. $^{x}[y_1,y_2] = [{}^{x}y_1, y_2] + [y_1, {}^{x}y_2]$
3. $^{cx}y = c({}^{x}y)$
4. $^{x}(cy) = c({}^{x}y) + \big(\zeta(x)(c)\big)y,$

for all $x, x' \in \mathcal{X}$*,* $y, y_1, y_2 \in Y$ *and* $c \in C$*.*

Definition 7. *Let* $\mathcal{X}$ *be a L-R algebra,* Y *be a Lie* C *-algebra and* $\mathcal{X}$ *acts on* Y*. Then* $\mathcal{X} \rtimes Y$ *is a Lie* r*-algebra with the Lie bracket*

$$[(x,y),(x',y')] = \big([x,x'], [y,y'] + {}^{x}y' - {}^{x'}y\big).$$

This construction is called semi-direct product of $\mathcal{X}$ and Y. If we define

$$\tilde{\gamma}: \mathcal{X} \rtimes Y \to Der(C),\ \tilde{\gamma}(x, y) = \gamma(x)$$

then the pair $(\mathcal{X} \rtimes Y, \tilde{\gamma})$ is a L-R algebra.

Definition 8. *Let Y be an abelian Lie C-algebra and $\mathcal{X}$ be a L-R C-algebra. If $\mathcal{X}$ acts on Y then we call Y as a L-R module over $\mathcal{X}$ or shortly $(\mathcal{X}, C)$-module .*

Crossed Modules

Crossed modules are a fundamental concept in algebraic topology and homological algebra, providing a way to study the interplay between group theory and homotopy theory. Put forward by Whitehead in (Whitehead, 1949), crossed modules generalize the notion of groups and their actions, allowing for a more nuanced understanding of how different algebraic structures can interact. This setup allows us to capture the idea of a group acting on another group, where the action is governed by the homomorphism. One of the key motivations for studying crossed modules is their role in the classification of 2-categories and the study of higher-dimensional algebra. They provide a framework for understanding how groups can be built up from simpler components, similar to how topological spaces can be constructed from simplices. Following Whitehead's definition, many researchers have investigated the properties of crossed modules by defining them on various algebraic structures (Aytekin & Şahan, 2022; Gürmen, 2023; Odabaş et al., 2016; Şahan, 2019).

Now, we will recall the definition of the Lie-Rinehart crossed module (or shortly called L-R crossed module). The examples, remarks and propositions for L-R crossed modules similar to crossed modules of commutative algebras and Lie algebras given in many different papers in the references. But many parts of the proofs are different for L-R algebra case.

Definition 9. *Let $\mathcal{X}$ be a L-R algebra, Y be a Lie C -algebra and $\mathcal{X}$ acts on Y. Lie r-algebra homomorphism $\partial: Y \to \mathcal{X}$ is called L-R crossed module such that the following identities hold*

1. $\partial({}^{l}y) = ({}^{l}\,\partial(y))$
2. $[\partial(y'), y] = [y', y]$
3. $\partial(cy) = c\,\partial(y)$
4. $\partial(y)(c) = 0,$

for all $y, y' \in Y, x \in \mathcal{X}$ and $c \in C$. This structure is denoted by $(Y, \mathcal{X}, \partial)$.

As indicated in (Casas et al., 2004), the third condition says that ∂ is an C-module homomorphism and the fourth condition says that the composition

$$Y \overset{\partial}{\to} \mathcal{X} \overset{\gamma}{\to} Der(C)$$

is zero.

Example 10. *Let $\mathcal{X}$ be a L-R algebra and I is an ideal of $\mathcal{X}$. With homomorphim*

$$\begin{array}{rcl} i: I & \to & \mathcal{L} \\ l & \mapsto & l \end{array}$$

and action

$$\begin{array}{rcl} \mathcal{X} \times I & \to & I \\ (x, t) & \mapsto & [x, t] \end{array}$$

for all $t \in \mathrm{I}, x \in \mathcal{X}$, $(I, \mathcal{X}, i)$, *is a L-R crossed module.*

Example 11. *Let* Y *be a* $(\mathcal{X}, C)$*-module. Then* $0: Y \to \mathcal{X}$ *is a L-R crossed module.*

Example 12. *Let* $\beta: Y \to Y'$ *be a homomorphism of* $(\mathcal{X}, C)$*-modules. We define an action of* $\mathcal{X} \rtimes Y'$ *on* Y *with* $(x, y') \cdot y = xy$ *for all* $x \in \mathcal{X}$, $y \in Y$ *and* $y' \in Y'$. *Define*

$$\begin{array}{rcl} \partial: Y & \to & \mathcal{X} \rtimes Y' \\ y & \mapsto & \big(0, \beta(y)\big), \end{array}$$

then $(Y, \mathcal{X} \rtimes Y', \partial)$ *is a L-R crossed module.*

Definition 13. *Let* $(Y, \mathcal{X}, \partial)$ *and* $(Y', \mathcal{X}', \partial')$ *be L-R crossed modules. The homomorphism between these two crossed modules is the pair* (f, ϕ) *of Lie* r*-algebra homomorphism* $f: Y \to Y'$ *and L-R algebra homomorphism* $\phi: \mathcal{X} \to \mathcal{X}'$ *such that*

$$f(x \cdot y) = \phi(x) \cdot f(y),\ \partial' f(y) = \phi\, \partial(y).$$

Thus, the category of L-R crossed modules whose objects are L-R crossed modules and whose morphisms are homomorphism pairs is defined and this category is denoted as $\mathfrak{Xmod}(\mathfrak{LR})$.

Now, we will give some basic functorial properties of this category. Obviously, we can easily define some forgetful functors as follows;

$$\begin{array}{rcl} U_1: \mathfrak{Xmod}(\mathfrak{LR}) & \to & \mathfrak{LR}(C) \\ (Y, \mathcal{X}, \partial) & \mapsto & \mathcal{X} \end{array}$$

$$\begin{array}{rcl} U_2: \mathfrak{Xmod}(\mathfrak{LR}) & \to & \mathfrak{L}(C) \\ (Y, \mathcal{X}, \partial) & \mapsto & Y \end{array}$$

where $\mathfrak{L}(C)$ represents the category of Lie algebras. Also, if we denote the category of Lie r-algebras by $\mathfrak{Xmod}(\mathfrak{L}\mathrm{ie})$ then we have

$$U_3: \mathfrak{Xmod}(\mathfrak{LR}) \to \mathfrak{Xmod}(\mathfrak{L}\mathrm{ie})$$

which forgets the C-module structure.

Simplicial L-R Algebras

Simplicial algebras are a branch of mathematics that arises from the interplay between algebraic structures and topological concepts, particularly in the study of simplicial sets and simplicial complexes. At its core, simplicial algebra provides a framework for understanding how algebraic operations can be performed on geometric objects, enabling mathematicians to explore relationships between topology, homotopy theory, and category theory. The fundamental building blocks of simplicial algebras are simplices geometric objects such as points, line segments, triangles, and higher-dimensional analogs. These simplices are organized into higher-dimensional structures known as simplicial complexes, which serve as a way to study spaces through combinatorial and algebraic methods. One of the key contributions of simplicial algebra is its ability to capture homotopical properties of topological spaces through algebraic invariants, such as homology and cohomology groups. This provides for a wide understanding of the shape and connectivity of spaces, bridging the gap between algebraic and geometric perspectives.

Now, we recall the definition of simplicial object in (Goerss & Jardine, 2009). Let $\mathfrak{C}$ be a category with all finite colimits and consider $s\mathfrak{C}$, the category of simplicial objects in $\mathfrak{C}$. A simplicial object in $\mathfrak{C}$ is defined as a contravariant functor $\triangle^{op} \to \mathfrak{C}$ from the ordinal number category $\triangle$. In this context, a simplicial L-R algebra (or shortly called SL-R algebra) $\mathbf{X}$ is a sequence of L-R algebras

$$\mathbf{X} = \{X_0, X_1, \ldots, X_n, \ldots\}$$

together with face and degeneracy maps

$$\begin{array}{lll} d_i^n: & X_n & \to X_{n-1} \\ s_i^n: & X_n & \to X_{n+1} \end{array}$$

for all $0 \leq i \leq n, n \neq 0$ which are L-R homomorphisms satisfying the general simplicial identities.

The Moore Complex

The Moore complex $\mathbf{NX}$ of a simplicial L-R algebra $\mathbf{X}$ is the complex

$$\mathbf{NX}: \ldots \mathrm{N}X_n \xrightarrow{\partial_n} \mathrm{N}X_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} \mathrm{N}X_1 \xrightarrow{\partial_1} \mathrm{N}X_0$$

where $NX_0 = X_0$, $NX_n = \bigcap\limits_{i=0}^{n-1} Kerd_i$ and ∂_n is the restriction of d_n to NX_n.

We express that the Moore complex $\mathbf{NX}$ of a SL-R algebra $\mathcal{X}$ is of length k if $NX_n = 0$ for all $n \geq k+1$. In this situation since the kernel of a L-R homomorphism is a Lie C-algebra, so NX_n is a Lie C-algebra for $n \geq 1$. So, it can be defined the category of SL-R denoted by $\mathfrak{Smp}_{\leq n}(\mathfrak{LR})$ whose objects are SL-R algebras with Moore complex of length n and the morphisms are families of L-R homomorphisms suitable with face and degeneracy maps.

Truncated SL-R Algebras

Details of the group case can be found in (Curtis, 1971). For each $k \geq 0$ we have a subcategory of $\triangle$, denoted as $\triangle_{\leq k}$ obtained by the objects $[j]$ of $\triangle$ with $j \leq k$. A truncated SL-R algebra is $\triangle_{\leq k}^{op} \to \mathfrak{LR}(C)$. As a consequence a truncated SL-R algebra is a family of L-R algebras $\{X_0, X_1, \ldots, X_k\}$ and homomorphism $d_i: X_n \to X_{n-1}$, $s_i: X_n \to X_{n+1}$ for each $0 \leq i \leq n$ which previously mentioned. We denote the category of k -truncated SL-R algebras by $\mathfrak{Tr}_k\mathfrak{Smp}(\mathfrak{LR})$. Also, there is the functor $tr_k: \mathfrak{Smp}(\mathfrak{LR}) \to \mathfrak{Tr}_k\mathfrak{Smp}(\mathfrak{LR})$ and the following relationship exists between it and the st_k and $cost_k$ functors;

$$\mathfrak{Tr}_k\mathfrak{Smp}(\mathfrak{LR}) \quad \underset{cost_k}{\overset{tr_k}{\leftarrows}} \quad \mathfrak{Smp}(\mathfrak{LR}) \quad \underset{st_k}{\overset{tr_k}{\rightleftarrows}} \quad \mathfrak{Tr}_k\mathfrak{Smp}(\mathfrak{LR}).$$

For the definitions of the functors $cost_k$ and st_k see (Curtis, 1971).

Theorem 14. *The category $\mathfrak{Xmod}(\mathfrak{LR})$ is naturally equivalent to the category $\mathfrak{Smp}_{\leq 1}(\mathfrak{LR})$.*

Proof. Let X be an object of $\mathfrak{Smp}_{\leq 1}(\mathfrak{LR})$. Take $Y = \ker d_0$ (so Y is a Lie C -algebra) and ∂ is the restriction of d_1 to Y. Action of X on Y is defined as

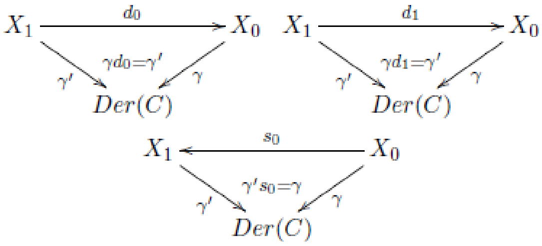
$$\begin{array}{ccc} X_0 \times Y & \to & Y \\ (x, y) & \mapsto & {}^x y = [s_0 x, y] \end{array}$$

for $x \in X$ and $y \in Y$. It is easy to check the conditions of the action.

Since the maps

$$X_1 \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} X_0$$

are L-R algebra homomorphism, we have the following commutative diagrams



By using the commutativity of diagrams and some simplicial identities we have

CM 1:

$$\begin{aligned} \partial({}^{x}y) &= \partial[s_0x, y] \\ &= [d_1s_0x, d_1y] \\ &= [x, \partial y] \qquad (\because\ d_1s_0 = id). \end{aligned}$$

CM 2: Employing similar manner we provide the following identity

$$^{\partial(y')}y \quad = \quad [y', y].$$

CM 3:

$\partial(cy) = c\,\partial(y)$ (since ∂, is a C -module homomorphism)

CM 4: Taking advantgre of the definitions and properties, we obtain the following

$$\partial(y)(c) \quad = \quad 0.$$

The homomorphism $\partial: Y \to X$ is a L-R crossed module . So we obtain a functor

$$N_1: \mathfrak{Smp}_{\leq 1}(\mathfrak{LR}) \to \mathfrak{Xmod}(\mathfrak{LR})$$

Conversely let $\partial: Y \to X$ be a L-R crossed module. With the action of X on Y we can create

$$X_1 = X \rtimes Y = \{(x, y): x \in X, y \in Y\}.$$

For all $c \in C, x, x' \in X_0$ and $y, y' \in Y$ the scalar multiplication, sum and multiplication defined as

$$\begin{aligned} c(x, y) &= (cx, cy), \\ (x, y) + (x', y') &= (x + x', y + y'), \\ [(x, y), (x', y')] &= \left([x, x'], [y, y'] + {}^{x}y' - {}^{x'}y\right), \end{aligned}$$

respectively. We have the homomorphisms

$$\begin{aligned} d_0: &\quad X \rtimes Y \to X \\ &\quad (x, y) \mapsto x \\ d_1: &\quad X \rtimes Y \to X \\ &\quad (x, y) \mapsto (\partial y) + x \\ s_0: &\quad X \to X \rtimes Y \\ &\quad x \mapsto (x, 0) \end{aligned}$$

and these maps satisfy the simplicial identities.

Now we must show that these maps are L-R homomorphisms. Since

$$\gamma d_0 = \tilde{\gamma}, \gamma d_1 = \tilde{\gamma} \text{ and } \tilde{\gamma} s_0 = \gamma$$

we have

$$\gamma d_0 = \tilde{\gamma} = \gamma, \gamma d_1 = \tilde{\gamma} = \gamma \text{ and } \tilde{\gamma} s_0 = \gamma s_0 = \gamma$$

so the maps are L-R homomorphisms (other conditions are omitted as they easy to check). Finally

$$X_1 \rightrightarrows X_0$$

is an object of $\mathfrak{Tr}_1\mathfrak{Smp}(\mathfrak{LR})$. So, we have

$$M: \mathfrak{Xmod}(\mathfrak{LR}) \to \mathfrak{Tr}_1\mathfrak{Smp}(\mathfrak{LR}).$$

On the other hand we have the functor

$$st_1: \mathfrak{Tr}_1\mathfrak{Smp}(\mathfrak{LR}) \to \mathfrak{Smp}_{\leq 1}(\mathfrak{LR}).$$

Thus, with the combination of the st_1 and M functors we obtain

$$N_2: \mathfrak{Xmod}(\mathfrak{LR}) \to \mathfrak{Smp}_{\leq 1}(\mathfrak{LR}).$$

So, thanks to the functors N_1 and N_2, the natural equivalence between the two categories is shown.

Cat^1 L-R Algebras

Cat^1 algebras emerge from the intersection of category theory and algebra, offering a powerful framework for studying algebraic structures through categorical concepts. By viewing algebraic entities such as groups, rings, and modules as objects within a category, Cat^1 algebras enable a deeper exploration of their relationships and transformations. In this context, a Cat^1 algebra can be understood as a category with a single object, where morphisms correspond to the algebraic operations defined on that object. This perspective allows for the analysis of algebraic properties in a more general setting, facilitating the investigation of homomorphisms, automorphisms, and extensions in a unified manner. One of the main advantages of Cat^1 algebras is their ability to capture the essence of algebraic structures while retaining the flexibility of categorical operations. This approach not only enhances the understanding of traditional algebraic concepts but also bridges connections to other areas of mathematics, such as topology and homological algebra.

Cat^1 groups were defined by Loday in (Loday, 1982). He developed this concept to explore the connections between categorical structures and group theory. Cat^1 groups provide an important framework for understanding how groups are structured within a category and how relationships between groups can be studied. Following Loday's definition, many researchers have investigated the properties of this structure on various algebraic structures (Alp, 1998; Alp & Gürmen, 2003; Arvasi & Odabaş, 2016; Şahan & Kendir, 2023, Temel,2019).

Definition 15. *A cat^1 L-R algebra is a triple $(\mathcal{X}, s, t)$ where $\mathcal{X}$ is a L-R algebra, s and t are L-R homomorphisms such that;*

$i)$ $st = t$ and $ts = s$

$ii)$ $[\ker s, \ker t] = 0.$

Obviously we can form the category of cat^1 L-R algebras where the morphisms are L-R algebra homomorphisms suitable with the source and target maps. We will denote this category by $\mathfrak{Cat}^1(\mathfrak{LR})$.

We refer (Arvasi, 1997) for commutative algebra case of below proposition.

Proposition 16. *The category $\mathfrak{Xmod}(\mathfrak{LR})$ is naturally equivalent to the category $\mathfrak{Cat}^1(\mathfrak{LR})$.*

Proof. Let $(\mathcal{X}, s, t)$ be a cat^1 L-R algebra, $M = \ker s, \mathrm{N} = \mathrm{Im} s$ and $\partial = t|_M$. First of all we have $\gamma s = \gamma$ and $\gamma t = \gamma$, so $\gamma s = \gamma t$. Define the action of N on M by ${}^n m = [n, m]$. Since M is a kernel L-R homomorphism so is a Lie A-algebra. The first two conditions are trivial as indicated in for commutative algebra case. In the fourth condition we must use the fact that $\alpha s = \alpha t$ which is different from the commutative algebra case.

CM 4: Since s and t are L-R algebra homomorphisms, $\gamma s = \gamma t = \gamma$. So for $m \in M$ and $c \in C$ we have

$$\begin{aligned} \partial(m)(c) &= \big(\gamma(\partial m)\big)(c) \\ &= \big(\gamma(tm)\big)(c) \\ &= \big(\gamma(sm)\big)(c) \quad (\gamma t = s = \gamma s) \\ &= \big(\gamma(0)\big)(c) \quad (\text{ Since } m \in \ker s = M) \\ &= 0 \end{aligned}$$

From above calculations $\partial: M \to N$ is a L-R crossed module.

Conversely, given a crossed module for L-R algebra $(\mathcal{X}, Y, \partial)$. $Y \ltimes \mathcal{X}$ is a L-R algebra as proved in (Casas et al., 2004).

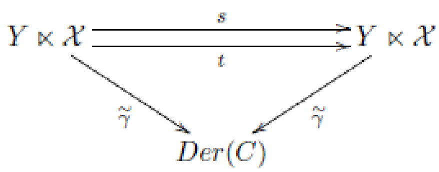
Define s, t as;

$$\begin{aligned} s: \quad Y \ltimes \mathcal{X} &\to Y \ltimes \mathcal{X} \\ (y, x) &\mapsto (0, x) \end{aligned}$$

and

$$\begin{aligned} t: \quad Y \ltimes \mathcal{X} &\to Y \ltimes \mathcal{X} \\ (y, x) &\mapsto (0, \partial y + x) \end{aligned}$$

It is easy to check that s and t are C-module and Lie r-algebra homomorphisms. Here we will show the commutativity of the diagrams



Since

$$\begin{aligned} \big(\tilde{\gamma}(s)\big)(y, x) &= \tilde{\gamma}(0, x) \\ &= \gamma(x) \\ &= \tilde{\gamma}(y, x), \end{aligned}$$

$$\begin{aligned} \big(\tilde{\gamma}(t)\big)(y, x) &= \tilde{\gamma}(0, \partial y + x) \\ &= \gamma(\partial y + x) \\ &= \gamma\, \partial y + \gamma x \\ &= 0 + \alpha x \\ &= \tilde{\gamma}(y, x), \end{aligned}$$

we have that s and t are L-R homomorphisms. Also $st = t$ and $ts = s$, that is,

$$\begin{aligned} s\big(t(y,x)\big) &= s(0,\partial y + x) \\ &= (0,\partial y + x) \\ &= t(y,x) \end{aligned}$$

and

$$\begin{aligned} t\big(s(y,x)\big) &= t(0,x) \\ &= (0,x) \\ &= s(y,x). \end{aligned}$$

So, we have $\ker s = \{(y,0) \mid y \in Y\}$ and $\ker t = \{(y,-\partial y) \mid y \in Y\}$. We have $[\ker s, \ker t] = 0$, since

$$\begin{aligned} [(y,0),\big(y',-\partial(y')\big)] &= \big([y,y'] -{}^{-\partial y'}y + {}^{0}y', [0,-\partial y']\big) \\ &= \big([y,y'] + {}^{\partial y'}y + 0,0\big) \\ &= ([y,y'] + [y',y],0) \\ &= (0,0). \end{aligned}$$

Conclusion and Suggestions

In this study, simplicial Lie-Rinehart algebras and Lie-Rinehart cat^1-algebras are defined. Also, the relationships between Lie-Rinehart crossed module, cat^1-algebra and simplicial Lie-Rinehart algebra categories are mentioned. Similar studies can be done using crossed modules on different algebraic structures.

Author Contribution

The authors co-wrote, read and approved the manuscript.

Ethics

There are no ethical issues regarding the publication of this article.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Kaynaklar

Alp, M. (1998). Pullbacks of crossed modules and cat1-groups. *Turkish Journal of Mathematics, 22*, 273-281. https://journals.tubitak.gov.tr/math/vol22/iss3/2

Alp, M., & Gürmen Alansal, Ö. (2003). Pushouts of profinite crossed module and cat1 profinite groups. *Turkish Journal of Mathematics*, *27*(4), 539–548. https://journals.tubitak.gov.tr/math/vol27/iss4/6

Arvasi, Z. (1997). Crossed squares and 2 crossed modules of commutative algebras . *Theory and Applications of Categories, 3*, 160–181. ftp://ftp.tac.mta.ca/pub/tac/html/volumes/1997/n7/n7

Arvasi, Z., & Akça, İ. İ. (2002). Simplicial and Crossed Lie Algebras. *Homology Homotopy and Applications, 4*(1), 43–57. ftp://ftp.rmi.acnet.ge/pub/hha/volumes/2002/n1a4/v4n1a4

Arvasi, Z., & Odabaş, A. (2016). Computing 2 dimensional algebras Crossed modules and Cat1-algebras. *Journal of Algebra and Its Applications, 15*(10), 1650185–0. https://doi.org/10.1142/S0219498816501851

Aytekin Arıcı, G., & Şahan, T. (2022). Coverings and liftings of generalized crossed modules. *Categories and General Algebraic Structures with Applications*, *17*(1), 117-140. https://doi.org/10.52547/cgasa.17.1.117

Casas, J. M. (2011). Obstructions to Lie–Rinehart Algebra Extensions. In *Algebra Colloquium*, *18*(01),83-104. https://doi.org/10.1142/S1005386711000046

Casas, J. M., Ladra, M., & Pirashvili, T. (2004). Crossed modules for Lie–Rinehart algebras. *Journal of Algebra, 274*(1), 192-201. https://doi.org/10.1016/j.jalgebra.2003.10.001

Casas, J. M., Ladra, M., & Pirashvili, T. (2005). Triple cohomology of Lie–Rinehart algebras and the canonical class of associative algebras. *Journal of Algebra, 291*(1), 144-163. https://doi.org/10.1016/j.jalgebra.2005.05.018

Curtis, E. B. (1971). Simplicial homotopy theory. *Advances in Mathematics, 6*(2), 107-209.

Goerss, P. G., & Jardine, J. F. (2009). *Simplicial homotopy theory*. Springer Science & Business Media.

Gürmen Alansal, Ö. (2023). Ideal and factor conditions for crossed modules and algebra-algebroids. *Hacettepe Journal of Mathematics and Statistics, 52*(3), 698-707. https://doi.org/10.15672/hujms.1131802

Herz, J. C. (1953). Pseudo-algebres de Lie. 1. *Comptes Rendus Hebdomadaıres Des Seances De l Academıe Des Scıences, 236*(20), 1935-1937.

Huebschmann, J. (1990). Poisson cohomology and quantization. *Journal Für Die Reine und Angewandte Mathematik, 408*, 57-113.

Loday, J. L. (1982). Spaces with finitely many non-trivial homotopy groups. *Journal of Pure and Applied Algebra, 24*, 179 – 202.

Mackenzie, K. (1987). *Lie groupoids and Lie algebroids in differential geometry* (Vol. 124). Cambridge University press.

Odabaş, A., Uslu, E. Ö., & Ilgaz Çağlayan, E. (2016). Isoclinism of crossed modules. *Journal of Symbolic Computation, 74*, 408–424. https://doi.org/10.1016/j.jsc.2015.08.006

Şahan, T. (2019). Further remarks on liftings of crossed modules. *Hacettepe Journal of Mathematics and Statistics, 48*(3), 743–752. https://doi.org/10.15672/hjms.2018.554

Şahan, T., & Kendir, E. (2023). Çaprazlanmış Cat1-Modüller. *Journal of the Institute of Science and Technology, 13*(4), 2958-2972. https://doi.org/10.21597/jist.1303212

Temel, S. (2019). Crossed semimodules and cat^1-monoids. *The Korean Journal of Mathematics, 27*(2), 535–545. https://doi.org/10.11568/kjm.2019.27.2.535

Whitehead, J. H. C. (1949). Combinatorial homotopy. *II. Bulletin of the American Mathematical Society*, 55, 213 – 245.