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Research Article

Some Results on Almost Contact Manifolds with B-Metric

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Abstract: In this work, almost contact B-metric manifolds and almost complex manifolds with Norden metric are considered. Almost complex manifolds with a Norden metric are obtained by the product of almost contact B-metric manifolds with $\mathbb{R}$, where almost complex structure and metric on the product manifold depend on two functions of $\mathbb{R}$. The relations between two classes of almost contact manifolds with B-metric (the classes $\mathcal{F}_4$ and $\mathcal{F}_5$) and classes of almost complex manifolds with a Norden metric are investigated.

Keywords: Almost complex manifold with a Norden metric, almost contact manifold, almost contact manifold with B-metric.

1. Introduction

Differentiable manifolds having special tensors are an object of interest in differential geometry. There are several studies on this area, for example, see [2, 4–8, 10, 11, 13–16, 19–21]. Differential manifolds having special tensor structure have been classified by considering the covariant derivative of their tensor structure [2, 4–8, 10, 11, 13, 21].

Manifolds with B-metric have been studied in the last 30 years by various researchers [7, 9, 10, 16, 20]. Recently, many differential geometers and theoretical physicists have been investigating Ricci solitons and η-Ricci solitons on manifolds with special structures, such as almost contact metric manifolds, almost paracontact metric manifolds, manifolds with B-metric, Norden manifolds, etc. [1, 3, 12, 17, 18]. In this investigations, classes of almost contact B-metric manifolds and almost complex manifolds with a Norden metric also gain importance.

In this study, we obtain an infinite number of Kaehlerian manifolds with a Norden metric in Theorem 3.3 and complex manifolds with a Norden metric (the class $W_1 \oplus W_2$) in Thoerem 3.5. In particular, we consider the classification of almost contact manifolds with B-metric and almost complex manifolds with a Norden metric given by [6, 7], respectively. We generalize the metric and

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the complex structure on the product manifold given in [9] by considering two functions. In [9], Sasaki-like manifolds which are subclasses of $\mathcal{F}_4$ of almost contact B-metric manifolds are studied. In this work, almost complex Norden metric manifolds are obtained from almost contact manifolds with B-metric M with product of $\mathbb{R}$ and an almost complex structure and a metric are defined on the product manifold $M \times \mathbb{R}$ depending on two functions σ and μ which are functions of t. Some relations between classes of almost complex manifolds with a Norden metric and the classes $\mathcal{F}_4$ and $\mathcal{F}_5$ of almost contact manifolds with B-metric are obtained.

2. Preliminaries

First, we introduce almost contact B-metric manifolds. A manifold M with odd dimension has an almost contact structure (φ, ξ, η), if it admits a vector field ξ, a map φ, and a 1-form η satisifying the following relations:

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi. \tag{1}$$

Here I is identity map. From (1),

$$\varphi(\xi) = 0, \quad \eta \circ \varphi = 0 \tag{2}$$

follow. In addition to an almost contact structure (φ, ξ, η), if there is a metric tensor g satisfying

$$g(\varphi(a), \varphi(b)) = -g(a,b) + \eta(a)\eta(b) \tag{3}$$

for all vector fields a, b, then M is said to be an almost contact manifold with B-metric. The Equation (3) yields

$$g(a,\xi) = \eta(a), g(\varphi(a), b) = g(a, \varphi(b)). \tag{4}$$

Assume ∇ is the Levi-Civita covariant derivative of g. We denote

$$\Gamma(a,b,c) = g((\nabla_a \varphi) b, c). \tag{5}$$

Γ has the following properties:

$$\begin{array}{rcl} \Gamma(a,b,c) & = & \Gamma(a,c,b), \\ \Gamma(a,\varphi(b),\varphi(c)) & = & \Gamma(a,b,c) - \eta(b)\Gamma(a,\xi,c) - \eta(c)\Gamma(a,b,\xi), \\ \Gamma(a,\xi,\xi) & = & 0 \end{array} \tag{6}$$

for all a, b, c vector fields. The 1-forms θ, θ^* and ω related with Γ are introduced as

$$\theta(a) = g^{ij}\Gamma(f_i, f_j, a), \quad \theta^*(a) = g^{ij}\Gamma(f_i, \varphi(f_j), a), \quad \omega(a) = \Gamma(\xi, \xi, a). \tag{7}$$

Here $\{f_1, \cdots, f_{2n}, \xi\}$ is a local frame, the inverse matrix of (g_{ij}) is denoted by (g^{ij}) and $a \in \chi(M)$ [7].

Using properties (6), the space of Levi-Civita connections of the endomorphism φ are defined as

$$\mathcal{F} = \{\Gamma \in \otimes_3^0 : \Gamma(a,b,c) = \Gamma(a,c,b) = \Gamma(a,\varphi(b),\varphi(c)) + \eta(b)\Gamma(a,\xi,c) + \eta(c)\Gamma(a,b,\xi)\}.$$

The space $\mathcal{F}$ is decomposed as

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_{11}.$$

The subspaces $\mathcal{F}_i$ are invariant and orthogonal with respect to the action of $G \times I$, where $G = GL(n,\mathbb{C}) \cap O(n,n)$, i.e., G is the group of real matrices $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ which belong to $O(n,n)$, A and B are $n \times n$ matrices [7].

Any almost contact manifold with B-metric belongs to a subclass $\mathcal{F}_{i_1} \oplus \cdots \oplus \mathcal{F}_{i_k}$ for $1 \le i_1 \le \cdots \le i_k \le 11$ of $\mathcal{F}$. The defining rules of classes we use are [7]:

$$\mathcal{F}_4 : \Gamma(a,b,c) = -\frac{\theta(\xi)}{2n}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right), \tag{8}$$

$$\mathcal{F}_5 : \Gamma(a,b,c) = -\frac{\theta^*(\xi)}{2n}\left(\eta(b)g(\varphi(a),c) + \eta(c)g(\varphi(a),b)\right). \tag{9}$$

An even-dimensional semi-Riemannian manifold N having an almost complex structure J and a semi-Riemannian metric h such that $h(J(a),J(b)) = -h(a,b)$ is called an almost complex manifold with a Norden metric. $G = GL(n,\mathbb{C}) \cap O(n,n)$ is the structure group of N, where $GL(n,\mathbb{C}) \cap O(n,n)$ is the group of real matrices

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

which are in $O(n,n)$ (A and B are $n \times n$ matrices) [6].

Almost complex manifolds with Norden metric are classified by considering the Levi-Civita connection ∇J of J. The following notation is used

$$\Upsilon(a,b,c) := h\left((\nabla_a J)\,b,c\right).$$

Υ satisfies

$$\Upsilon(a,b,c) = \Upsilon(a,c,b) \text{ and } \Upsilon(a,J(b),J(c)) = \Upsilon(a,b,c).$$

The 1-form Θ related with Υ is given by

$$\Theta(a) = h^{ij}\Upsilon(f_i,f_j,a) \tag{10}$$

for all $a \in \chi(N)$, where $\{f_1, f_2, \cdots, f_{2n}\}$ is a local frame on N and (h^{ij}) is the inverse matrix of h. The tensor Υ belongs to the space

$$W = \left\{\Upsilon \in \otimes_3^0 \; : \; \Upsilon(a,b,c) = \Upsilon(a,c,b) = \Upsilon(a, J(b), J(c))\right\},$$

which splits into a direct sum of three subspaces W_i, $i = 1,2,3$ [5]. Defining relations of almost complex manifolds with a Norden metric are:

1. Kaehlerian Norden metric manifolds: $\Upsilon(a,b,c) = 0$ for all $a,b,c \in \chi(N)$.

2. Class W_1 (Conformally Kaehlerian manifolds with a Norden metric):

$$\begin{aligned} \Upsilon(a,b,c) &= \frac{1}{2n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &\left. +h(a,J(b))\Theta(J(c)) + h(a,J(c))\Theta(J(b))\right). \end{aligned} \tag{11}$$

3. Class W_2 (Special complex manifolds with a Norden metric):

$$\Upsilon(a,b,J(c)) + \Upsilon(b,c,J(a)) + \Upsilon(c,a,J(b)) = 0, \tag{12}$$

$$\Theta = 0. \tag{13}$$

4. Class W_3 (Quasi-Kaehlerian manifolds with a Norden metric):

$$\Upsilon(a,b,c) + \Upsilon(b,c,a) + \Upsilon(c,a,b) = 0. \tag{14}$$

5. Class $W_1 \oplus W_2$ (Complex manifolds with a Norden metric):

$$\Upsilon(a,b,J(c)) + \Upsilon(b,c,J(a)) + \Upsilon(c,a,J(b)) = 0.$$

6. Class $W_1 \oplus W_3$:

$$\begin{aligned} \Upsilon(a,b,c) + \Upsilon(b,c,a) + \Upsilon(c,a,b) &= \frac{1}{n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &+ h(b,c)\Theta(a) + h(a,J(b))\Theta(J(c)) \\ &+ \left. h(b,J(c))\Theta(J(a)) + h(c,J(a))\Theta(J(b))\right) \end{aligned} \tag{15}$$

7. Class $W_2 \oplus W_3$ (Semi-Kaehlerian manifolds with a Norden metric):

$$\Theta = 0.$$

8. Class $W_1 \oplus W_2 \oplus W_3$ (No relation):

Any $\Upsilon \in W$ can be written as $\Upsilon = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 \in W$, where $\Upsilon_i \in W_i$. The projections Υ_i are given below [6]:

$$\begin{aligned} \Upsilon_1(a,b,c) &= \frac{1}{2n}\left(h(a,b)\Theta(c) + h(a,c)\Theta(b)\right. \\ &\left. +h(a,J(b))\Theta(J(c)) + h(a,J(c))\Theta(J(b))\right), \end{aligned} \tag{16}$$

$$\begin{aligned}\Upsilon_2(a,b,c) &= -\frac{1}{2n}\left(h(a,b)\Theta(c)+h(a,c)\Theta(b)\right.\\ &\quad +h(a,J(b))\Theta(J(c))+h(a,J(c))\Theta(J(b)))\\ &\quad +\frac{1}{4}\left(2\Upsilon(a,b,c)+\Upsilon(b,c,a)+\Upsilon(c,a,b)\right.\\ &\quad \left.-\Upsilon(J(b),c,J(a))+\Upsilon(J(c),a,J(b))\right),\end{aligned} \tag{17}$$

$$\begin{aligned}\Upsilon_3(a,b,c) &= \frac{1}{4}\left(2\Upsilon(a,b,c)-\Upsilon(b,c,a)-\Upsilon(c,a,b)\right.\\ &\quad \left.+\Upsilon(J(b),c,J(a))-\Upsilon(J(c),a,J(b))\right).\end{aligned} \tag{18}$$

3. Almost Complex Manifolds with Norden Metric from Almost Contact Manifolds with B-Metric

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric, $\dim M = 2n+1$. Consider a vector field $\left(a,\alpha\frac{d}{dt}\right)$ on $M\times\mathbb{R}$, where a is a vector field on M, t is the coordinate of $\mathbb{R}$ and α is a C^∞ function on $M\times\mathbb{R}$. On $M\times\mathbb{R}$ we define an almost complex structure with a Norden metric $(\tilde{J},\tilde{h})$ with respect to the functions σ and μ on $M\times\mathbb{R}$, where σ and μ depend only on t as

$$\tilde{J}\left(a,\alpha\frac{d}{dt}\right) := \left(\varphi(a)-\alpha e^{-(\sigma+\mu)}\xi, e^{(\sigma+\mu)}\eta(a)\frac{d}{dt}\right), \tag{19}$$

$$\tilde{h}\left(\left(a,\alpha\frac{d}{dt}\right),\left(b,\beta\frac{d}{dt}\right)\right) := e^{2\sigma}g(a,b)+e^{2\sigma}(e^{2\mu}-1)\eta(a)\eta(b)-\alpha\beta. \tag{20}$$

In this study, we use the notation a,b,c for vector fields on M. In addition, we use A,B,C to denote vector fields on M such that $A,B,C\in Ker\eta$.

Using the Kozsul formula, we evaluate the components of Levi-Civita covariant derivative $\tilde{\nabla}$ of $\tilde{h}$ which are different than zero as

$$\begin{aligned}
\tilde{h}(\tilde{\nabla}_A B,C) &= e^{2\sigma}g(\nabla_A B,C),\\
\tilde{h}(\tilde{\nabla}_A B,\xi) &= e^{2\sigma}g(\nabla_A B,\xi)-e^{2\sigma}(e^{2\mu}-1)d\eta(A,B),\\
\tilde{h}(\tilde{\nabla}_A B,\frac{d}{dt}) &= -e^{2\sigma}\frac{d\sigma}{dt}g(A,B),\\
\tilde{h}(\tilde{\nabla}_A \xi,C) &= e^{2\sigma}g(\nabla_A\xi,C)+e^{2\sigma}(e^{2\mu}-1)d\eta(A,C),\\
\tilde{h}(\tilde{\nabla}_A \tfrac{d}{dt},C) &= e^{2\sigma}\frac{d\sigma}{dt}g(A,C),\\
\tilde{h}(\tilde{\nabla}_\xi B,C) &= e^{2\sigma}g(\nabla_\xi B,C)+e^{2\sigma}(e^{2\mu}-1)d\eta(B,C),\\
\tilde{h}(\tilde{\nabla}_\xi B,\xi) &= e^{2(\sigma+\mu)}g(\nabla_\xi B,\xi),\\
\tilde{h}(\tilde{\nabla}_\xi \xi,C) &= e^{2(\sigma+\mu)}g(\nabla_\xi\xi,C),\\
\tilde{h}(\tilde{\nabla}_\xi \xi,\frac{d}{dt}) &= -e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}),\\
\tilde{h}(\tilde{\nabla}_\xi \tfrac{d}{dt},\xi) &= e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}),\\
\tilde{h}(\tilde{\nabla}_{\frac{d}{dt}} B,C) &= e^{2\sigma}\frac{d\sigma}{dt}g(B,C),\\
\tilde{h}(\tilde{\nabla}_{\frac{d}{dt}} \xi,\xi) &= e^{2(\sigma+\mu)}(\frac{d\sigma}{dt}+\frac{d\mu}{dt}).
\end{aligned}$$

Then, we write down the non-zero components of $\tilde{\nabla}\tilde{J}$ as

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),C)=e^{2\sigma}g((\nabla_A\varphi)(B),C), \tag{21}$$

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),\xi) = e^{2\sigma}\Big(g(\nabla_A\varphi(B),\xi)+e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,B) -(e^{2\mu}-1)d\eta(A,\varphi(B))\Big), \tag{22}$$

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(B),\frac{d}{dt}) = -e^{2\sigma}\frac{d\sigma}{dt}g(A,\varphi(B))+e^{\sigma-\mu}g(\nabla_A B,\xi) -e^{\sigma-\mu}(e^{2\mu}-1)d\eta(A,B), \tag{23}$$

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(\xi),C) = e^{3\sigma+\mu}\frac{d\sigma}{dt}g(A,C)-e^{2\sigma}g(\nabla_A\xi,\varphi(C)) -e^{2\sigma}(e^{2\mu}-1)d\eta(A,\varphi(C)), \tag{24}$$

$$\tilde{h}((\tilde{\nabla}_A\tilde{J})(\frac{d}{dt}),C) = -e^{\sigma-\mu}g(\nabla_A\xi,C)-e^{\sigma-\mu}(e^{2\mu}-1)d\eta(A,C) -e^{2\sigma}\frac{d\sigma}{dt}g(A,\varphi(C)), \tag{25}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),C) = e^{2\sigma}g((\nabla_\xi\varphi)(B),C) +e^{2\sigma}(e^{2\mu}-1)\left(d\eta(\varphi(B),C)-d\eta(B,\varphi(C))\right), \tag{26}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),\xi) = e^{2(\sigma+\mu)}g(\nabla_\xi\varphi(B),\xi), \tag{27}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(B),\frac{d}{dt}) = e^{\sigma+\mu}g(\nabla_\xi B,\xi), \tag{28}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\xi),C) = e^{2(\sigma+\mu)}g(\nabla_\xi\xi,\varphi(C)), \tag{29}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\frac{d}{dt}),C) = -e^{\sigma+\mu}g(\nabla_\xi\xi,C), \tag{30}$$

$$\tilde{h}((\tilde{\nabla}_\xi\tilde{J})(\xi),\xi) = 2e^{3(\sigma+\mu)}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{31}$$

$$\tilde{h}\left((\tilde{\nabla}_\xi\tilde{J})\left(\frac{d}{dt}\right),\frac{d}{dt}\right) = 2e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{32}$$

$$\tilde{h}\left((\tilde{\nabla}_{\frac{d}{dt}}\tilde{J})(\xi),\frac{d}{dt}\right) = e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right), \tag{33}$$

$$\tilde{h}\left((\tilde{\nabla}_{\frac{d}{dt}}\tilde{J})(\frac{d}{dt}),\xi\right) = -e^{\sigma+\mu}\left(\frac{d\sigma}{dt}+\frac{d\mu}{dt}\right). \tag{34}$$

Then, we have the Theorem 3.1.

Theorem 3.1 $\tilde{\nabla}\tilde{J} = 0$ *if and only if relations below are satisfied*

$$\Gamma(A,B,C) = \Gamma(\xi,\xi,C) = 0, \tag{35}$$

$$\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0, \tag{36}$$

$$\Gamma(\xi,B,C) = 0, \tag{37}$$

$$\Gamma(A,B,\xi) = -e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,B) \tag{38}$$

for all $A,B,C \in Ker\eta$.

Proof Let $\tilde{\nabla}\tilde{J} = 0$. From Equations (21), (27)-(34), we get Equations (35), (36) and $\tilde{\nabla}_{\xi}\xi = 0$. Also, from Equation (25), we obtain

$$g(\nabla_A\xi, C) = -\left(e^{2\mu} - 1\right)d\eta(A,C) - e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,\varphi(C)). \tag{39}$$

Then, Equation (39) implies $d\eta = 0$. In addition, from Equation (26), we obtain $\beta(\xi,B,C) = 0$. Also, Equation (22) gives the relation (38). The converse of proof is clear. □

Now, we state Theorem 3.2 which is used to prove Theorem 3.3.

Theorem 3.2 *Assume* (M,φ,ξ,η,g) *is an almost contact manifold with B-metric. The followings are equivalent:*

(i) (M,φ,ξ,η,g) *satisfies the Equations (35), (37) and (38).*

(ii) (M,φ,ξ,η,g) *satisfies*

$$\Gamma(a,b,c) = e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right) \tag{40}$$

for all $a,b,c \in \chi(M)$.

Proof Let (M,φ,ξ,η,g) satisfy (35), (37) and (38). Take

$$\begin{array}{lcll} a &=& a-\eta(a)\xi+\eta(a)\xi = A+\eta(a)\xi, & A = a-\eta(a)\xi \\ b &=& b-\eta(b)\xi+\eta(b)\xi = B+\eta(b)\xi, & B = b-\eta(b)\xi \\ c &=& c-\eta(c)\xi+\eta(c)\xi = C+\eta(c)\xi, & C = c-\eta(c)\xi, \end{array}$$

where $A,B,C \in Ker\eta$. Then, we obtain

$$\begin{array}{rcl} \Gamma(a,b,c) &=& \Gamma\left(A+\eta(a)\xi, B+\eta(b)\xi, C+\eta(c)\xi\right) \\ &=& \Gamma(A,B,C)+\eta(c)\Gamma(A,B,\xi)+\eta(b)\Gamma(B,C,\xi) \\ && \eta(a)\Gamma(\xi,B,C)+\eta(a)\eta(c)\Gamma(\xi,\xi,B)+\eta(a)\eta(b)\Gamma(\xi,\xi,C) \\ &=& \eta(c)\Gamma(A,B,\xi)+\eta(b)\Gamma(A,C,\xi) \\ &=& -e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(c)g(A,B)+\eta(b)g(A,C)\right) \\ &=& e^{\sigma+\mu}\frac{d\sigma}{dt}\left(\eta(c)g(\varphi(a),\varphi(b))+\eta(b)g(\varphi(a),\varphi(c))\right). \end{array} \tag{41}$$

The proof of converse is trivial. □

Consider the defining relation of $\mathcal{F}_4$ of almost contact manifold with B-metric

$$\Gamma(a,b,c) = -\frac{\theta(\xi)}{2n}\left(\eta(b)g(\varphi(a),\varphi(c)) + \eta(c)g(\varphi(a),\varphi(b))\right).$$

Choose functions σ and μ so that

$$-\frac{\theta(\xi)}{2n} = e^{\sigma+\mu}\frac{d\sigma}{dt}. \tag{42}$$

Then, M is in $\mathcal{F}_4$. However, the Equation (42) has a solution if $\theta(\xi)$ is a constant real number. Consequently, the Theorem 3.3 is stated.

Theorem 3.3 *Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric. $(M\times\mathbb{R},\tilde{J},\tilde{h})$ is Kaehlerian manifold with Norden metric iff the manifold M is of the class $\mathcal{F}_4$, $\theta(\xi)$ is a real number and following equalities are satisfied*

$$e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n}, \quad \frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0. \tag{43}$$

Proof If $M\times\mathbb{R}$ is a Kaehlerian Norden metric manifold, from Theorem 3.1, we have Equations (35) - (38). Also from Theorem 3.2, we get the Equation (40). If functions σ and μ are chosen to satisfy

$$e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n},$$

then M is of the class $\mathcal{F}_4$ since $\theta(\xi)$ is constant.

On the contrary, if M is of the class $\mathcal{F}_4$, $\theta(\xi)$ is constant and Equation (43) holds, then we have

$$\sigma(t) + \mu(t) = c, \quad c \in \mathbb{R}.$$

In addition, the differential equation $e^{\sigma+\mu}\frac{d\sigma}{dt} = -\frac{\theta(\xi)}{2n}$ has the solutions

$$\sigma(t) = -\frac{\theta(\xi)}{2n}e^{-c}t + c_1, \quad \mu(t) = c + \frac{\theta(\xi)}{2n}e^{-c}t - c_1, \quad c_1 \in \mathbb{R}. \tag{44}$$

If σ and μ are chosen as in (44), then $(M\times\mathbb{R},\tilde{J},\tilde{h})$ is in trivial class. In fact, we obtain an infinite number of Kaehlerian manifolds with a Norden metric depending on c and c_1. □

Example 3.4 *Assume G is a five dimensional Lie group, take a basis $\{x_0,x_1,x_2,x_3,x_4\}$ of left-invariant vector fields such that the non-zero Lie brackets are*

$$[x_0,x_1] = \lambda x_2 + x_3 + \mu x_4, \quad [x_0,x_2] = -\lambda x_1 - \mu x_3 + x_4,$$

$$[x_0, x_3] = -x_1 - \mu x_2 + \lambda x_4, \quad [x_0, x_4] = \mu x_1 - x_2 - \lambda x_3,$$

where λ and μ are constants. Let g be the metric satisfying

$$g(x_0, x_0) = g(x_1, x_1) = g(x_2, x_2) = 1, \quad g(x_3, x_3) = g(x_4, x_4) = -1,$$

$$g(x_i, x_j) = 0, \quad i, j \in \{0, 1, \cdots, 4\}, i \neq j.$$

If we take $\xi = x_0$, $\varphi(x_1) = x_3$ and $\varphi(x_2) = x_4$, then (ξ, η, φ, g) is an almost contact structure with B-metric, where η is dual 1-form of x_0. From the Kozsul formula, we evaluate the non-zero Levi-Civita covariant derivative as

$$\nabla_{x_0} x_1 = \lambda x_2 + \mu x_4, \quad \nabla_{x_0} x_2 = -\lambda x_1 - \mu x_3,$$

$$\nabla_{x_0} x_3 = -\mu x_2 + \lambda x_4, \quad \nabla_{x_0} x_4 = \mu x_1 - \lambda x_3,$$

$$\lambda_{x_1} x_0 = -x_3, \quad \lambda_{x_2} x_0 = -x_4, \quad \lambda_{x_3} x_0 = x_1, \quad \lambda_{x_4} x_0 = x_2,$$

$$\lambda_{x_1} x_3 = \lambda_{x_2} x_4 = \lambda_{x_3} x_1 = \lambda_{x_4} x_2 = -x_0.$$

$(G, \varphi, \xi, \eta, g)$ is of class $\mathcal{F}_4$ with $\theta(\xi) = -2n$ [9]. If we take $\sigma(t) = e^{-c} t + c_1$, $\mu(t) = c - e^{-c} t - c_1$, where c and c_1 are arbitrary real numbers, then $G \times \mathbb{R}$ is a Kaehlerian manifold with a Norden metric.

Let $\{f_1, \cdots, f_n, \varphi(f_1), \cdots, \varphi(f_n), \xi\}$ be an orthonormal frame on open set U of M such that

$$g(f_i, f_i) = 1, \; g(\varphi(f_i), \varphi(f_i)) = -1, \quad g(\xi, \xi) = 1, \; 1 \leq i \leq n,$$

$$g(f_i, f_j) = g(\varphi(f_i), \varphi(f_j)) = g(f_i, \varphi(f_j)) = 0 \text{ for } i \neq j, \; 1 \leq i, j \leq n.$$

Then,

$$\left\{ (e^{-\sigma} f_1, 0), (e^{-\sigma} f_2, 0), \cdots, (e^{-\sigma} f_n, 0), (e^{-\sigma} \varphi(f_1), 0), \cdots, (e^{-\sigma} \varphi(f_n), 0), \left(e^{-(\sigma+\mu)} \xi, 0\right), \left(0, \frac{d}{dt}\right) \right\}$$

is an orthonormal frame of $\tilde{h}$ on the open subset $U \times \mathbb{R}$ of $M \times \mathbb{R}$. By using this frame, $\tilde{\Theta}\left(a, \alpha \frac{d}{dt}\right)$ is obtained by direct calculation:

$$\begin{aligned} \tilde{\Theta}\left(a, \alpha \frac{d}{dt}\right) &= \theta(a) - \alpha e^{-(\sigma+\mu)} \theta^*(\xi) + 2n e^{\sigma+\mu} \eta(a) \frac{d\sigma}{dt} \\ &\quad + 3e^{\sigma+\mu} \left(\frac{d\sigma}{dt} + \frac{d\mu}{dt} \right) \eta(a) + g\left(\nabla_\xi \xi, \varphi(a)\right). \end{aligned} \tag{45}$$

Let M be in $\mathcal{F}_5$. We investigate the class of $M \times \mathbb{R}$.

Theorem 3.5 *If (M,φ,ξ,η,g) is in $\mathcal{F}_5$ and $\frac{d\sigma}{dt}+\frac{d\mu}{dt}=0$, then $(M\times\mathbb{R},\tilde{J},\tilde{h})$ belongs to $W_1\oplus W_2$.*

Proof Since M is in $\mathcal{F}_5$, Equation (9) is satisfied. In the class $\mathcal{F}_5$, we have

$$\nabla_a\xi=-\frac{\theta^*(\xi)}{2n}\varphi^2(a),\quad d\eta=0.$$

In addition, since $\frac{d\sigma}{dt}+\frac{d\mu}{dt}=0$, the only components of Levi-Civita covariant derivative of $\tilde{J}$ which do not vanish are

$$\begin{aligned}
\tilde{g}\left((\tilde{\nabla}_A J)(B),\xi\right) &= -e^{2\sigma}\left(\frac{\theta^*(\xi)}{2n}g(A,\varphi(B))-e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,B)\right),\\
\tilde{g}\left((\tilde{\nabla}_A J)(B),\frac{d}{dt}\right) &= -e^{2\sigma}\left(\frac{d\sigma}{dt}g(A,\varphi(B))+e^{-(\sigma+\mu)}\frac{\theta^*(\xi)}{2n}g(A,B)\right),\\
\tilde{g}\left((\tilde{\nabla}_A J)(\xi),C\right) &= e^{2\sigma}\left(e^{\sigma+\mu}\frac{d\sigma}{dt}g(A,C)-\frac{\theta^*(\xi)}{2n}g(A,\varphi(C))\right),\\
\tilde{g}\left((\tilde{\nabla}_A J)\left(\frac{d}{dt}\right),C\right) &= -e^{2\sigma}\left(e^{-(\sigma+\mu)}\frac{\theta^*(\xi)}{2n}g(A,C)+\frac{d\sigma}{dt}g(A,\varphi(C))\right).
\end{aligned}$$

Also, by direct calculation we have

$$\tilde{\Theta}\left(a,\alpha\frac{d}{dt}\right)=-\alpha e^{-(\sigma+\mu)}\theta^*(\xi)+2ne^{\sigma+\mu}\eta(a)\frac{d\sigma}{dt}. \tag{46}$$

In addition, since

$$\Upsilon_1\left(\left(0,\frac{d}{dt}\right),(\xi,0),(\xi,0)\right)=\frac{1}{n}e^{\sigma+\mu}\theta^*(\xi)\neq 0 \tag{47}$$

and

$$\Upsilon_2\left(\left(0,\frac{d}{dt}\right),(\xi,0),(\xi,0)\right)=-\frac{1}{n}e^{\sigma+\mu}\theta^*(\xi)\neq 0, \tag{48}$$

the projections α_1,α_2 are non-zero. By direct calculation

$$\Upsilon_3\left(\left(a,\alpha\frac{d}{dt}\right),\left(b,\beta\frac{d}{dt}\right),\left(c,\gamma\frac{d}{dt}\right)\right)=0. \tag{49}$$

Hence, $M\times\mathbb{R}$ is of the class $W_1\oplus W_2$. □

Example 3.6 *Let $\mathbb{R}^{2n+2}=\{(a_1,\cdots,a_{n+1},b_1,\cdots,b_{n+1})\ :\ a_i,b_i\in\mathbb{R}\}$. Consider the canonical complex structure*

$$J\left(\frac{\partial}{\partial a_i}\right)=\frac{\partial}{\partial b_i},\quad J\left(\frac{\partial}{\partial b_i}\right)=-\frac{\partial}{\partial a_i},\quad 1\leq i\leq n+1$$

and

$$g(u,u) = -\delta_{ij}x_i x_j + \delta_{ij}y_i y_j,$$

where $u = x_i \frac{\partial}{\partial a_i} + y_i \frac{\partial}{\partial b_i}$. *Identify the point* $p = (a_1, \cdots, a_{n+1}, b_1, \cdots, b_{n+1})$ *in* $\mathbb{R}^{2n+2}$ *with its position vector* P. *Let* M *be the hypersurface of* $\mathbb{R}^{2n+2}$ *determined by*

$$M = \left\{ P \in \mathbb{R}^{2n+2} \ : \ g(P, J(P)) = 0, \ g(P,P) > 0 \right\}.$$

Define vector field ξ *as*

$$\xi = -\frac{1}{\cosh t} P,$$

where $t \in (-\pi/2, \pi/2)$. *For any vector field* u, *we can define* φ *with regard to the unique decomposition*

$$J(u) = \varphi(u) + \frac{1}{\cosh t}\eta(u) J(P).$$

$(M, \varphi, \xi, \eta, g)$ *is in* $\mathcal{F}_5$ *[7]. From the Theorem 3.5, by choosing the functions* σ *and* μ *to satisfy* $\frac{d\sigma}{dt} + \frac{d\mu}{dt} = 0$, $M \times \mathbb{R}$ *is of the class* $W_1 \oplus W_2$.

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Nülifer Özdemir]: Thought and designed the research/problem, contributed to research method or evaluation of data, collected the data, wrote the manuscript (%50).

Author [Elanur Eren]: Collected the data, contributed to completing the research and solving the problem (%50).

Conflicts of Interest

The authors declare no conflict of interest.

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