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LATTICE STRUCTURES OF MULTI-FUZZY SOFT SETS

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Abstract: The multi-fuzzy soft set theory has recently been introduced and it has started to be applied in some fields such as decision making and medical diagnosis. In this paper, algebraic structure of multi-fuzzy soft sets is studied. Several related properties of some operations on multi-fuzzy soft sets are investigated. Two lattice structures of multi-fuzzy soft sets are constructed. It is shown that these lattices are distributive and whence modular. Additionally, the ordering relations on the lattices of multi-fuzzy soft sets are presented. Moreover, by giving an example, it is indicated that some pairs of operations on multi-fuzzy soft sets do not satisfy the absorption rule which is necessary to form a lattice. So it is proved that a lattice structure cannot be constructed by using these operations.

Keywords: Lattice, Multi-fuzzy set, Multi-fuzzy soft set

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1. Introduction

A big part of our lives is full of uncertainty and vagueness. Uncertainties are problems waiting to be solved for us. Traditional tools are not always successful to solve these problems. While probability theory, fuzzy set theory (Zadeh, 1965), rough set theory (Pawlak, 1982) and other mathematical tools are well-known and often useful approaches to describe uncertainty, each of these theories has its inherent difficulties as pointed out in Molodtsov (1999)'s paper that introduced the notion of soft set to deal with uncertainty. From then on, the soft set model has been combined with other mathematical models. Sun et al. (2008) proposed the notion of soft modules and studied their basic properties. Jun (2008) introduced the concept of soft BCK/BCI-algebras. Feng et al. (2008) initiated the study of soft semirings, soft ideals on soft semirings and idealistic soft semirings. Kazancı et al. (2010) introduced the concepts of soft BCH-algebra and soft BCH-subalgebra. They discussed some of their properties and structural characteristics. Qin and Hong (2010) gave the lattice structures of soft sets and introduced the concept of soft equality.

Fuzzy set theory was initiated by Zadeh (1965). Maji et al. (2001) presented the definition of fuzzy soft set, which is a combination of fuzzy set and soft set, and they studied its properties. Further, Aygünoğlu and Aygün (2009) introduced the concept of fuzzy soft group and discussed some of their properties. Majumdar and Samanta (2010) generalized the notion of fuzzy soft sets as introduced by Maji et al. (2001). The definition of generalized fuzzy soft set is more practical than the definition of fuzzy soft set as it adds one more degree to the parametrization of the fuzzy set. Yang (2011) presented the notions of fuzzy soft semigroup and fuzzy soft ideal.

Birkhoff's work in 1930s started the general development of lattice theory (Birkhoff, 1984). Because of the lattices are one of the algebraic structures widely used and discussed in mathematics and its applications, many authors focused on studying lattice structures of algebraic systems. For example, Shao and Qin (2012) applied the notion of fuzzy soft set to lattice theory and they investigated the algebraic structure of fuzzy soft lattices.

The concept of multi-fuzzy sets presents a new method that contributes to explaining some problems that are difficult to represent with fuzzy set theory. Sebastian and Ramakrishnan (2011a) proposed the concept of multi-fuzzy set which is a more general fuzzy set using ordinary fuzzy sets. Yang et al. (2013) combined the multi-fuzzy set and soft set models. They introduced the concept of multi-fuzzy soft sets and defined some operations on multi-fuzzy soft sets. Akın (2021) applied the multi-fuzzy soft sets to the theory of groups and form a new algebraic structure which is called a multi-fuzzy soft group as an extension of multi-fuzzy sets. Kazancı et al. (2022) combined the multi-fuzzy soft set and polygroup structure, from which they obtain a new soft structure called the multi-fuzzy soft polygroup. The organization of this paper is as follows: In Section 2, some basic definitions and theorems of the lattice theory, soft set theory, fuzzy soft sets, multi-fuzzy sets and multi-fuzzy soft sets are introduced. In Section 3, the lattice structure of multi-fuzzy soft sets is studied. Some algebraic structures of multi-fuzzy soft sets are presented. In Section 4, the results obtained in this work are summarized.

2. Preliminaries

In this section, we present some basic definitions and facts related to lattices, soft sets, fuzzy soft sets, multi-fuzzy sets and multi-fuzzy soft sets; see the references (Birkhoff, 1984; Molodtsov, 1999; Skornjakov, 1977; Gratzer, 1978; Zadeh, 1965; Maji et al., 2001; Sebastian and Ramakrishnan, 2011 a; Sebastian and Ramakrishnan, 2011 b; Yang et al., 2013; Kazancı et al., 2022).

By a partly ordered set is meant a system X in which a binary relation " $\leq$ " is reflexive, antisymmetric and transitive. An upper bound of a subset X of a partially ordered set P is an element $a \in P$ which is greater than every $x \in X$. A least upper bound (l.u.b.) is an upper bound lesser than every other upper bound of X. The notions of a lower bound and greatest lower bound *(g.l.b.)* are defined dually (Birkhoff, 1984).

Definition 2.1. A *lattice* is a partly ordered set in which any pair of elements a, b have a g.l.b. or "meet" $a \wedge b$ and a l.u.b. or "join" $a \vee b$ (Birkhoff, 1984).

Theorem 2.2. For a lattice L, the following identities are satisfied, for all $a, b, c \in L$ (Skornjakov, 1977):

(i) $a \vee a = a, a \wedge a = a$,

(ii) $a \vee b = b \vee a, a \wedge b = b \wedge a$,

(iii) $a \vee (b \vee c) = (a \vee b) \vee c$,
$a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

(iv) $a = a \vee (a \wedge b), a = a \wedge (a \vee b)$

Let L be the set of propositions, $\vee$ denote the connective "or" and $\wedge$ denote the connective "and". Then (i) to (iv) are well-known properties from propositional logic.

Let L be a set endowed with two binary operations $'' \wedge ''$ and $'' \vee ''$ which satisfy the identities given in Theorem 2.2. Then if we set

$a \leq b \Leftrightarrow a \wedge b = a$ (or $a \leq b \Leftrightarrow a \vee b = b$),

then L is a lattice with the ordering relation $'' \leq ''$ (Gratzer, 1978).

Definition 2.3. The algebra $(L, \circ)$ is a *semilattice* iff " $\circ$ " is idempotent, commutative and associative (Gratzer, 1978).

Definition 2.4. Let L be a lattice and L' be a subset of L such that for every pair of elements a, b in L' both $a \vee b$ and $a \wedge b$ are in L', then we say that L' with the same operations is a *sublattice e* of L (Birkhoff, 1984).

For a lattice L, $\emptyset$ is considered as a sublattice of L.

Definition 2.5. Let L be a partially ordered set. L is a *complete lattice* in which every subset had a least upper bound and a greatest upper bound (Birkhoff, 1984).

Theorem 2.6. Let L be a partly ordered set with 1 and every non-void subset of L have a g.l.b., then L is a complete lattice (Birkhoff, 1984).

Definition 2.7. A *distributive lattice* is a lattice which satisfies either of the distributive laws, for all $a, b, c \in L$,

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

One can see a lattice L satisfies (i) if and only if satisfies (ii) (Birkhoff, 1984).

Definition 2.8. A lattice is called a *modular lattice* if and only if its elements satisfy the condition,

if $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Any distributive lattice is modular (Birkhoff, 1984).

Definition 2.9. Let U be an initial universe set and E be a set of parameters. $P(U)$ denotes the power set of U and $A \subseteq E$. A pair (F, A) is called a *soft set* over U, where F is a set-valued function $F: A \to P(U)$ can be defined as

$F(x) = \{y \in P(U): (x, y) \in R\}$, for all $x \in A$

and R will refer to an arbitrary binary relation between an element of A and an element of U, that is, R is a subset of $A \times U$. In fact, a soft set over U is a parameterized family of subsets of universe U (Molodtsov,1999).

Definition 2.10. Let X be a non-empty set. A *fuzzy subset* μ of X is a function $\mu: X \to [0,1]$ (Zadeh, 1965).

Definition 2.11. Let $\tilde{P}(U)$ be the set of all fuzzy subsets of U. A pair (F, A) is called *fuzzy soft set* over U, where F is a mapping given by $F: A \to \tilde{P}(U)$. That is, for each $a \in A$,
$F(a) = F_a: U \to [0,1]$ is a fuzzy set on U (Maji et al., 2001).

Definition 2.12. Let k be a positive integer. A *multi-fuzzy set $\tilde{A}$* in U is a set of ordered sequences

$\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)): u \in U\}$,

where $\mu_i \in \tilde{P}(U)$, $i = 1,2,\ldots,k$. The function $\mu_{\tilde{A}} = (\mu_1, \ldots, \mu_k)$ is called the multi membership function of multi-fuzzy set $\tilde{A}$ denoted by $MM_{\tilde{A}}$, k is called a dimension of $\tilde{A}$. The set of all multi-fuzzy sets of dimension k in U is denoted by $M^kF^S(U)$ (Sebastian and Ramakrishnan, 2011a).

Definition 2.13. Let $\tilde{A} \in M^kF^S(U)$. If
$\tilde{A} = \{u/(0, 0, \ldots, 0): u \in U\}$, then $\tilde{A}$ is called the *null multi-fuzzy* set of dimension k, denoted by $\tilde{\emptyset}_k$. If
$\tilde{A} = \{u/(1, 1, \ldots, 1): u \in U\}$, then $\tilde{A}$ is called the *absolute multi-fuzzy* set of dimension k, denoted by $\tilde{1}_k$ (Sebastian and Ramakrishnan, 2011a).

Definition 2.14. Let
$\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \ldots, \mu_k(u)): u \in U\}$,
$\tilde{B} = \{u/(\gamma_1(u), \gamma_2(u), \ldots, \gamma_k(u)): u \in U\} \in M^kF^s(U)$.
We define the following relations and operations.

(i) $\tilde{A} \sqsubseteq \tilde{B}$ if and only if $MM_{\tilde{A}} \leq MM_{\tilde{B}}$, i.e $\mu_i(u) \leq \gamma_i(u), \forall u \in U$ and $1 \leq i \leq k$.

(ii) $\tilde{A} = \tilde{B}$ if and only if $MM_{\tilde{A}} = MM_{\tilde{B}}$, i.e $\mu_i(u) = \gamma_i(u), \forall u \in U$ and $1 \leq i \leq k$.

(iii) $\tilde{A} \sqcup \tilde{B} = \{u / (\mu_1(u) \vee \gamma_1(u), \ldots, \mu_k(u) \vee \gamma_k(u)): u \in U\}$. That is $MM_{\tilde{A} \sqcup \tilde{B}} = MM_{\tilde{A}} \vee MM_{\tilde{B}}$.

(iv) $\tilde{A} \sqcap \tilde{B} = \{u / (\mu_1(u) \wedge \gamma_1(u), \dots, \mu_k(u) \wedge \gamma_k(u)) : u \in U\}$. That is $MM_{\tilde{A}\sqcap\tilde{B}} = MM_{\tilde{A}} \wedge MM_{\tilde{B}}$.

(v) $\tilde{A}^c = \{u/(\mu_1{}^c, \mu_2{}^c, \dots, \mu_k{}^c) : u \in U\}$

(Sebastian and Ramakrishnan, 2011 a).

Theorem 2.15. Let $\tilde{A}, \tilde{B}, \tilde{C} \in M^k F^s(U)$. Then

(i) $\tilde{A} \sqcup \tilde{A} = \tilde{A}, \tilde{A} \sqcap \tilde{A} = \tilde{A}$,

(ii) $\tilde{A} \sqsubseteq \tilde{A} \sqcup \tilde{B}, \tilde{B} \sqsubseteq \tilde{A} \sqcup \tilde{B}$,

$\tilde{A} \sqcap \tilde{B} \sqsubseteq \tilde{A}, \tilde{A} \sqcap \tilde{B} \sqsubseteq \tilde{B}$.

(iii) $\tilde{A} \sqsubseteq \tilde{B}$ if and only if $\tilde{A} \sqcup \tilde{B} = \tilde{B}$ and $\tilde{A} \sqcap \tilde{B} = \tilde{A}$

(Sebastian and Ramakrishnan, 2011b).

Theorem 2.16. Let $\tilde{A}, \tilde{B}, \tilde{C} \in M^k F^s(U)$. Then

(i) $\tilde{A} \sqcup \tilde{B} = \tilde{B} \sqcup \tilde{A}, \tilde{A} \sqcap \tilde{B} = \tilde{B} \sqcap \tilde{A}$,

(ii) $\tilde{A} \sqcup (\tilde{B} \sqcup \tilde{C}) = (\tilde{A} \sqcup \tilde{B}) \sqcup \tilde{C}$,

$\tilde{A} \sqcap (\tilde{B} \sqcap \tilde{C}) = (\tilde{A} \sqcap \tilde{B}) \sqcap \tilde{C}$,

(iii) $\tilde{B} \sqsubseteq \tilde{C}$ implies

$\tilde{A} \sqcup \tilde{B} \sqsubseteq \tilde{A} \sqcup \tilde{C}$ and $\tilde{A} \sqcap \tilde{B} \sqsubseteq \tilde{A} \sqcap \tilde{C}$

(Sebastian and Ramakrishnan, 2011a).

Proposition 2.17. $\tilde{A}, \tilde{B} \in M^k F^s(U)$. Then

(i) $(\tilde{A} \sqcup \tilde{B}) \sqcap \tilde{A} = \tilde{A}$,

(ii) $(\tilde{A} \sqcap \tilde{B}) \sqcup \tilde{A} = \tilde{A}$.

Proof. (i) Let $\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \dots, \mu_k(u)) : u \in U\}$, $\tilde{B} = \{u/(\gamma_1(u), \gamma_2(u), \dots, \gamma_k(u)) : u \in U\}$. Then

$(\tilde{A} \sqcup \tilde{B}) \sqcap \tilde{A}$

$= \{u/(\mu_1(u) \vee \gamma_1(u), \dots, \mu_k(u) \vee \gamma_k(u)) : u \in U\}$

$\sqcap \{u/(\mu_1(u), \mu_2(u), \dots, \mu_k(u)) : u \in U\}$

$= \Big\{u/ \Big(\big(\mu_1(u) \vee \gamma_1(u)\big) \wedge \mu_1(u), \dots,$

$\big(\mu_k(u) \vee \gamma_k(u)\big) \wedge \mu_k(u)\Big) : u \in U\Big\}$

$= \{u/(\mu_1(u), \mu_2(u), \dots, \mu_k(u)) : u \in U\} = \tilde{A}$.

(ii) The proof is similar to (i).

Proposition 2.18. Let $\tilde{A}, \tilde{B}, \tilde{C} \in M^k F^s(U)$. Then,

(i) $\tilde{A} \sqcup (\tilde{B} \sqcap \tilde{C}) = (\tilde{A} \sqcup \tilde{B}) \sqcap (\tilde{A} \sqcup \tilde{C})$,

(ii) $\tilde{A} \sqcap (\tilde{B} \sqcup \tilde{C}) = (\tilde{A} \sqcap \tilde{B}) \sqcup (\tilde{A} \sqcap \tilde{C})$.

Proof. (i) Let $\tilde{A} = \{u/(\mu_1(u), \mu_2(u), \dots, \mu_k(u)) : u \in U\}$, $\tilde{B} = \{u/(\gamma_1(u), \gamma_2(u), \dots, \gamma_k(u)) : u \in U\}$ and $\tilde{C} = \{u/(\theta_1(u), \theta_2(u), \dots, \theta_k(u)) : u \in U\}$. Then

$\tilde{A} \sqcup (\tilde{B} \sqcap \tilde{C})$

$= \{u/(\mu_1(u), \mu_2(u), \dots, \mu_k(u)) : u \in U\} \sqcup$

$\{u/(\gamma_1(u) \wedge \theta_1(u), .., \gamma_k(u) \wedge \theta_k(u)) : u \in U\}$

$= \Big\{u/ \Big(\mu_1(u) \vee (\gamma_1(u) \wedge \theta_1(u)), \dots, \mu_k(u) \vee (\gamma_k(u) \wedge \theta_k(u))\Big) : u \in U\Big\}$

$= \Big\{u/ \Big(\big((\mu_1(u) \vee \gamma_1(u)) \wedge (\mu_1(u) \vee \theta_1(u))\big), \dots,$

$\big((\mu_k(u) \vee \gamma_k(u)) \wedge (\mu_k(u) \vee \theta_k(u))\big)\Big) : u \in U\Big\}$

$= \{u/(\mu_1(u) \vee \gamma_1(u), \dots, \mu_k(u) \vee \gamma_k(u)) : u \in U\} \sqcap$

$= \{u/(\mu_1(u) \vee \theta_1(u), \dots, \mu_k(u) \vee \theta_k(u)) : u \in U\}$

$= (\tilde{A} \sqcup \tilde{B}) \sqcap (\tilde{A} \sqcup \tilde{C})$.

(ii) The proof is similar to (i).

Definition 2.19. A pair $(\tilde{F}, A)$ is called a *multi-fuzzy soft set* of dimension k over U, where $\tilde{F}$ mapping given by $\tilde{F}: A \to M^k F^S(U)$. That is, for each $a \in A$, $\tilde{F}(a) = MM_{\tilde{F}(a)} \in M^k F^S(U)$. For $a \in A$, $\tilde{F}(a)$ may be considered a set of a-approximate elements of the multi-fuzzy soft set $(\tilde{F}, A)$. Let $A \subseteq E$, denote the set of all multi-fuzzy soft sets of dimension k over U by $M^k F_S^S(U)$ (Yang et al., 2013).

Definition 2.20. Let $A, B \in E$, $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two multi-fuzzy soft sets of dimension k over U. $(\tilde{F}, A)$ is said to be a *multi-fuzzy soft subset* of $(\tilde{G}, B)$ if, $A \subseteq B$ and for each $a \in A$, $\tilde{F}(a) \sqsubseteq \tilde{G}(a)$. In this case, we write $(\tilde{F}, A) \mathrel{\tilde{\sqsubseteq}} (\tilde{G}, B)$ (Yang et al., 2013).

Definition 2.21. A multi-fuzzy soft set $(\tilde{F}, A)$ of dimension k over U is said to be *null multi-fuzzy soft set*, denoted by $\widetilde{\emptyset_A}^k$ if $\tilde{F}(a) = \widetilde{\emptyset_A}^k$ for all $a \in A$. A multi-fuzzy soft set $(\tilde{F}, A)$ of dimension k over U is said to be *absolute multi-fuzzy soft set*, denoted by $\widetilde{U_A}^k$ if $\tilde{F}(a) = \widetilde{U_A}^k$ for all $a \in A$ (Yang et al., 2013).

Definition 2.22. The *extended union* of two multi-fuzzy soft sets sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of dimension k over U is the multi-fuzzy soft set $(\tilde{H}, C)$, where $C = A \cup B$ and for all $x \in C$, $\tilde{H}(x) = \tilde{F}(x)$ if $x \in A - B$, $\tilde{H}(x) = \tilde{G}(x)$ if $c \in B - A$ and $\tilde{H}(x) = \tilde{F}(x) \sqcup \tilde{G}(x)$ if $x \in A \cap B$. We write $(\tilde{F}, A) \mathrel{\tilde{\sqcup}_\varepsilon} (\tilde{G}, B) = (\tilde{H}, C)$ (Yang et al., 2013).

Definition 2.23. The *restricted union* of two multi-fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of dimension k over U is the multi-fuzzy soft set $(\tilde{H}, C)$, where $C = A \cap B$ and for all $x \in C, \tilde{H}(x) = \tilde{F}(x) \sqcup \tilde{G}(x)$. We write $(\tilde{F}, A) \mathrel{\tilde{\sqcup}_R} (\tilde{G}, B) = (\tilde{H}, C)$.

Definition 2.24. The *extended intersection* of two multi-fuzzy soft sets sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of dimension k over U is the multi-fuzzy soft set $(\tilde{H}, C)$, where $C = A \cup B$ and for all $x \in C$, $\tilde{H}(x) = \tilde{F}(x)$ if $x \in A - B$, $\tilde{H}(x) = \tilde{G}(x)$ if $x \in B - A$ and $\tilde{H}(x) = \tilde{F}(x) \sqcap \tilde{G}(x)$ if $x \in A \cap B$. We write $(\tilde{F}, A) \mathrel{\tilde{\sqcap}_\varepsilon} (\tilde{G}, B) = (\tilde{H}, C)$ (Kazancı et al., 2022).

We define the restricted intersection of two multi-fuzzy soft sets as follows. Note that our definition is different from the definition given by Yang at al. (Yang et al., 2013). Because even if $A \cap B = \emptyset$ we can still define $(\tilde{F}, A) \mathrel{\tilde{\sqcap}_R} (\tilde{G}, B) = (\tilde{H}, \emptyset)$, where $\tilde{H} = \emptyset: \emptyset \to M^k F^S(U)$.

Definition 2.25. The *restricted intersection* of two multi-fuzzy soft sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of dimension k over U is the multi-fuzzy soft set $(\tilde{H}, C)$, where $C = A \cap B$ for all $x \in C, \tilde{H}(x) = \tilde{F}(x) \sqcap \tilde{G}(x)$. We write $(\tilde{F}, A) \mathrel{\tilde{\sqcap}_R} (\tilde{G}, B) = (\tilde{H}, C)$.

3. Lattice Structure of Multi-Fuzzy Soft Sets

In this section, we investigate some properties of the operations given on multi-fuzzy soft sets. Then we give the lattice structure of multi-fuzzy soft sets.

Proposition 3.1. $(M^k F_S^S(U), \tilde{\sqcup}_\varepsilon)$ is a semilattice.

Proof. (i) Let $(\tilde{F}, A) \mathrel{\tilde{\sqcup}_\varepsilon} (\tilde{F}, A) = (\tilde{H}, C)$, where $C = A \cup A = A$ and for all $c \in A$, $\tilde{H}(c) = \tilde{F}(c)$. Therefore,

$(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{F},A)=(\tilde{F},A)$.

(ii) Let $(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)=(\tilde{K},C)$ and $(\tilde{G},B)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{F},A)=(\tilde{L},D)$. Then, $C=A\cup B$ and $D=B\cup A$.

Since $A\cup B=B\cup A$, then $C=D$. If $x\in A-B$, $\tilde{K}(x)=\tilde{F}(x)$ and $\tilde{L}(x)=\tilde{F}(x)$. If, $x\in B-A$, $\tilde{K}(x)=\tilde{G}(x)$ and $\tilde{L}(x)=\tilde{G}(x)$. If, $x\in A\cap B$, $\tilde{K}(x)=\tilde{F}(x)\sqcup\tilde{G}(x)$ and $\tilde{L}(x)=\tilde{G}(x)\sqcup\tilde{F}(x)$. Therefore, $(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)=(\tilde{G},B)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{F},A)$.

(iii) Let $(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)=(\tilde{K},A\cup B)$ and $(\tilde{K},A\cup B)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{H},C)=(\tilde{L},(A\cup B)\cup C)$. Similarly, $(\tilde{G},B)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{H},C)=(\tilde{M},B\cup C)$ and $(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{M},B\cup C)=\left(\tilde{N},A\cup(B\cup C)\right)$.

It is clear that $(A\cup B)\cup C=A\cup(B\cup C)=A\cup B\cup C$. Let $x\in A\cup B\cup C$. Then $x\in A$ or $x\in B$ or $x\in C$. Without loss of generality, we can assume that $x\in C$. Then,

a) If, $x\notin A$ and $x\notin B$, $\tilde{L}(x)=\tilde{H}(x)=\tilde{N}(x)$.
b) If, $x\in A$ and $x\notin B$, $\tilde{L}(x)=\tilde{F}(x)\sqcup\tilde{H}(x)=\tilde{N}(x)$.
c) If, $x\notin A$ and $x\in B$, $\tilde{L}(x)=\tilde{G}(x)\sqcup\tilde{H}(x)=\tilde{N}(x)$.
d) If, $x\in A$ and $x\in B$,
$$\tilde{L}(x)=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcup\tilde{H}(x)=\tilde{F}(x)\sqcup\left(\tilde{G}(x)\sqcup\tilde{H}(x)\right)=\tilde{N}(x).$$

Therefore,
$$\left((\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)\right)\tilde{\sqcup}_{\varepsilon}\,(\tilde{H},C)=(\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\left((\tilde{G},B)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{H},C)\right).$$

Proposition 3.2. $(M^kF_S^S(U),\tilde{\sqcup}_R)$ is a semilattice.

Proof. (i) Let $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{F},A)=(\tilde{H},C)$, where $C=A\cap A=A$ and for all $x\in A$, $\tilde{H}(x)=\tilde{F}(x)$. Therefore, $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{F},A)=(\tilde{F},A)$.

(ii) Let $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)=(\tilde{K},C)$ and $(\tilde{G},B)\,\tilde{\sqcup}_R\,(\tilde{F},A)=(\tilde{L},D)$. Then, $C=A\cap B=B\cap A=D$. For any $x\in A\cap B$, $\tilde{K}(x)=\tilde{F}(x)\sqcup\tilde{G}(x)=\tilde{G}(x)\sqcup\tilde{F}(x)=\tilde{L}(x)$. Therefore, $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)=(\tilde{G},B)\,\tilde{\sqcup}_R\,(\tilde{F},A)$.

(iii) Let $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)=(\tilde{K},A\cap B)$ and $(\tilde{K},A\cap B)\,\tilde{\sqcup}_R\,(\tilde{H},C)=(\tilde{L},(A\cap B)\cap C)$. Similarly, $(\tilde{G},B)\,\tilde{\sqcup}_R\,(\tilde{H},C)=(\tilde{M},B\cap C)$ and $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{M},B\cap C)=\left(\tilde{N},A\cap(B\cap C)\right)$. It is clear that $(A\cap B)\cap C=A\cap(B\cap C)=A\cap B\cap C$.

Let $x\in A\cap B\cap C$, we have $x\in A$ and $x\in B$ and $x\in C$.
$$\tilde{L}(x)=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcup\tilde{H}(x)=\tilde{F}(x)\sqcup\left(\tilde{G}(x)\sqcup\tilde{H}(x)\right)=\tilde{N}(x).$$

Therefore,
$$\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)\tilde{\sqcup}_R\,(\tilde{H},C)=(\tilde{F},A)\,\tilde{\sqcup}_R\left((\tilde{G},B)\,\tilde{\sqcup}_R\,(\tilde{H},C)\right).$$

Proposition 3.3 $(M^kF_S^S(U),\tilde{\sqcap}_{\varepsilon})$ is a semilattice.

Proof. (i) Let $(\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{F},A)=(\tilde{H},C)$, where $C=A\cup A=A$ and for all $x\in A$, $\tilde{H}(x)=\tilde{F}(x)$. Therefore, $(\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{F},A)=(\tilde{F},A)$.

(ii) Let $(\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{G},B)=(\tilde{K},C)$ and $(\tilde{G},B)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{F},A)=(\tilde{L},D)$. Therefore, $C=A\cup B$ and $D=B\cup A$. Since $A\cup B=B\cup A$, then $C=D$.
If $x\in A-B$, $\tilde{K}(x)=\tilde{F}(x)$ and $\tilde{L}(x)=\tilde{F}(x)$. If $x\in B-A$, $\tilde{K}(x)=\tilde{G}(x)$ and $\tilde{L}(x)=\tilde{G}(x)$. If $x\in A\cap B$, $\tilde{K}(x)=\tilde{F}(x)\sqcap\tilde{G}(x)$ and $\tilde{L}(x)=\tilde{G}(x)\sqcap\tilde{F}(x)$. Therefore, $(\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{G},B)=(\tilde{G},B)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{F},A)$.

(iii). Let $\left((\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{G},B)\right)\tilde{\sqcap}_{\varepsilon}\,(\tilde{H},C)=(\tilde{K},(A\cup B)\cup C)$ and $(\tilde{F},A)\,\tilde{\sqcap}_{\varepsilon}\left((\tilde{G},B)\,\tilde{\sqcap}_{\varepsilon}\,(\tilde{H},C)\right)=\left(\tilde{L},A\cup(B\cup C)\right)$. It is obvious that $(A\cup B)\cup C=A\cup(B\cup C)=A\cup B\cup C$. For all $\in A\cup B\cup C$, $x\in A$ or $x\in B$ or $x\in C$. Without loss of generality, we can suppose that $x\in C$. Then,

a) If, $x\notin A$ and $x\notin B$, $\tilde{K}(x)=\tilde{H}(x)=\tilde{L}(x)$.
b) If, $x\in A$ and $x\notin B$, $\tilde{K}(x)=\tilde{F}(x)\sqcap\tilde{H}(x)=\tilde{L}(x)$.
c) If, $x\notin A$ and $x\in B$, $\tilde{K}(x)=\tilde{G}(x)\sqcap\tilde{H}(x)=\tilde{L}(x)$.
d) If, $x\in A$ and $x\in B$, $\tilde{K}(x)=\left(\tilde{F}(x)\sqcap\tilde{G}(x)\right)\sqcap\tilde{H}(x)=\tilde{F}(x)\sqcap\left(\tilde{G}(x)\sqcap\tilde{H}(x)\right)=\tilde{L}(x)$.

Proposition 3.4. $(M^kF_S^S(U),\tilde{\sqcap}_R)$ is a semilattice.

Proof. (i) Let $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{F},A)=(\tilde{H},C)$, where $C=A\cap A=A$ and for all $x\in A$, $\tilde{H}(x)=\tilde{F}(x)$. Thus, $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{F},A)=(\tilde{F},A)$.

(ii) Let $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)=(\tilde{K},C)$ and $(\tilde{G},B)\,\tilde{\sqcap}_R\,(\tilde{F},A)=(\tilde{L},D)$. Therefore, $C=A\cap B=B\cap A=D$. If $x\in A\cap B$, $\tilde{K}(x)=\tilde{F}(x)\sqcap\tilde{G}(x)=\tilde{G}(x)\sqcap\tilde{F}(x)=\tilde{L}(x)$. So, $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)=(\tilde{G},B)\,\tilde{\sqcap}_R\,(\tilde{F},A)$.

(iii) Let $\left((\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)\right)\tilde{\sqcap}_R\,(\tilde{H},C)=(\tilde{K},(A\cap B)\cap C)$ and $(\tilde{F},A)\,\tilde{\sqcap}_R\left((\tilde{G},B)\,\tilde{\sqcap}_R\,(\tilde{H},C)\right)=\left(\tilde{L},A\cap(B\cap C)\right)$. We know that $(A\cap B)\cap C=A\cap(B\cap C)=A\cap B\cap C$.
If $x\in A\cap B\cap C$, then $x\in A$ and $x\in B$ and $x\in C$. Therefore,
$$\tilde{K}(x)=\left(\tilde{F}(x)\sqcap\tilde{G}(x)\right)\sqcap\tilde{H}(x)=\tilde{F}(x)\sqcap\left(\tilde{G}(x)\sqcap\tilde{H}(x)\right)=\tilde{L}(x).$$

Proposition 3.5. Let $(\tilde{F},A)$,$(\tilde{G},B)\in M^kF_S^S(U)$. Then,

(i) $\left((\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)\right)\tilde{\sqcap}_R\,(\tilde{F},A)=(\tilde{F},A)$,
(ii) $\left((\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)\right)\tilde{\sqcup}_{\varepsilon}\,(\tilde{F},A)=(\tilde{F},A)$.

Proof.

(i) Let $\left((\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)\right)=(\tilde{H},A\cup B)$ and $\left((\tilde{F},A)\,\tilde{\sqcup}_{\varepsilon}\,(\tilde{G},B)\right)\tilde{\sqcap}_R\,(\tilde{F},A)=(\tilde{K},(A\cup B)\cap A)=(\tilde{K},A)$.
Let $x\in A$.

a) If $x\in B$, then $\tilde{K}(x)=\tilde{H}(x)\sqcap\tilde{F}(x)=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcap\tilde{F}(x)=\tilde{F}(x)$.
b) If $x\notin B$, then $\tilde{K}(x)=\tilde{H}(x)\sqcap\tilde{F}(x)=\tilde{F}(x)\sqcap\tilde{F}(x)=\tilde{F}(x)$.

(ii) Let $\left((\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)\right)=(\tilde{H},A\cap B)$ and

$\left((\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)\right)\tilde{\sqcup}_\mathcal{E}\,(\tilde{F},A)=(\tilde{K},(A\cap B)\cup A)$
$=(\tilde{K},A)$.

Let $x\in A$.

a) If $x\in B$, then, $\tilde{K}(x)=\tilde{H}(x)\sqcup\tilde{F}(x)$
$=\left(\tilde{F}(x)\sqcap\tilde{G}(x)\right)\sqcup\tilde{F}(x)=\tilde{F}(x)$.

b) If $x\notin B$, then, $\tilde{K}(x)=\tilde{H}(x)\sqcup\tilde{F}(x)$
$=\tilde{F}(x)\sqcup\tilde{F}(x)=\tilde{F}(x)$.

Proposition 3.6. Let $(\tilde{F},A)$,$(\tilde{G},B)\in M^kF_S^S(U)$. Then,

(i) $\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)\tilde{\sqcap}_\mathcal{E}\,(\tilde{F},A)=(\tilde{F},A)$,

(ii) $\left((\tilde{F},A)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{G},B)\right)\tilde{\sqcup}_R\,(\tilde{F},A)=(\tilde{F},A)$.

Proof. **(i)** Let $\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)=(\tilde{H},A\cap B)$ and $\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)\tilde{\sqcap}_\mathcal{E}\,(\tilde{F},A)$
$=(\tilde{K},(A\cap B)\cup A)=(\tilde{K},A)$.

Let $x\in A$.

a) If $x\in B$, then $\tilde{K}(x)=\tilde{H}(x)\sqcap\tilde{F}(x)$
$=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcap\tilde{F}(x)=\tilde{F}(x)$.

b) If $x\notin B$, then we have $x\notin A\cap B$.
Therefore $\tilde{K}(x)=\tilde{F}(x)$.

(ii) Let $\left((\tilde{F},A)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{G},B)\right)=(\tilde{H},A\cup B)$ and

$\left((\tilde{F},A)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{G},B)\right)\tilde{\sqcup}_R\,(\tilde{F},A)$

$=(\tilde{K},(A\cup B)\cap A)=(\tilde{K},A)$.

Let $x\in A$.

a) If $x\in B$, then $\tilde{K}(x)=\tilde{H}(x)\sqcup\tilde{F}(x)$
$=\left(\tilde{F}(x)\sqcap\tilde{G}(x)\right)\sqcup\tilde{F}(x)=\tilde{F}(x)$.

b) If $x\notin B$, then $\tilde{K}(x)=\tilde{F}(x)$.

Corollary 3.7. $(M^kF_S^S(U)$,$\tilde{\sqcap}_R$,$\tilde{\sqcup}_\mathcal{E})$ is a complete lattice.

Proof. It is straightforward from Proposition 3.1, Proposition 3.4, Proposition 3.5 and Theorem 2.6.

Theorem 3.8. Let $\tilde{\sqsubseteq}_1$ be the ordering relation in the lattice $(M^kF_S^S(U)$,$\tilde{\sqcap}_R$,$\tilde{\sqcup}_\mathcal{E})$ and $(\tilde{F},A),(\tilde{G},B)\in M^kF_S^S(U)$. Then $(\tilde{F},A)\,\tilde{\sqsubseteq}_1\,(\tilde{G},B)$ if and only if $A\subseteq B$ and $\tilde{F}(x)\sqsubseteq\tilde{G}(x)$ for all $x\in A$.

Proof. Let $(\tilde{F},A)\,\tilde{\sqsubseteq}_1\,(\tilde{G},B)$. Therefore,
$(\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{G},B)=(\tilde{G},B)$ and $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)=(\tilde{F},A)$. Using Definition 2.22 and Definition 2.25, we have
$A\cup B=B$ and $A\cap B=B$. Then $A\subseteq B$. Therefore,
for each $x\in A$, $\tilde{F}(x)\sqcup\tilde{G}(x)=\tilde{G}(x)$ and $\tilde{F}(x)\sqsubseteq\tilde{G}(x)$.
Conversely, let $A\subseteq B$ and $\tilde{F}(x)\sqsubseteq\tilde{G}(x)$, for all $x\in A$. Therefore, $(\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{G},B)=(\tilde{G},B)$ and $(\tilde{F},A)\,\tilde{\sqcap}_R\,(\tilde{G},B)=(\tilde{F},A)$. Thus $(\tilde{F},A)\,\tilde{\sqsubseteq}_1\,(\tilde{G},B)$.

Theorem 3.9. The complete lattice $\left(M^kF_S^S(U)\text{,}\tilde{\sqcap}_R\text{,}\tilde{\sqcup}_\mathcal{E}\right)$ is distributive.

Proof. Let $(\tilde{F},A),(\tilde{G},B),(\tilde{H},C)\in M^kF_S^S(U)$. It is enough to prove that
$(\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\left((\tilde{G},B)\,\tilde{\sqcap}_R\,(\tilde{H},C)\right)$
$=\left((\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{G},B)\right)\tilde{\sqcap}_R\left((\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{H},C)\right)$.

Let $(\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\left((\tilde{G},B)\,\tilde{\sqcap}_R\,(\tilde{H},C)\right)=\left(\tilde{K},A\cup(B\cap C)\right)$ and $\left((\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{G},B)\right)\tilde{\sqcap}_R\left((\tilde{F},A)\,\tilde{\sqcup}_\mathcal{E}\,(\tilde{H},C)\right)$
$=\left(\tilde{L},(A\cup B)\cap(A\cup C)\right)=\left(\tilde{L},A\cup(B\cap C)\right)$.

For all $x\in A\cup(B\cap C)$, we have that $x\in A$ or $x\in B\cap C$.

a) If $x\in A$ and $x\in B\cap C$, then
$\tilde{K}(x)=\tilde{F}(x)\sqcup\left(\tilde{G}(x)\sqcap\tilde{H}(x)\right)$
$=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcap\left(\tilde{F}(x)\sqcup\tilde{H}(x)\right)=\tilde{L}(x)$.

b) If $x\in A$ and $x\notin B\cap C$, then $x\notin B$ or $x\notin C$.

- If $x\notin B$ and $x\in C$, $\tilde{K}(x)=\tilde{F}(x)=\tilde{F}(x)\sqcap\left(\tilde{F}(x)\sqcup\tilde{H}(x)\right)=\tilde{F}(x)=\tilde{L}(x)$.
- If $x\in B$ and $x\notin C$, $\tilde{K}(x)=\tilde{F}(x)=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcap\tilde{F}(x)=\tilde{F}(x)$
$=\tilde{L}(x)$.
- If $x\notin B$ and $x\notin C$, $\tilde{K}(x)=\tilde{F}(x)=\tilde{F}(x)\sqcap\tilde{F}(x)=\tilde{F}(x)=\tilde{L}(x)$.

c) If $x\notin A$ and $x\in B\cap C$, we have that $x\in B$ and $x\in C$, $\tilde{K}(x)=\tilde{G}(x)\sqcap\tilde{H}(x)=\tilde{L}(x)$.

Corollary 3.10. The complete lattice $\left(M^kF_S^S(U)\text{,}\tilde{\sqcap}_R\text{,}\tilde{\sqcup}_\mathcal{E}\right)$ is modular.

Proof. Since every distributive lattice is modular, the proof easily comes from Theorem 3.9.

Corollary 3.11. $(M^kF_S^S(U)$,$\tilde{\sqcap}_\mathcal{E}$,$\tilde{\sqcup}_R)$ is a complete lattice.

Proof. It easily comes from Proposition 3.2, Proposition 3.3, Proposition 3.6 and Theorem 2.6.

Theorem 3.12. Let $\tilde{\sqsubseteq}_2$ be ordering the relation in the lattice $(M^kF_S^S(U)$,$\tilde{\sqcap}_\mathcal{E}$,$\tilde{\sqcup}_R)$ and $(\tilde{F},A),(\tilde{G},B)\in M^kF_S^S(U)$. Then, $(\tilde{F},A)\,\tilde{\sqsubseteq}_2\,(\tilde{G},B)$ if and only if $B\subseteq A$ and
$\tilde{F}(x)\sqsubseteq\tilde{G}(x)$ for all $x\in A$.

Proof. Let $(\tilde{F},A)\,\tilde{\sqsubseteq}_2\,(\tilde{G},B)$. Thus $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)=(\tilde{G},B)$ and $(\tilde{F},A)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{G},B)=(\tilde{F},A)$. According to Definition 2.23 and Definition 2.24, we get that $A\cap B=B$ and $A\cup B=A$, so $B\subseteq A$. For all $x\in B$, $\tilde{F}(x)\sqcup\tilde{G}(x)=\tilde{G}(x)$ and $\tilde{F}(x)\sqcap\tilde{G}(x)=\tilde{F}(x)$. Therefore, $\tilde{F}(x)\sqsubseteq\tilde{G}(x)$.
Conversely, let $B\subseteq A$ and $\tilde{F}(x)\sqsubseteq\tilde{G}(x)$ for all $x\in B$. Therefore, $(\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)=(\tilde{G},B)$ and $(\tilde{F},A)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{G},B)=(\tilde{F},A)$. Then, $(\tilde{F},A)\,\tilde{\sqsubseteq}_2\,(\tilde{G},B)$.

Theorem 3.13. The complete lattice $(M^kF_S^S(U)$,$\tilde{\sqcap}_\mathcal{E}$,$\tilde{\sqcup}_R)$ is distributive.

Proof. Let $(\tilde{F},A),(\tilde{G},B),(\tilde{H},C)\in M^kF_S^S(U)$. It is enough to prove that
$(\tilde{F},A)\,\tilde{\sqcup}_R\left((\tilde{G},B)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{H},C)\right)$
$=\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)\tilde{\sqcap}_\mathcal{E}\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{H},C)\right)$.

Let $(\tilde{F},A)\,\tilde{\sqcup}_R\left((\tilde{G},B)\,\tilde{\sqcap}_\mathcal{E}\,(\tilde{H},C)\right)=\left(\tilde{K},A\cap(B\cup C)\right)$ and $\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{G},B)\right)\tilde{\sqcap}_\mathcal{E}\left((\tilde{F},A)\,\tilde{\sqcup}_R\,(\tilde{H},C)\right)$
$=\left(\tilde{L},(A\cap B)\cup(A\cap C)\right)=\left(\tilde{L},A\cap(B\cup C)\right)$.

We need to show that $\tilde{K}(x)=\tilde{L}(x)$ for all $x\in A\cap(B\cup C)$. Let $x\in A\cap(B\cup C)$. Then we have $x\in A$ and $x\in B\cup C$.

Let $x\in A$.

a) If $x\in B$ and $x\in C$,
$\tilde{K}(x)=\tilde{F}(x)\sqcup\left(\tilde{G}(x)\sqcap\tilde{H}(x)\right)$
$=\left(\tilde{F}(x)\sqcup\tilde{G}(x)\right)\sqcap\left(\tilde{F}(x)\sqcup\tilde{H}(x)\right)=\tilde{L}(x)$.

b) If $x \in B$ and $x \notin C$, $\widetilde{K}(x) = \tilde{F}(x) \sqcup \tilde{G}(x) = \tilde{L}(x)$.

c) If $x \notin B$ and $x \in C$, $\widetilde{K}(x) = \tilde{F}(x) \sqcup \widetilde{H}(x) = \tilde{L}(x)$.

Corollary 3.14. The complete lattice $(M^k F_S^S(U), \tilde{\sqcap}_\varepsilon, \tilde{\sqcup}_R)$ is modular.

Proof. Since every distributive lattice is modular, the proof easily comes from Theorem 3.13.

Let $(\tilde{F}, A), (\tilde{G}, B) \in M^k F_S^S(U)$. Note that the following equalities don't hold in general:

(i) $\left((\tilde{F}, A) \tilde{\sqcup}_\varepsilon (\tilde{G}, B)\right) \tilde{\sqcap}_\varepsilon (\tilde{F}, A) = (\tilde{F}, A)$,

(ii) $\left((\tilde{F}, A) \tilde{\sqcap}_R (\tilde{G}, B)\right) \tilde{\sqcup}_R (\tilde{F}, A) = (\tilde{F}, A)$.

Example 3.15. Let L be a lattice with the diagram in Figure 1. Let $A = \{e_1, e_2\}$ and $B = \{e_2, e_3, e_4\}$ be the set of parameters. We define the multi-fuzzy soft set

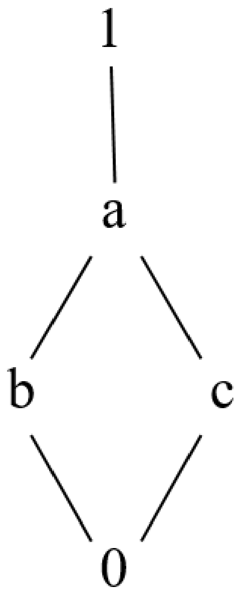

Figure 1. Lattice diagram.

$\tilde{F}: A \to M^2FS(L)$ as follows.

$\tilde{F}(e_1) = \{0/(0.4,0.7), a/(0.4,0.8), b/(0.7,0.2), c/(0.3,0.6), 1/(0.5,0.9)\}$,

$\tilde{F}(e_2) = \{0/(1.0,0.3), a/(0.4,0.5), b/(0.9,0.3), c/(0.2,0.1), 1/(0.6,0.7)\}$.

We define the multi-fuzzy soft set

$\tilde{G}: B \to M^2FS(L)$ as follows.

$\tilde{G}(e_2) = \{0/(0.5,0.8), a/(0.4,0.7), b/(0.5,0.7), c/(0.3,0.6), 1/(0.4,0.6)\}$,

$\tilde{G}(e_3) = \{0/(0.3,0.6), a/(0.8,0.6), b/(0.7,0.5), c/(0.2,0.3), 1/(0.1,0.1)\}$,

$\tilde{G}(e_4) = \{0/(0.8,1.0), a/(0.3,0.1), b/(0.8,0.4), c/(0.1,0.1), 1/(0.9,0.6)\}$.

(i) Let $\left((\tilde{F}, A) \tilde{\sqcup}_\varepsilon (\tilde{G}, B)\right) \tilde{\sqcap}_\varepsilon (\tilde{F}, A)$ $= (\widetilde{H}, (A \cup B) \cup A) = (\widetilde{H}, A \cup B)$. Then $\widetilde{H}(e_1) = \tilde{F}(e_1)$, $\widetilde{H}(e_2) = \tilde{F}(e_2)$, $\widetilde{H}(e_3) = \tilde{G}(e_3)$, $\widetilde{H}(e_4) = \tilde{G}(e_4)$. Therefore, $\left((\tilde{F}, A) \tilde{\sqcup}_\varepsilon (\tilde{G}, B)\right) \tilde{\sqcap}_\varepsilon (\tilde{F}, A) \neq (\tilde{F}, A)$.

(ii) Let $\left((\tilde{F}, A) \tilde{\sqcup}_R (\tilde{G}, B)\right) \tilde{\sqcap}_R (\tilde{F}, A) = (\widetilde{K}, (A \cap B) \cap A)$ $= (\widetilde{K}, A \cap B)$. $\widetilde{K}(e_2) = \tilde{F}(e_2)$. Therefore, $\left((\tilde{F}, A) \tilde{\sqcup}_R (\tilde{G}, B)\right) \tilde{\sqcap}_R (\tilde{F}, A) \neq (\tilde{F}, A)$.

4. Conclusion

In this paper, we deal with the lattice structure of multi-fuzzy soft sets. We give two lattice constructions on multi-fuzzy soft sets and investigate some related properties.

Author Contributions

The percentages of authors' contributions are presented below. All authors reviewed and approved the final version of the manuscript.

	R.İ.	Ş.Y.
C	30	70
D	55	45
S	30	70
DCP	70	30
DAI	55	45
L	60	40
W	80	20
CR	40	60
SR	45	55
PM	55	45
FA	55	45

C=Concept, D=Design, S=Supervision, DCP=Data Collection and/or Processing, DAI=Data analysis and/or interpretation, L=Literature search, W= Writing, CR=Critical Rewiew, SR=Submission and Revision, PM=Project Management, FA=Funding Acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

References

Akın C. 2021. Multi-fuzzy soft groups. Soft Comput, 25: 137-145.

Aygünoğlu A, Aygün H. 2009. Introduction to fuzzy soft groups. Comput Math Appl, 58: 1279-1286.

Birkhoff G. 1984. Lattice theory. American Math Soc, Providence, US.

Feng F, Jun YB, Zhao XZ. 2008. Soft semirings. Comput Math Appl, 56: 2621-2628.

Gratzer G. 1978. General lattice theory. Birkhauser Verlag, Basel, Switzerland.

Jun YB. 2008. Soft BCK/BCI-algebras. Comput Math Appl, 56: 1408-1413.

Kazancı O, Mayerova S, Davvaz B. 2022. Algebraic hyperstructure of multi-fuzzy soft sets related to polygroups. Mathematics, 10: 2178.

Kazancı O, Yılmaz Ş, Yamak S. 2010. Soft sets and soft BCH-algebras. Hacettepe J Math Stat, 39(2): 205 – 217.

Maji PK, Biswas R, Roy AR. 2001. Fuzzy soft sets. J Fuzzy Math, 9: 589-602.

Majumdar P, Samanta SK. 2010. Generalised fuzzy soft sets. Comput Math Appl, 59: 1425-1432.

Molodtsov D. 1999. Soft set theory-first results. Comput Math Appl, 37: 19-31.

Pawlak Z. 1982. Rough sets. Int J Info Comput Sci, 11: 341-356.

Qin K, Hong Z. 2010. On soft equality. J Comput Appl Math, 234: 1347-1355.

Sebastian S, Ramakrishnan TV. 2011a. Multi-fuzzy sets: an extension of fuzzy sets. Fuzzy Info Eng, 1: 35-43.

Sebastian S, Ramakrishnan TV. 2011b. Multi-fuzzy subgroups. Int J Contemp Math Sci, 8: 365-372.

Shao Y, Qin K. 2012. Fuzzy soft sets and fuzzy soft lattices. Int J Comput Intel Syst, 5 (6): 1135-1147.

Skornjakov LA. 1977. Elements of lattice theory. Hindustan Publishing Corporation, New Delhi, India.

Sun QM, Zhang ZL, Liu J. 2008. Soft sets and soft modules. Lecture Notes Comput Sci, 5009: 403-409.

Yang C. 2011. Fuzzy soft semigroups and fuzzy soft ideals. Comput Math Appl, 61: 255-261.

Yang Y, Tan X, Meng C. 2013. The multi-fuzzy soft set and its application in decision making. Appl Math Model, 37: 4915-4923.

Zadeh LA. 1965. Fuzzy sets. Info Cont, 8: 338-353.