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# Deferred Statistical $r$-Convergence of Sequences of Sets

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## Abstract

In the present paper, we introduce and study the concept of Wijsman deferred statistical $r$-convergence of sequences of sets and have its characterization in terms of deferred statistically dense subsequences. Beside this, we explore the concept of strongly deferred Cesàro summability and its relation with the newly introduced notion of Wijsman deferred statistical $r$-convergence.

***Keywords:*** *Statistical convergence, Deferred statistical convergence, Wijsman convergence, Wijsman statistical convergence.*
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## 1. Introduction

In 1951, Fast [9] and Stienhaus [24] introduced the idea of statistical convergence which is, in fact, a generalization of usual notion of convergence. Later on, Buck [6] studied this concept as "convergence in density" in 1953. It was also a part of monograph by Zygmund [32] and referred as almost convergence. Schoenberg [23] introduced and studied this concept independently in connection with summability of sequences in 1959.

The notion of statistical convergence has its main pillar as natural density, which is defined as

**Definition 1.1.** *For $K \subseteq \mathbb{N}$, the natural density is denoted by $\delta(K)$ and defined as*

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \in K : m \leq n\}),$$

*provided the limit exists. It is easily verified that $\delta(K) = 0$, for finite subset $K$ of $\mathbb{N}$ and $\delta(K) + \delta(\mathbb{N} - K) = 1$ for every $K \subseteq \mathbb{N}$, whenever $\delta(K)$ exists.*

Using the notion of natural density, statistical convergence is defined as

**Definition 1.2.** *A real valued sequence $(z_m)$ is statistically convergent to $\ell \in \mathbb{R}$ if for each $\varepsilon > 0$,*

$$\delta(\{m \in \mathbb{N} : |z_m - \ell| \geq \varepsilon\}) = 0,$$

$$i.e., \lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \leq n : |z_m - \ell| \geq \varepsilon\}) = 0,$$

*and $\ell$ is referred as statistical limit of $(z_m)$. We write $z_m \xrightarrow{S} \ell$ and by $S(c)$ we denote the set of all statistically convergent real sequences.*

In the last two decades, there has been growing interest in the study of rough convergence theory which arises after the introduction of the notion of rough convergence by Phu [20] as follows

**Definition 1.3.** *A sequence $(z_m)$ in normed linear space $(Z, \|\cdot\|)$ is said to be rough convergent to $z_0 \in Z$ for some $r \geq 0$ if for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that $\|z_m - z_0\| < r + \varepsilon$ for all $m \geq m_0$.*

S. Aytar [3] utilized the notion of rough convergence to establish the statistical analog about this concept as rough statistical convergence as follows

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**Definition 1.4.** *A sequence $(z_m)$ in normed linear space $(Z, \|\cdot\|)$ is said to be rough statistical convergent to $z_0 \in Z$ for some $r \geq 0$ if for every $\varepsilon > 0$,*

$$\delta(\{m \in \mathbb{N} : \|z_m - z_0\| \geq r + \varepsilon\}) = 0,$$

*holds. That is $\lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \in \mathbb{N} : \|z_m - z_0\| \geq r + \varepsilon\}) = 0$ and $z_0$ is identified as r-statistical limit of $(z_m)$.*

R.P. Agnew [1] in 1932, introduced the notion of deferred Cesàro mean of scalar sequences $z = (z_m)$ as

$$(D_{p,q}z)_n = \frac{1}{q_n - p_n} \sum_{m=p_n+1}^{q_n} z_m, \qquad (n = 1, 2, 3, \dots) \tag{1.1}$$

where $p = \{p_n : n \in \mathbb{N}\}$ and $q = \{q_n : n \in \mathbb{N}\}$ are the sequences of non-negative integer satisfying

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty.$$

Agnew also showed that besides being regular, the method given by (1.1) has many more important properties, (see in [1]).

Motivating from the work of Agnew, the notions of deferred density and deferred statistical convergence were given by Küçükaslan and Yılmaztürk [17, 31] as follows:

Let $K \subseteq \mathbb{N}$ and denote the set $\{m \in K : p_n < m \leq q_n\}$ by $K_{p,q}(n)$.

**Definition 1.5.** *The deferred density of $K \subseteq \mathbb{N}$ is defined as*

$$\delta_D(K) = \lim_{n\to\infty} \frac{1}{q_n - p_n}\text{card}(K_{p,q}(n)),$$

*provided that the limit exists. If $q_n = n$ and $p_n = 0$ for all $n \in \mathbb{N}$, then deferred density coincides with the natural density.*

In view of deferred density the following is obvious

**Remark 1.6.** *Every finite subset of $\mathbb{N}$ has zero deferred density.*

**Definition 1.7.** *A scalar sequence $(z_m)$ is said to be deferred statistically convergent to $\ell \in \mathbb{R}$, if for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n}\text{card}(\{m \in \mathbb{N} : p_n < m \leq q_n,\ |z_m - \ell| \geq \varepsilon\}) = 0,$$

*and $\ell$ is referred as deferred statistical limit of $(z_m)$. We write $S_D - \lim z_m = \ell$ and by $S_D$ we denote the set of all deferred statistically convergent sequences. If $p_n = 0$, $q_n = n$, then deferred statistical convergence coincides with statistical convergence.*

A large amount of research work regarding various generalizations/extensions of statistical convergence, i.e., lacunary statistical convergence, rough convergence, weighted statistical rough convergence, $\lambda$-statistical convergence, statistical convergence by modulus and Orlicz function etc. has been carried out by many more researchers for scalar as well as for vector valued sequences. One may refer to [4, 10, 11, 12, 14, 18, 21] where many more references can be found.

In 1902, Painleve introduced the limit of sequences of sets. Although sets convergence introduced by him, has long mathematical history, it is only during the last four decades that this concept has become so much wider and vigour.

Before proceeding further, we pause here to recall some definitions/notations related to convergence of sets sequences (called Wijsman convergence which is due to Wijsman [29, 30]).

Let $(Z, d)$ be a metric space. The distance $d(z, F)$ from a point $z$ to a non-empty subset $F$ of $(Z, d)$ is defined as

$$d(z, F) = \inf_{f\in F} d(z, f).$$

**Definition 1.8.** *([29, 30]) Let $(F_m)$ be a sequence of non-empty closed subsets of a metric space $(Z, d)$ and $F$ be a non-empty closed subset of $Z$.*

*(a) $(F_m)$ is said to be Wijsman convergent to $F$, if for each $z \in Z$, $\lim_{m\to\infty} d(z, F_m) = d(z, F)$, i.e., the sequence $(d(z, F_m))_{m\in\mathbb{N}}$ of reals is convergent to $d(z, F)$. In this case, we write $W - \lim F_m = F$ or $F_m \xrightarrow{W} F$. The set of all Wijsman convergent sequences is denoted by $W$.*

*(b) $(F_m)$ is said to be bounded (Wijsman bounded) if $\sup_m |d(z, F_m)| < \infty$ for each $z \in Z$, i.e., for $z \in Z$, there exist real number $M_z$ such that $d(z, F_m) \leq M_z$ for all $m \in \mathbb{N}$.*

Adding the flavour of statistical convergence to the Wijsman convergence, Nuray and Rhoades [19] introduced the Wijsman statistical convergence as follows

**Definition 1.9.** *Let $(F_m)_{m\in\mathbb{N}}$ be a sequence of non-empty closed subsets of a metric space $(Z, d)$ and $F$ be a non-empty closed subset of $Z$. Then, $(F_m)_{m\in\mathbb{N}}$ is said to be Wijsman statistically convergent to $F$ (denoted by $F_m \xrightarrow{WS} F$), if for each $z \in Z$, $(d(z, F_m))_{m\in\mathbb{N}}$ is statistically convergent to $d(z, F)$, i.e., for each $z \in Z$ and for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \leq n : |d(z, F_m) - d(z, F)| \geq \varepsilon\}) = 0,$$

*holds. The set of all Wijsman statistically convergent sequences is denoted by $WS$.*

Using the concepts of Wijsman convergence and deferred statistical convergence, Et and Yilmazer [8] in 2020, introduced and studied the concept of Wijsman deferred statistical convergence and Wijsman deferred strongly Cesàro summability as follows

**Definition 1.10.** *Let $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ be sequences of non-negative integers satisfying $p_n < q_n$ and $q_n \to \infty$ as $n \to \infty$. A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman deferred statistically convergent to a non-empty closed set $F (F \subseteq Z)$ if*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \text{card}(\{p_n < m \le q_n : |d(z,F_m) - d(z,F)| \ge \varepsilon\}) = 0$$

*is satisfied for each $\varepsilon > 0$ and for each $z \in Z$. And we write $F_m \xrightarrow{WS_D} F$ or $WS_D - \lim F_m = F$. The set $WS_D$ will denote the collection of all Wijsman deferred statistical convergent sequences. If $q_n = n$, $p_n = 0$ for all $n \in \mathbb{N}$, then the notion of Wijsman deferred statistical convergence reduces to Wijsman statistical convergence and we write $WS$ in place of $WS_D$.*

**Definition 1.11.** *If $(F_m)_{m\in\mathbb{N}}$ is a sequence of sets from Z such that $(F_m)_{m\in\mathbb{N}}$ satisfies property P for all m, except a set of deferred density zero, then we say $(F_m)_{m\in\mathbb{N}}$ satisfies property P for "deferred almost all m (d.a.a. m)".*

In view of Definition 1.11, Wijsman deferred statistical convergence may be redefined as

**Definition 1.12.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman deferred statistically convergent to a non-empty closed set $F (F \subseteq Z)$ if*

$$|d(z,F_m) - d(z,F)| < \varepsilon \;\; d.a.a.\, m$$

*for each $\varepsilon > 0$ and for each $z \in Z$.*

**Definition 1.13.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman strongly deferred Cesàro summable to a non-empty closed set F of Z, if for each $z \in Z$,*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{m=p_n+1}^{q_n} |d(z,F_m) - d(z,F)| = 0.$$

*By $NW_D$ we denote the set of all Wijsman strongly deferred Cesàro summable sequences.*

By considering the notion of Wijsmann deferred statistical convergence Altınok et al. [2] introduced the concept of deferred statistical equivalence of sets and established some inclusion relations under strict restrictions.

For various generalizations of Wijsman convergence one may refer to [5, 7, 13, 15, 16, 22, 25, 26, 27, 28] where many more references can be found.

In the present paper, using the tools of deferred density, convergence of set sequences, statistical equivalence of sequences of sets and rough convergence, we availed an opportunity to introduce and explore the notions of Wijsmann deferred statistical $r$-convergence and deferred statistical $r$-equivalence of sequences of sets which proved to be generalization of some existing notions in literature for $r = 0$. It is observed that Wijsmann deferred statistical $r$-convergent sequences are precisely those which have a deferred statistically dense, Wijsmann $r$-convergent subsequence. Apart from this, a statistical analog of squeeze principle for sequences of sets has been established.

## 2. Wijsman deferred statistical $r$-convergence

In this section, we introduce the notion of Wijsman deferred statistical $r$-convergence and have a characterization of it in terms of Wijsman $r$-convergence.

**Definition 2.1.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman r-convergent to a non-empty closed set $F (\subseteq Z)$ for $r \ge 0$ if for given $\varepsilon > 0$ and for $z \in Z$, there exist $n_0 \in \mathbb{N}$ such that*

$$|d(z,F_m) - d(z,F)| < r + \varepsilon$$

*for all $m \ge n_0$ and we write $F_m \xrightarrow{W^r} F$ or $W^r - \lim F_m = F$. The set of all Wijsman r-convergent sequences is denoted by $W^r$.*

**Definition 2.2.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman r-Cauchy for $r \ge 0$ if for given $\varepsilon > 0$ and for $z \in Z$, there exist $n_0 \in \mathbb{N}$ such that*

$$|d(z,F_m) - d(z,F_n)| < r + \varepsilon \text{ for all } m,n \ge n_0.$$

**Definition 2.3.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman statistical r-convergent to a non-empty closed set F of Z for $r \ge 0$ if for each $z \in Z$ and for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \text{card}(\{m \le n : |d(z,F_m) - d(z,F)| \ge r + \varepsilon\}) = 0.$$

*We write $F_m \xrightarrow{WS^r} F$ and the set of all Wijsman statistically r-convergent sequences is denoted by $WS^r$.*

**Remark 2.4.** *It is to be noted that for $r = 0$, the definition of Wijsman r-convergence reduces to Wijsman convergence and Wijsman statistical r-convergence reduces to Wijsman statistical convergence.*

**Remark 2.5.** *Every Wijsman convergent (and hence Wijsman r-convergent) sequence is Wijsman bounded.*

**Definition 2.6.** *For a non-negative real number r, we say sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z to be Wijsman deferred statistically r-convergent to a non-empty closed set F of Z provided that*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\text{card}(\{p_n<m\le q_n:|d(z,F_m)-d(z,F)|\ge r+\varepsilon\})=0,$$

$$i.e.,\ |d(z,F_m)-d(z,F)|<r+\varepsilon\ \ d.a.a.\,m$$

*for each $\varepsilon>0$ and for each $z\in Z$. And we write $F_m\xrightarrow{WS_D^r}F$ or $WS_D^r-\lim F_m=F$. Here $WS_D^r$ denote the set of all Wijsman deferred statistically r-convergent sequences.*

**Definition 2.7.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is said to be Wijsman deferred statistically bounded if for each $z\in Z$, there exist $M_z>0$ such that*

$$\delta_D(\{m\in\mathbb{N}:d(z,F_m)>M_z\})=0,$$

$$i.e.,\ d(z,F_m)\le M_z\ \ d.a.a.\,m.$$

**Theorem 2.8.** *Every Wijsman r-convergent sequence is Wijsman deferred statistically r-convergent (Wijsman statistically r-convergent) but converse is not true, in general.*

*Proof.* The result follows in view of the Remark 1.6. For converse part, consider $Z=\mathbb{R}$, with usual metric $d(z,z')=|z-z'|$, $z,z'\in Z$ and

$$F_m=\begin{cases}\{m\} & \text{for } m=q_n\\ \{(-1)^m\} & otherwise.\end{cases}\qquad m\in(p_n,q_n],\ m\in\mathbb{N}.$$

Then it is easy to see that the sequence $(F_m)$ is not Wijsman convergent. Even more, this sequence has no Wijsman $r$-limit for any $r$ as it is unbounded from above. However, it is Wijsman deferred statistically $r$-convergent to $F=\{-1,1\}$ for $r\ge 2$. □

**Theorem 2.9.** *Every Wijsman deferred statistically r-convergent sequence is Wijsman deferred statistically bounded. Converse need not be true.*

*Proof.* Let $(F_m)_{m\in\mathbb{N}}$ be a Wijsman deferred statistically $r$-convergent sequence to $F$, so for each $z\in Z$ and $\varepsilon>0$

$$|d(z,F_m)-d(z,F)|<r+\varepsilon\quad d.a.a\,m.$$

This is also true for $\varepsilon=1$. Then, we have

$$\begin{aligned}|d(z,F_m)| &\le |d(z,F_m)-d(z,F)|+|d(z,F)|\\ &< r+1+|d(z,F)|\quad d.a.a.\ m\\ &= M_z,\\ i.e.,\quad |d(z,F_m)| &\le M_z\quad d.a.a.\ m.\end{aligned}$$

For converse part, consider $Z=\mathbb{R}$, with usual metric $d(z,z')=|z-z'|$, $z,z'\in Z$. Take

$$F_m=\begin{cases}\{q_n\} & \text{if } m=q_n\\ \{(-1)^m\} & \text{otherwise}\end{cases}\qquad m\in(p_n,q_n],\ m\in\mathbb{N},$$

and $F=\{2\}$.

Then $d(z,F_m)=|z+1|$ or $|z-1|$ for $m\ne q_n$ and $d(z,F)=|z-2|$. Thus $|d(z,F_m)-d(z,F)|\le 3$ for all $z\in Z$, $m\ne q_n$. This implies

$$|d(z,F_m)-d(z,F)|\not< r+\varepsilon\quad d.a.a.\ m\quad (0\le r<3),$$

i.e., $(F_m)_{m\in\mathbb{N}}$ is not Wijsman deferred statistically $r$-convergent for $r\in[0,3)$. But for $z\in\mathbb{R}$, $|d(z,F_m)|\le M_z$ for all $m\ne q_n$ (Take $M_z=|z|+3$) and hence Wijsman deferred statistically bounded. □

The following theorem is the statistical analog of the Squeeze principle.

**Theorem 2.10.** *Let $(F_m)_{m\in\mathbb{N}}$, $(E_m)_{m\in\mathbb{N}}$ and $(G_m)_{m\in\mathbb{N}}$ are non-empty closed set sequences of Z such that $F_m\subseteq E_m\subseteq G_m$ for every $m\in\mathbb{N}$. If $F_m\xrightarrow{WS_D^r}F$ and $G_m\xrightarrow{WS_D^r}F$ then $E_m\xrightarrow{WS_D^r}F$.*

*Proof.* Since $F_m\subseteq E_m\subseteq G_m$, then

$$d(z,G_m)\le d(z,E_m)\le d(z,F_m)$$

holds for each $z\in Z$ and $m\in\mathbb{N}$. Then, for every $\varepsilon>0$ and $z\in Z$, we have

$$\begin{aligned}\{p_n<m\le q_n:|d(z,E_m)-d(z,F)|\ge r+\varepsilon\} &= \{p_n<m\le q_n:d(z,E_m)-d(z,F)\ge r+\varepsilon\}\cup\{p_n<m\le q_n:d(z,E_m)\le d(z,F)-(r+\varepsilon)\}\\ &\subseteq \{p_n<m\le q_n:d(z,F_m)\ge d(z,F)+r+\varepsilon\}\cup\{p_n<m\le q_n:d(z,G_m)\le d(z,F)-(r+\varepsilon)\}\\ &\subseteq \{p_n<m\le q_n:|d(z,F_m)-d(z,F)|\ge r+\varepsilon\}\cup\{p_n<m\le q_n:|d(z,G_m)-d(z,F)|\ge r+\varepsilon\}.\end{aligned}$$

Now the result follows from the above inequality and the fact $F_m,G_m\xrightarrow{WS_D^r}F$. □

**Theorem 2.11.** *For $r_1 > r_2$ we have $WS_D^{r_1} \subseteq WS_D^{r_2}$.*

*Proof.* The proof is easy and a routine verification, hence omitted. □

**Definition 2.12.** *A subset $B$ of $\mathbb{N}$ is said to be deferred statistically dense if $\delta_D(B) = 1$.*

**Theorem 2.13.** *$WS_D^r - \lim F_m = F$ iff there exists an increasing index sequence $(m_k)_{k\in\mathbb{N}}$ such that the related set $M = \{m_1 < m_2 < m_3 < \ldots\}$ has $\delta_D(M) = 0$ and $W^r - \lim_{m\in\mathbb{N}-M} F_m = F$.*

*Proof.* **Step-1** For each $j \in \mathbb{N}$, let us consider $M_j = \left\{m \in \mathbb{N} : |d(z,F_m) - d(z,F)| \geq r + \frac{1}{j}\right\}$ such that $\delta_D(M_j) = 0$ holds. It is to be noted that $M_j \subset M_{j+1}$ for all $j \in \mathbb{N}$. If all $M_j{}'s$ are empty, then result is trivially established. Let us consider the case when some of the $M_j{}'s$ are non-empty. Without loss of generality we may assume that $M_1 \neq \emptyset$. Take any $n_1 \in M_1$. Then using the the fact $M_1 \subseteq M_2$ and $\delta_D(M_2) = 0$, there exist a number $n_2 \in M_2$ such that $n_2 > n_1$ and

$$\frac{1}{q_n - p_n}\text{card}\left(\left\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \frac{1}{2}\right\}\right) < \frac{1}{2}$$

for all $n \geq n_2$. Similarly we choose a number $n_3$ from $M_3$ such that $n_3 > n_2$ and

$$\frac{1}{q_n - p_n}\text{card}\left(\left\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \frac{1}{3}\right\}\right) < \frac{1}{3}$$

for all $n \geq n_3$. Continuing in this way, we have an increasing sequence $\{n_j\}$, $j = 1,2,3,\ldots$, of natural numbers such that $n_j \in M_j$ and

$$\frac{1}{q_n - p_n}\text{card}\left(\left\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \frac{1}{j}\right\}\right) < \frac{1}{j}$$

for all $n \geq n_j$.

**Step-2** Construct a set $M$ of increasing indices as follows

$$M = ([n_1,n_2) \cap M_1) \cup ([n_2,n_3) \cap M_2) \cup ([n_3,n_4) \cap M_3) \ldots$$

Then $\{m \in M : p_n < m \leq q_n\} \subseteq \{m \in M_j : p_n < m \leq q_n\}$ for some $j$. Now

$$\begin{aligned}\frac{1}{q_n - p_n}\text{card}\left(\{m \in M : p_n < m \leq q_n\}\right) &\leq \frac{1}{q_n - p_n}\text{card}\left(\{m \in M_j : p_n < m \leq q_n\}\right)\\ &= \frac{1}{q_n - p_n}\text{card}\left(\left\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \frac{1}{j}\right\}\right) < \frac{1}{j} \text{ for all } n \geq n_j\end{aligned}$$

and so $\delta_D(M) = 0$.

**Step-3** Let $\varepsilon > 0$. Then, there exists $j_0 \in \mathbb{N}$ such that $\frac{1}{j_0} < \varepsilon$ and for all $j \geq j_0$ we have $\frac{1}{j} < \varepsilon$. Now, for $m \in \mathbb{N} - M$ and $m \geq n_j$, then there exists $s \geq j$ such that $n_s < m \leq n_{s+1}$. As $m \notin M_s$ so

$$|d(z,F_m) - d(z,F)| < r + \frac{1}{s} \leq r + \frac{1}{j} < r + \varepsilon$$

Thus, $W^r - \lim_{m\in\mathbb{N}-M} F_m = F$.

Conversely, let $\varepsilon > 0$ be given. Then, there exists $m_0 \in \mathbb{N}$ such that $|d(z,F_m) - d(z,F)| \leq r + \varepsilon$ for all $m \in \mathbb{N} - M$ with $m \geq m_0$. The result now follows in view of the following inclusion

$$\{m \in \mathbb{N} : |d(z,F_m) - d(z,F)| \geq r + \varepsilon\} \subseteq M \cup \{1,2,3,\ldots,m_0 - 1\}.$$

So, the proof is ended. □

**Corollary 2.14.** *A sequence $(a_m)_{m\in\mathbb{N}}$ of reals is deferred statistically r-convergent to $\ell \in \mathbb{R}$ iff there exist a deferred statistical dense set $B = \{m_1 < m_2 < m_3 < \ldots < m_k < \ldots\} \subseteq \mathbb{N}$ such that $r - \lim a_{m_k} = \ell$.*

*Proof.* By considering $F_m = \{a_m\}$, $m \in \mathbb{N}$ in Theorem 2.13, the result follows. □

In analog to the notion of Wijsman deferred $r$-convergence we define

**Definition 2.15.** *A sequence $(a_m)_{m\in\mathbb{N}}$ of reals is said to be deferred statistical r-convergent to $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n}\text{card}(\{p_n < m \leq q_n : |a_m - \ell| \geq r + \varepsilon\}) = 0$$

*holds. That is $|a_m - \ell| < r + \varepsilon$ d.a.a. $m$.*

**Theorem 2.16.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of $Z$ is Wijsman deferred statistically r-convergent to a non-empty closed subset $F$ of $Z$ iff there exist a sequence $(G_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of $Z$ which is Wijsman r-convergent to $F$ and $G_m = F_m$ d.a.a. $m$.*

*Proof.* Let $F_m \xrightarrow{WS_D^r} F$, then $\delta_D(A) = 0$ where $A = \{m \in \mathbb{N} : |d(z,F_m) - d(z,F)| \geq r + \varepsilon\}$. Take

$$G_m = \begin{cases} F_m, & \text{for } m \in \mathbb{N} - A, \\ F, & \text{for } m \in A. \end{cases}$$

Clearly, $G_m = F_m$ $d.a.a.$ $m$ and

$$|d(z,G_m) - d(z,F)| = \begin{cases} |d(z,F_m) - d(z,F)| & \text{for } m \in \mathbb{N} - A \\ 0 & \text{for } m \in A. \end{cases}$$

Thus $|d(z,G_m) - d(z,F)| < r + \varepsilon$ for all $m \in \mathbb{N}$, for all $z \in Z$. Hence $(G_m)_{m\in\mathbb{N}}$ is Wijsman $r$-convergent to $F$.

Conversely, it is given that there exists $(G_m)_{m\in\mathbb{N}}$ such that $G_m = F_m$ $d.a.a$ $m$ and $(G_m)_{m\in\mathbb{N}}$ is Wijsman $r$-convergent to $F$. Let $A = \{m \in \mathbb{N} : G_m \neq F_m\}$ then $\delta_D(A) = 0$. Also for $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$|d(z,G_m) - d(z,F)| < r + \varepsilon \text{ for all } m \geq N_0.$$

The result follows in view of

$$\{m : |d(z,G_m) - d(z,F)| \geq r + \varepsilon\} \subseteq \{1, 2, 3, \ldots, N_0 - 1\} \cup A$$

and the fact that every finite subset of $\mathbb{N}$ has null deferred density. □

**Definition 2.17.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of $Z$ is called Wijsman deferred statistically $r$-Cauchy if for every $\varepsilon > 0$ and $z \in Z$, there exist $m_0 \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \text{card}\left(\{p_n < m \leq q_n : |d(z,F_m) - d(z,F_{m_0})| \geq r + \varepsilon\}\right) = 0,$$

*holds. That is $|d(z,F_m) - d(z,F_{m_0})| < r + \varepsilon$ $d.a.a.$ $m$.*

**Theorem 2.18.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of $Z$ is deferred statistically $r$-Cauchy iff there exist an increasing index sequence $M = \{m_1 < m_2 < m_3 < \ldots\} \subseteq \mathbb{N}$ such that $\delta_D(M) = 0$ and $(F_m)_{m\in\mathbb{N}-M}$ is Wijsman $r$-Cauchy.*

*Proof.* The proof runs on the similar lines as in Theorem 2.13 and hence omitted. □

**Theorem 2.19.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of $Z$ is Wijsman deferred statistically $r$-convergent sequence for some $r$ iff sequence $(F_m)_{m\in\mathbb{N}}$ is a Wijsman deferred statistically $2r$-Cauchy sequence.*

*Proof.* Let $WS_D^r - \lim F_m = F$. Then for any $\varepsilon > 0$, and $z \in Z$ we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \text{card}\left(\left\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \frac{\varepsilon}{2}\right\}\right) = 0.$$

Choose $m_0$ such that

$$|d(z,F_{m_0}) - d(z,F)| < r + \frac{\varepsilon}{2}.$$

Then, we have

$$\begin{aligned} |d(z,F_m) - d(z,F_{m_0})| &\leq |d(z,F_m) - d(z,F)| + |d(z,F) - d(z,F_{m_0})| \\ &< r + \frac{\varepsilon}{2} + r + \frac{\varepsilon}{2} = 2r + \varepsilon \end{aligned}$$

holds, $d.a.a.$ $m$. Thus $(F_m)$ is a Wijsman deferred statistically $2r$-Cauchy.

Conversely, as $(F_m)_{m\in\mathbb{N}}$ is a Wijsman deferred statistically $2r$-Cauchy, so by Theorem 2.18, there exists a Wijsman $2r$-Cauchy subsequence $(F_{m_k})_{k\in\mathbb{N}}$ of $(F_m)_{m\in\mathbb{N}}$ with $\delta_D(\{m_k : k \in \mathbb{N}\}) = 1$. Following on similar lines as in (Theorem 1 [19]), we have $(F_{m_k})_{k\in\mathbb{N}}$ is Wijsman $r$-convergent to some $F \subseteq Z$. The result now follows in view of Theorem 2.13. □

**Theorem 2.20.** *If $\lim_n \frac{q_n - p_n}{n} > 0$ and $q_n < n$, then every $WS^r$-convergent sequence is also $WS_D^r$-convergent to same limit set.*

*Proof.* Let $WS^r - \lim F_m = F$. Then, for a given $\varepsilon > 0$, we have

$$\begin{aligned} \frac{1}{n}\text{card}\left(\{m \leq n : |d(z,F_m) - d(z,F)| \geq r + \varepsilon\}\right) &\geq \frac{1}{n}\text{card}\left(\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \varepsilon\}\right) \\ &= \frac{q_n - p_n}{n} . \frac{1}{q_n - p_n}\text{card}\left(\{p_n < m \leq q_n : |d(z,F_m) - d(z,F)| \geq r + \varepsilon\}\right). \end{aligned}$$

This inequality implies that $F_m \xrightarrow{WS_D^r} F$ because of $WS^r - \lim F_m = F$. □

**Theorem 2.21.** *Every Wijsman strongly deferred Cesàro summable sequence is Wijsman deferred statistically $r$-convergent, but the converse is not true, in general.*

*Proof.* The proof is easy and a routine verification, hence omitted.

For converse part, let $Z=\mathbb{R}$ with usual metric $d(z,z')=|z-z'|$ ; $z,z'\in Z$ and $r=0$. Let $p_n=3^{n-1}$ and $q_n=3^n$, $n\in\mathbb{N}$. Define $(F_m)$ as

$$F_m=\begin{cases}\{3^{n-1}\} & \text{if } m=3^{n-1}+1,\ m\in(p_n,q_n]\\ \{0\} & \text{otherwise}\end{cases} \quad \text{and} \quad F=\{0\}.$$

Clearly $(F_m)$ is not bounded sequence for each $z\in\mathbb{R}$, we have

$$\begin{aligned}\frac{1}{q_n-p_n}\sum_{m=p_{n+1}}^{q_n}|d(z,F_m)-d(z,F)| &= \frac{1}{q_n-p_n}\sum_{m=p_{n+1}}^{q_n}|d(z,F_m)-d(z,\{0\})|\\ &= \frac{1}{3^n-3^{n-1}}\sum_{m=3^{n-1}+1}^{3^n}|d(z,F_m)-d(z,\{0\})|\\ &= \frac{1}{2.3^{n-1}}\left|d\left(z,\{3^{n-1}\}\right)-d(z,\{0\})\right|\\ &= \frac{1}{2.3^{n-1}}\left||z-3^{n-1}|-|z|\right|\\ &= \frac{1}{2.3^{n-1}}.3^{n-1}=\frac{1}{2} \quad \text{for } z>3^{n-1}.\end{aligned}$$

But

$$\begin{aligned}\frac{1}{q_n-p_n}\operatorname{card}(\{p_n<m\le q_n:|d(z,F_m)-d(z,F)|\ge r+\varepsilon\}) &= \frac{1}{3^n-3^{n-1}}\operatorname{card}\left(\left\{3^{n-1}<m\le 3^n:|d(z,F_m)-d(z,\{0\})|\ge r+\varepsilon\right\}\right)\\ &= \frac{1}{2.3^{n-1}}.1\to 0 \text{ as } n\to\infty\end{aligned}$$

Thus $(F_m)_{m\in\mathbb{N}}\in WS_D^r$ and $(F_m)_{m\in\mathbb{N}}$ is not Wijsman deferred strongly $r$-Cesàro summable for $r=0$. □

## 3. Wijsman deferred statistical $r$-convergence via dense subsequences

In this section, we will focus on deferred statistical dense subsequences of a sequence and we here avail an opportunity to establish a relation between Wijsman deferred statistical $r$-convergent sequence and its deferred statistically dense subsequences.

**Definition 3.1.** *A subsequence $(F_{m_k})_{k\in\mathbb{N}}$ of $(F_m)_{m\in\mathbb{N}}$ is said to be deferred statistically dense if $\delta_D(B)=1$, where $B=\{m_1,m_2,m_3,\ldots\}$.*

**Remark 3.2.** *Unlike usual convergence, a subsequence of a Wijsman deferred statistically r-convergent sequence need not be Wijsman deferred statistically r-convergent.*
*For this, consider $Z=\mathbb{R}$ with usual metric $d(z,z')=|z-z'|$ for all $z,z'\in Z$. Let $(F_m)_{m\in\mathbb{N}}$ be a sequence of subsets of Z defined as follows:*

$$F_m=\begin{cases}\{m\} & \text{for } m=q_n,\ m\in(p_n,q_n]\\ \{1\} & \text{otherwise}\end{cases} \quad \text{and} \quad F=\{1\}.$$

*Here $\{p_n<m\le q_n:|d(z,F_m)-d(z,F)|>r+\varepsilon\}\subseteq\{1\}$ and so $F_m\xrightarrow{WS_D^r}F$. Now $\left(F_{q_n}\right)=(\{q_n\})$ is a subsequence of $(F_m)_{m\in\mathbb{N}}$ which is not Wijsman deferred statistically r-convergent.*

The following theorem characterizes Wijsman deferred statistically $r$-convergent sequence in terms of its deferred statistically dense subsequences.

**Theorem 3.3.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is Wijsman deferred statistically r-convergent to some non-empty closed $F\subseteq Z$ iff every deferred statistically dense subsequence of $(F_m)_{m\in\mathbb{N}}$ is Wijsman deferred statistically r-convergent to the same limit F.*

*Proof.* Let $(F_m)_{m\in\mathbb{N}}$ is Wijsman deferred statistically $r$-convergent to some $F\subseteq Z$ and $(F_{m_k})$ is a subsequence of $(F_m)_{m\in\mathbb{N}}$ which is not Wijsman deferred statistically $r$-convergent to $F$. Then for $\varepsilon>0$ and $z\in Z$, we have

$$\liminf_n\frac{1}{q_n-p_n}\operatorname{card}(\{p_n<m_k\le q_n:|d(z,F_{m_k})-d(z,F)|\ge r+\varepsilon\})=\lambda, \tag{3.1}$$

where $\lambda\in(0,1)$. As $(F_{m_k})_{k\in\mathbb{N}}$ is a subsequence of $(F_m)_{m\in\mathbb{N}}$, so we have

$$\{p_n<m\le q_n:|d(z,F_m)-d(z,F)|\ge r+\varepsilon\}\supseteq\{p_n<m_k\le q_n:|d(z,F_{m_k})-d(z,F)|\ge r+\varepsilon\}$$

In view of (3.1), we have

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\operatorname{card}(\{p_n<m\le q_n:|d(z,F_m)-d(z,F)|\ge r+\varepsilon\})\neq 0$$

a contradiction as $F_m\xrightarrow{WS_D^r}F$.

Converse part follows in view of the fact that every sequence is a deferred statistically dense subsequence of itself. □

A partial answer to Remark 3.2 may be given in the form of following

**Corollary 3.4.** *A deferred statistically dense subsequence of a Wijsman deferred statistically r-convergent sequence is Wijsman deferred statistically r-convergent.*

**Theorem 3.5.** *A sequence $(F_m)_{m\in\mathbb{N}}$ of non-empty closed subsets of Z is Wijsman deferred statistically r-convergent to some non-empty closed $F\subseteq Z$ iff $(F_m)_{m\in\mathbb{N}}$ has a deferred statistically dense subsequence which is Wijsman r-convergent to F.*

*Proof.* Let $(F_m)_{m\in\mathbb{N}}$ is Wijsman deferred statistically $r$-convergent to $F$. Then for each $\varepsilon>0$ and $z\in Z$,

$$\delta_D(\{m\in\mathbb{N}:|d(z,F_m-d(z,F)|\geq r+\varepsilon\})=0,$$

i.e., $\delta_D(\{m\in\mathbb{N}:|a_m-0|\geq r+\varepsilon\})=0$ where $a_m=d(z,F_m)-d(z,F)$. Thus $(a_m)_{m\in\mathbb{N}}$ is a sequence of reals which is deferred statistically $r$-convergent to 0. Using Corollary 2.14, there exists deferred statistically dense set $B=\{m_1<m_2<m_3<\ldots<m_k<\ldots\}\subseteq\mathbb{N}$ such that $r-\lim a_{m_k}=0$, i.e., $|a_{m_k}-0|<r+\varepsilon$ for all $k\in\mathbb{N}$, i.e., $|d(z,F_{m_k})-d(z,F)|<r+\varepsilon$ for all $k\in\mathbb{N}$ and thus $(F_{m_k})_{k\in\mathbb{N}}$ is a deferred statistically dense subsequence of $(F_m)_{m\in\mathbb{N}}$ which is Wijsman $r$-convergent to $F$.

Conversely, let $(F_{m_k})_{k\in\mathbb{N}}$ is a deferred statistically dense subsequence of $(F_m)_{m\in\mathbb{N}}$ such that $F_{m_k}\xrightarrow{W^r}F$, i.e., $W^r-\lim F_{m_k}=F$. So for $\varepsilon>0$, there exist $N\in\mathbb{N}$ such that $|d(z,F_{m_k})-d(z,F)|<r+\varepsilon$ for all $k\geq N$ and $\delta_D(B)=1$ where $B=\{m_1<m_2<m_3<\ldots\}$. Now

$$\{m\in\mathbb{N}:p_n<m\leq q_n,|d(z,F_m)-d(z,F)|\geq r+\varepsilon\}\subseteq\{m_1,m_2,m_3,\ldots,m_{N-1}\}\cup(\mathbb{N}-B).$$

The proof follows in view of the fact that every finite subset of $\mathbb{N}$ has zero deferred density. □

**Definition 3.6.** *Let $(Z,d)$ be a metric space and $(F_m)_{m\in\mathbb{N}}$, $(G_m)_{m\in\mathbb{N}}$ are sequences of non-empty closed subsets of Z such that for each $z\in Z$, $d(z,F_m)>0$ and $d(z,G_m)>0$.*

*(a) We say that the sequences $(F_m)_{m\in\mathbb{N}}$ and $(G_m)_{m\in\mathbb{N}}$ are asymptotically Wijsman deferred r-statistical equivalent of multiple L if for each $\varepsilon>0$ and for each $z\in Z$,*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\text{card}\left(\left\{p_n<m\leq q_n:\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|\geq r+\varepsilon\right\}\right)=0,$$

*denoted by $F_m\sim G_m\left(WS_D^{L,r}\right)$ and simply asymptotically Wijsman deferred r-statistical equivalent if $L=1$, written as $F_m\sim G_m\,(WS_D^r)$.*

*(b) The sequences $(F_m)_{m\in\mathbb{N}}$ and $(G_m)_{m\in\mathbb{N}}$ are strongly Cesàro asymptotically deferred r-equivalent of multiple L if for each $\varepsilon>0$ and for each $z\in Z$,*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{m=p_n+1}^{q_n}\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|=r,$$

$$i.e.,\ \lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{m=p_n+1}^{q_n}\left[\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|-r\right]=0,$$

*and denoted by $F_m\sim G_m\left(WN_D^{L,r}\right)$.*

**Theorem 3.7.** *Let $(Z,d)$ be a metric space and $(F_m)_{m\in\mathbb{N}}$, $(G_m)_{m\in\mathbb{N}}$, be sequences of non-empty closed subsets of Z then $F_m\sim G_m\left(WN_D^{L,r}\right)$ implies $F_m\sim G_m\left(WS_D^{L,r}\right)$. Converse need not be true in general.*

*Proof.* Let $\varepsilon>0$ be given and $F_m\sim G_m\left(WN_D^{L,r}\right)$. For $z\in Z$, put $\Lambda_{m,r}=\left[\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|-r\right]$, then

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{m=p_n+1}^{q_n}\Lambda_{m,r}=0.$$

Now

$$\frac{1}{q_n-p_n}\sum_{m=p_n+1}^{q_n}\left[\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|-r\right]\geq\frac{1}{q_n-p_n}\sum_{\substack{m=p_n+1\\ \Lambda_{m,r}\geq\varepsilon}}^{q_n}\left[\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|-r\right]$$

$$\geq\varepsilon.\frac{1}{q_n-p_n}\text{card}\left(\left\{p_n<m\leq q_n:\left|\frac{d(z,F_m)}{d(z,G_m)}-L\right|\geq r+\varepsilon\right\}\right)$$

which gives the result.

Converse need not be true in general. For this take $(F_m)_{m\in\mathbb{N}}$ and $(G_m)_{m\in\mathbb{N}}$ two sequences as follows:

$$F_m=\begin{cases}\{m\} & \text{if } m=q_n\\ \{0\} & \text{otherwise}\end{cases}$$

and $G_m=\{0\}$ for all $m\in\mathbb{N}$. For $\varepsilon>0$ and for each $z\in Z$,

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\text{card}\left(\left\{p_n<m\leq q_n:\left|\frac{d(z,F_m)}{d(z,G_m)}-1\right|\geq r+\varepsilon\right\}\right)\leq\lim_{n\to\infty}\frac{1}{q_n-p_n}=0$$

so $(F_m)\sim WS_D^r(G_m)$. But

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{m=p_n+1}^{q_n}\left|\frac{d(z,F_m)}{d(z,G_m)}-1\right|\neq0.$$

Thus $(F_m)\nsim WN_D^r(G_m)$. It is noted that here $(F_m)_{m\in\mathbb{N}}$ is not bounded sequence. □

**Remark 3.8.** *It is an open problem to see the condition under which the converse of Theorem 3.7 holds.*

**Remark 3.9.** *It is an open problem to determine the condition which guarantee that subsequence of Wijsmann deferred statistically $r$-convergent sequence is Wijsmann deferred statistically convergent.*

# 4. Conclusion

In this concluding section, we once again reiterate the main achievements of the manuscript as well as possible directions for further study in the form of open problems. In the present paper, we availed an opportunity to introduce and explore the notion of Wijsmann deferred statistical $r$-convergence of sequences of sets, which is in fact a generalization of Wijsmann deferred statistical convergence of sequences of sets. A statistical analog of squeeze principle for closed set sequences is also obtained. Apart from this, we characterize Wijsmann deferred statistically $r$-convergent sequence in terms of its deferred statistically dense subsequences. Open problems in the form of Remak 3.8 and Remark 3.9 are included in the Section 3.

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# References

[1] R. P. Agnew, *On deferred Cesàro mean*, Ann. Math., **33**(1932), 413-421.
[2] M. Altınok, B. Inan and M. Küçükaslan, *On asymptotically Wijsman deferred statistical equivalence of sequence of sets* Thai J. Math., **18**(2) (2020), 803-817.
[3] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim., **29**(3-4) (2008), 291-303.
[4] E. Bayram, A. Aydin and M. Küçükaslan, *Weighted statistical rough convergence in normed spaces*, Maejo Int. J. Sci. Technol., **18**(2) (2024), 178-192.
[5] V. K. Bhardwaj and S. Dhawan, *Density by moduli and Wijsman lacunary statistical convergence of sequences of sets*, J. Ineq. Appl., **2017**, 1-20.
[6] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math., **75**(2) (1953), 335-346.
[7] A. Esi, N. L. Braha and A. Rushiti, *Wijsman $\lambda$-statistical convergence of interval numbers*, Bol. Soc. Parana. Mat., **35** (2017), 9-18.
[8] M. Et and M. C. Yilmazer, *On deferred statistical convergence of sequences of sets*, AIMS Mathematics, **5**(3) (2020), 2143-2152.
[9] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2**(3-4) (1951), 241-244.
[10] A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. Londan Math. Soc., **37**(3) (1978), 508-520.
[11] J. A. Fridy, *On statistical convergence*, Analysis, **5**(4) (1985), 301-314.
[12] J. A. Fridy and C. Orhan, *Lacunary statistical summability*, J. Math. Anal. Appl., **173**(2) (1993), 497-504.
[13] E. Gülle and U. Ulusu, *Wijsman deferred invariant statistical and strong p-deferred invariant equivalence of order $\alpha$*, Fundam. J. Math. Appl., **6**(4) (2023), 211-217.
[14] S. Gupta and V. K. Bhardwaj, *On deferred $f-$statistical convergence*, Kyungpook Math. J., **58** (2018), 91-103.
[15] B. Hazarika and A. Esi, *On $\lambda$-asymptotically Wijsman generalized statistical convergence of sequences of sets*, Tatra Mt. Math. Publ.-Number Theory, **56** (2013), 67-77.
[16] B. Hazarika, A. Esi and N. L. Braha, *On asymptotically Wijsman $\sigma$-statistical convergence of set sequences*, J. Math. Anal., **4**(3) (2013), 33-46.
[17] M. Küçükaslan and M. Yılmaztürk, *On deferred statistical convergence of sequences*, Kyungpook Math. J., **56** (2016) 357-366.
[18] M. Mursaleen, *$\lambda$-statistical convergence*, Math. Slovaca, **50** (2000), 111-115.
[19] F. Nuray and B. E. Rhoades, *Statistical convergence of sequence of sets*, Fasc. Math., **49** (2012), 87-99.
[20] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim., **22**(1-2) (2001), 199-222.
[21] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30**(2) (1980), 139-150.
[22] E. Savaş, *On I-lacunary statistical convergence of order $\alpha$ for sequences of sets*, Filomat, **29**(6) (2015), 1223-1229.
[23] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, The Amer. Math. monthly, **66**(5) (1959), 361-375.
[24] H. Steinhaus, *Sur la convergence ordinaireet la convergence asymptotique*, Colloq. Math., **2**(1) (1951), 73-74.
[25] Ö. Talo, Y. Sever and F. Başar, *On Statistically convergent sequences of closed sets*, Filomat, **30**(6) (2016), 1497–1509.
[26] U. Ulusu and E. Dündar, *I-lacunary statistical convergence of sequences of sets*, Filomat, **28**(8) (2014), 1567-1574.
[27] U. Ulusu and F. Nuray, *Lacunary statistical summability of sequences of sets*, Konuralp J. Math., **3**(2) (2015), 176-184.
[28] U. Ulusu and E. Savaş, *An extension of asymptotically lacunary statistical equivalence set sequences*, J. Ineq. Appl., **2014**, 1-8.
[29] R. A. Wijsman, *Convergence of sequence of convex sets, cones and functions*, Bull. Amer. Math. Soc., **70** (1964), 186-188.
[30] R. A. Wijsman, *Convergence of sequence of convex sets, cones and functions II*, Trans. Amer. Math. Soc., **123**(1) (1966), 32-45.
[31] M. Yılmaztürk and M. Küçükaslan, *On strongly deferred Cesaro summability and deferred statistical convergence of the sequences*, Bitlis Eren Univ. J. Sci. Technol., **3**(1) (2013) 22-25.
[32] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, UK 1979.