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Decision Problems in Queueing Theory: A Numeric Application

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Abstract

The application of general behavioral patterns obtained from stochastic processes has always played an important role in Queuing Theory. Since the first studies in which optimization techniques were used in the decision-making process, design and control procedures have been included in studies especially in the field of statistics and operations. In the first studies where queuing systems were modeled and their operability was optimized, performance measures such as the block probability and the average waiting times in the system were considered in the decision-making process. With the availability of performance measures based on probabilistic methods in modeling queuing systems, the decision-making process has begun to be based on such measures. In this study probabilistic calculations of some performance measures of a custom queueing system is given in order to make a decision for optimum parameters of the system. In addition, a numerical example is given to illustrate the case.

Keywords: Queueing theory; Stochastic process; Decision process; Optimization

1. *Introduction*

The application of general behavioral patterns obtained from stochastic processes has always played an important role in Queuing theory. Since the first studies in which optimization techniques were used in the decision-making process, design and control procedures have been included in studies especially in the field of statistics and operations.

On the other hand, the number of studies conducted constitutes a small part of the potential of the subject in terms of volume. Queuing Theory, which occupies a large part of the field of Stochastic processes, can be said to have started with the study conducted by A. K. Erlang (1917). The "equilibrium state of the system" behavior that inevitably occurs in most queuing systems was first studied by Polaczek (1965), in this study, the behavior of the system was analyzed and tried to be defined within a finite time interval. The first analyzed system in queuing theory is the M/M/1 system. Obtaining the equilibrium state equations of this system under some statistical assumptions and defining the limit distribution of the queue length are relatively simple and can be solved with iterative techniques. On the other hand, when the time parameter is taken into consideration, more complex mathematical calculations are needed. In this sense, the first solution methods were proposed by Bailey (1952). In addition, Ledermann and Reuter (1956) used spectral theory for solutions in their study. In the following studies, Laplace transform and techniques using Laplace transform and generating functions together were used as solution methods. Probabilistic methods were first used in the analysis of queuing systems by Kendall (1956), (1953). Stidham (1995) discussed the reasons for the inadequacy of studies on design and control in queuing theory. The decision-making process is generally based on two basic ideas: performance measures and decision problems. Decision problems are divided into two as design and control problems (T. B. Crabill et al. 1977). This study shows that a suitable process to be determined to optimize criteria such as cost, or profit is a design problem. The optimization of a control problem that we encounter in real life is dynamic. In other words, the functions of the system are in a constant change with time.

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2. *Performance Measures for Decision Making*

There is a "decision making" process inherent in the solution phase of any problem we encounter in real life. In the first studies where queuing systems were modeled and the operability of the systems was optimized, performance measures such as the probability of the system being blocked and the average waiting times in the system were considered in the decision-making process. Various charts were developed for these performance measures (Bhat, 2003). Today, since the competence of data visualization and graphic software has increased greatly, these charts have become almost unnecessary. As we have mentioned before, with the availability of performance measures based on probabilistic methods in modeling queuing systems, the decision-making process has begun to be based on such measures. In addition, the simulation support provided by computers has become a multiplier power in the decision-making process. As it is known, simulations determine the accuracy and best features of the model under certain conditions and situations.

3. *Results Design Problems in Decision Making*

In design problems, regardless of the system being modeled, the main idea is to optimize the parameters in a way that will optimize the operating performance of the system. The cost function is a part of the optimization process, and this function is used to obtain the optimum values of the optimal configurations. In this sense, cost functions can be based on monetary costs or performance measures depending on the model. Therefore, the problems mentioned can also be called "economic problems". Optimization of any problem is static, and the process is carried out using predefined and specific procedures. However, when it comes to modeling queuing systems and queuing system models can enter very complex situations, the procedures inherent in static optimization may not work. Statistical and numerical procedures will be much more appropriate for such situations. If we consider the problem of determining the optimum number of customers that should be accepted for service in an $M/M/1/N$ queueing system, we will need to balance the cost of service with the cost of losing customers. Let us assume that customer arrival is Poisson with rate λ and that there are service units with rate μ. In this case, where the cost per unit of time is $F\mu$ and the gross profit per single service is B, the net profit per unit of time is calculated as follows:

$$K = \frac{\lambda B(1-\rho^N)}{1-\rho^{N+1}} - F\mu \tag{1}$$

In the Equation (1), ρ is the traffic density of the system and is determined as $\rho = \lambda / \mu$ and N is the total number of customers in the system. If the derivative of this equation is taken with respect to the parameter μ and set equal to zero, the following equation is obtained for the maximum value of the parameter μ:

$$\rho^{N+1}\frac{N-(N+1)\rho+\rho^{N+1}}{(1-\rho^{N+1})^2} = \frac{F}{B} \tag{2}$$

The graph obtained from this equation can be used to determine the number of customers N that should be accepted into the queue system for the varying cost parameters and service rates of the system. In the case of infinite waiting space in the queue system, the optimum value of the service rate μ is obtained with the following cost function using the standard optimization approach:

$$Y\mu + XZ = Y\mu + \frac{X}{\mu-\lambda} \tag{3}$$

In this equation, X is the waiting cost per unit time, Y is the service cost per unit time, and Z is the average waiting time. If the derivative of this cost function is taken with respect to the parameter μ and set equal to zero:

$$\mu = \lambda + \sqrt{\frac{X}{Y}} \tag{4}$$

is obtained. On the other hand, in systems with multiple service units, optimization is performed by trial-and-error method to determine the optimum number of service units. Let's assume that all arrivals in a three-server queue system are Poisson and the queue discipline is First come First served (FCFS). In this case, the waiting cost will be proportional to the time elapsed in the system and the service cost will be a linear function of the number of servers. Again, three basic models are used to determine the optimum values for λ, μ and s in the cost functions, namely the arrival rate per server λ, the service rate per server μ and the number of servers s: the first model finds the s value from the cost function, the second model finds the λ and s values, and the third model finds the μ and s values, respectively. Due to the multi-server structure, it should also be considered that each server should have its own waiting line if the service time is not exponential.

4. Application: a Numeric example

Let's assume that customers arrive to a supermarket according to a Poisson distribution with a rate of λ. After customers receive their products, they will line up in front of the cash register to pay. The time spent in this process is assumed to be exponentially distributed. Let's determine the optimum value of the number of cash registers in the supermarket under the following cost function, where the mean of the distribution of the time it takes for each customer to go to the cash register is m_1 and the second moment is m_2. (i) X_1 is the cost per unit of time of a customer waiting (ii) X_2 is the cost per unit of time of a customer paying at the cash register. Since the time spent by the customer in choosing a product is exponentially distributed, the arrival process at the checkout can also be assumed to be Poisson. When the number of checkouts is s and customers are assumed to choose checkouts randomly, the arrival rate at each checkout is assumed to be Poisson with rate λ/s. Considering the waiting time of a customer in the $M/G/1$ queue system:

$$Z_q = \left(\frac{\lambda}{s}\right) m_2/2\left(1-\frac{\lambda m_1}{s}\right) = \frac{\lambda m_2}{2(s-\lambda m_1)} \tag{5}$$

s obtained. Let the total cost per unit time be X, then the average total cost is calculated with the following equation,

$$E(X) = \frac{\lambda m_2 X_1}{2s-2\lambda m_1} + SX_2 \tag{6}$$

If the cost function is minimized with respect to s, It turns out that the equation minimizes $E(X)$ as given in Equation (6). For a numerical example, if $\lambda = 2$ (2 customer arrivals per minute), the checkout time will be an exponential distribution with mean $m_1 = 3$ minutes. Thus, $m_2 = 9$. In addition, $X_1 = 0.5$, $X_2 = 2.5$ are calculated and if these values are written in the equation (6), the optimal value for the number of checkouts in the supermarket is obtained as $s = 7$.

5. Results and Discussion

There are plenty of papers given on optimum design and Control Theory on various real-world problem. As mentioned before design and control procedures are rarely applied in stochastic process. There are several reasons for this. First, Queueing Theory has a dynamic structure, that is the parameters of the system mostly depend on time which makes it a big challenge to apply optimization and control procedure. However as shown in this study there are some probabilistic methods to calculate some important performance measures of a queueing system. On the other hand, obtaining performance characteristics of more complex queueing systems via control and design procedures can be more challenging. In this manner for more studies in the field, computer aided techniques such as simulation and computational statistical methods can be applied for the optimization and design of queueing systems.

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