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TUBULAR SURFACES OF ADJOINT CURVES ACCORDING TO THE MODIFIED ORTHOGONAL FRAME

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Abstract: In this paper, we study tubular surfaces defined by adjoint curves which have a wide range of applications. In three-dimensional Euclidean space, we consider tubular surfaces in modified orthogonal frame generated by a curve β whose center curve is adjoint of a curve α . We give some characterizations for tubular surfaces constructed according to with curvature and with torsion modified orthogonal frames. Through these characterizations we obtain some important results. We also study asymptotic and geodesic curves as well as flat, minimal, Weingarten and linear-Weingarten surfaces using conventional differential geometry techniques. Finally, we present case examples for both versions of the frame to validate our theoretical results.

Keywords: Tubular surface, Adjoint curve, Modified orthogonal frame

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1. Introduction

The Frenet frame has not always been sufficient for solving all problems in studies involving curves and surfaces. Therefore, alternative frames have been developed to address such issues. While the Serret-Frenet frame is a useful tool for the analysis of curves, it cannot be defined at points where the second derivative of the curve is zero. Moreover, at these points, the principal normal vector may exhibit discontinuities, making computations more challenging. To address these problems, Sasai introduced the modified orthogonal frame as an alternative to the Frenet frame. In this frame, the curvature function is used to multiply each Frenet vector, resulting in a new set of vectors. This approach allows the application of a new formula corresponding to the Frenet differentiation equations for the aforementioned cases (Sasai, 1984).

In computer-aided design (CAGD), the envelope of a moving sphere with a changeable radius is called a canal surface, and it is commonly used for surface and solid modeling. The combination of the spheres that are determined by the radius function $r(s)$ and the center curve $\alpha(s)$ yields the canal surface $\psi(s,\theta)$. ψ canal surface can be parameterized as follows (equation 1):

$$\psi(s,\theta) = \alpha(s) - r'(s)r(s)t(s) \pm r(s)\sqrt{1-r'(s)^2}(\cos\theta n(s) + \sin\theta b(s)) \quad (1)$$

where $\alpha(s)$ is a unit speed curve parameterized by arc-lenght s. $\{t,n,b\}$ is the Frenet frame of $\alpha(s)$ (Xu et al., 2006). These canal surfaces are known as tubular surfaces if the radius function $r(s)=r$. In addition, it is useful for reconstructing shapes, planning robot motion, creating blending surfaces, and observing long, thin things like pipes, ropes, poles, even living intestines. Research on tube surfaces in various frames and spaces can be found in (Karacan et al., 2006; Karacan and Yayli, 2008; Yüksel et al., 2011; Karacan and Tuncer, 2013; Saad et al., 2024).

The theory of curves has been one of the main fields of study in differential geometry (Bükçü and Karacan, 2016; Mazlum et al., 2022; Yüksel et al., 2022). Involute-evolute curves, adjoint curves, Bertrand curves, Manheimm curves, and helices have been the most common intersecting curves in recent years. As stated in (Kühnel and Hunt, 2005), an adjoint curve is the integral of a binormal vector of a curve $\alpha(s)$ with any parameter s. In numerous applications, including number theory, coding theory, algebraic geometry, etc., the adjoint curves are crucial. Moreover, adjoint curves are the subject of numerous investigations in (Nurkan et al., 2019; Arıkan and Nurkan, 2020; Cakmak and Şahin, 2022; Nurkan and Güven, 2022).

This study examines tubular surfaces whose centers are adjoint curves defined with respect to the modified orthogonal frame in $\mathbb{E}^3$. First, the modified orthogonal frame is used to create the adjoint curve and establish the link between the Frenet vectors (Arıkan and Nurkan, 2020). Next, the geometric properties of these surfaces are obtained, along with some significant findings. Finally, using an example, we demonstrate the visualizations of provided surfaces and tubular surfaces.

2. Materials and Methods

The mathematical definitions for the curvature and torsion of a curve in E^3 are provided in (O'Neill, 1996). To begin, let us consider unit-speed curves. A key element in the differential geometry of a curve is the use of the Frenet frame field $\{t, n, b\}$. The Frenet differentiation formulas, constructed using these vectors, are expressed as follows:

$t' = \kappa n, n' = -\kappa t + \tau b, b' = -\tau n,$

where κ, τ represent the first and second curvature of the curve, respectively.

Let $\alpha: I \to E^3$ be a space curve. We assume that the curvature κ of α is not identically zero. As a result, the modified orthogonal frame $\{T, N, B\}$ with the curvature κ of the curve α can be defined. Now we define the modified orthogonal frame $\{T, N, B\}$ as follows:

$$T = \frac{d\alpha}{ds}, N = \frac{dT}{ds}, B = T \wedge N.$$

The following represents the relationships between the modified orthogonal frame $\{T, N, B\}$ and Frenet frame $\{t, n, b\}$ at non-zero positions of κ

$T = t, N = \kappa n, B = \kappa b.$

The modified orthogonal frame $\{T, N, B\}$ satisfies the below relations

$\langle N, T\rangle = \langle B, T\rangle = \langle B, N\rangle = 0,$

$\langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = \kappa^2,$

where $\langle , \rangle$ is the inner product. Due to these equations, the derivative equations of the modified orthogonal frame $\{T, N, B\}$ are given as

$$T' = N,$$

$$N' = -\kappa^2 T + \frac{\kappa'}{\kappa} N + \tau B,$$

$$B' = -\tau N + \frac{\kappa'}{\kappa} B,$$

where $\tau = \frac{det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of α (Bükçü and Karacan, 2016).

Now let's talk about the modified orthogonal modified frame with torsion.

The following represents the relationships between the modified orthogonal frame with torsion $\{T, N, B\}$ and Frenet frame $\{t, n, b\}$ at non-zero positions of

$T = t, N = \tau n, B = \tau b$

where

$\langle N, T\rangle = \langle B, T\rangle = \langle B, N\rangle = 0,$

$\langle T, T\rangle = 1, \langle N, N\rangle = \langle B, B\rangle = \tau^2.$

In this case, the following modified orthogonal frame with torsion hold:

$$T' = \frac{\kappa}{\tau} N,$$

$$N' = -\kappa\tau T + \frac{\tau'}{\tau} N + \tau B,$$

$$B' = -\tau N + \frac{\tau'}{\tau} B$$ (Bükçü and Karacan, 2016).

Let $\psi(s, \theta)$ a surface in $\mathbb{E}^3$ and $U(s, \theta)$ be the typical unit normal vector field on $\psi(s, \theta)$ defined by $U = \frac{\psi_s \times \psi_\theta}{\|\psi_s \times \psi_\theta\|}$ where $\psi_s = \frac{\partial \psi}{\partial s}$ and $\psi_\theta = \frac{\partial \psi}{\partial \theta}$ are the tangent vectors of $\psi(s, \theta)$. Then, the first fundamental form I of $\psi(s, \theta)$ is defined by

$I = g_{11} ds^2 + 2g_{12} ds d\theta + g_{22} d\theta^2$

where

$g_{11} = \langle \psi_s, \psi_s\rangle,\ g_{12} = \langle \psi_s, \psi_\theta\rangle, g_{22} = \langle \psi_\theta, \psi_\theta\rangle.$

The second fundamental form of $\psi(s, \theta)$ can be defined as follows:

$II = h_{11} ds^2 + 2h_{12} ds d\theta + h_{22} d\theta^2$

where

$h_{11} = \langle \psi_{ss}, U\rangle,\ h_{12} = \langle \psi_s, U\rangle, g_{22} = \langle \psi_{\theta\theta}, U\rangle$

The mean curvature H and the Gaussian curvature K are both represented as follows:

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, H = \frac{h_{11}g_{22} - 2g_{12}h_{12} + g_{11}h_{22}}{2(g_{11}g_{22} - g_{12}^2)}. \quad (1)$$

Definition 1; Let α be a unit speed curve in E^3 with $\tau \neq 0$ and the Frenet frame of α be $\{T_\alpha, N_\alpha, B_\alpha\}$. The adjoint curve of α is defined in as

$\beta(s) = \int_{s_0}^{s} B_\alpha(s) ds$ (Kühnel and Hunt, 2005).

Theorem 1; Let α be a curve with an arc length parameter s and β be the adjoint curve of α. If the Frenet vectors of α and β are $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$ the curvature and torsion are $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ respectively, then the following relations hold (Arıkan and Nurkan, 2020):

$$\begin{cases} T_\beta = B_\alpha, \\ N_\beta = -N_\alpha, \kappa_\beta = \tau_\alpha, \tau_\beta = \kappa_\alpha. \\ B_\beta = T_\alpha, \end{cases}$$

Theorem 2; Let α be a unit speed regular curve in E^3 and β is the adjoint curve of α according to the modified orthogonal frame with curvature. If the modified orthogonal frames of α and β are $\{T_\alpha, N_\alpha, B_\alpha\}$ and $\{T_\beta, N_\beta, B_\beta\}$, the curvature and torsion are $\{\kappa_\alpha, \tau_\alpha\}$ and $\{\kappa_\beta, \tau_\beta\}$ respectively, the following relations hold (Arıkan and Nurkan, 2020):

$$\begin{cases} T_\beta = \left(\frac{1}{\kappa_\alpha}\right) B_\alpha, \\ N_\beta = -\left(\frac{\tau_\alpha}{\kappa_\alpha^2}\right) N_\alpha, \kappa_\beta = \frac{\tau_\alpha}{\kappa_\alpha}, \tau_\beta = 1. \\ B_\beta = \left(\frac{\tau_\alpha}{\kappa_\alpha}\right) T_\alpha, \end{cases}$$

Definition 2; If □$(X, Y) = 0$, where the Jacobi function □ is defined as $X_S Y_\theta - Y_S X_\theta = 0$ then the pair $(X, Y), X \neq Y$ of the curvatures K, H of a tubular surface $\psi(s, \theta)$ is said to be a (X, Y)-Weingarten surface (Kim et al., 2016).

Definition 3; If $\psi(s, \theta)$ satisfies the following relation, then the pair $(X, Y), X \neq Y$ of the curvatures K, H of the tubular surface $\psi(s, \theta)$ is said to be a (X, Y)-linear Weingarten surface:

$a_1 X + a_2 Y = a_3$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $(a_1, a_2, a_3) \neq (0,0,0)$ (López, 2009).

3. Results

3.1. Tubular Surfaces Whose Center Curve is an Adjoint Curve in a Modified Orthogonal Frame with Curvature

The tubular surface whose center curve adjoint curve β of the α using to the modified orthogonal frame with curvature in E^3 is examined in this section. Concerning the modified orthogonal frame, the parameterization of the tubular surface exists.

$$\psi(s,\theta)=\alpha(s)+\frac{r}{\kappa(s)}(\cos\theta N(s)+\sin\theta B(s)) \tag{2}$$

where $r=const.$ and $\kappa\neq 0$. The center curve β of the curve α is considered to be the adjoint curve of this surface. So from equation 2 we get

$$\psi(s,\theta)=\beta(s)+\frac{r}{\kappa(s)}\left(\cos\theta N_\beta(s)+\sin\theta B_\beta(s)\right)$$

$$\psi(s,\theta)=\int_{s_0}^{s}B_\alpha(s)ds+r\left(-\tfrac{\cos\theta}{\kappa_\alpha(s)}N_\alpha(s)+\sin\theta T_\alpha(s)\right).$$

The derivatives according to s and θ concerning the tubular surface $\psi(s,\theta)$ are (equation 3)

$$\psi_s=(r\kappa_\alpha\cos\theta)T_\alpha+(r\sin\theta)N_\alpha+\left(1-\tfrac{r\tau_\alpha}{\kappa_\alpha}\cos\theta\right)B_\alpha\,,$$

$$\psi_\theta=(r\cos\theta)T_\alpha+\left(\tfrac{r}{\kappa_\alpha}\sin\theta\right)N_\alpha\,,$$

$$\psi_{ss}=(r\kappa_\alpha'\cos\theta-r\kappa_\alpha^2\sin\theta)T_\alpha+\left(\left(r\kappa_\alpha\cos\theta+r\frac{\kappa_\alpha'}{\kappa_\alpha}\sin\theta\right)-\left(\frac{\tau_\alpha}{\kappa_\alpha}\right)(\kappa_\alpha-r\tau_\alpha\cos\theta)\right)N_\alpha+\begin{pmatrix} r\tau_\alpha\sin\theta+\frac{r\cos\theta}{\kappa_\alpha^2}(\tau_\alpha\kappa_\alpha'-\kappa_\alpha\tau_\alpha') \\ +\frac{\kappa_\alpha'}{\kappa_\alpha^2}(\kappa_\alpha-r\tau_\alpha\cos\theta)\end{pmatrix}B_\alpha\,,$$

$$\psi_{\theta\theta}=(-r\sin\theta)T_\alpha+\left(\frac{r}{\kappa_\alpha}\cos\theta\right)N_\alpha. \tag{3}$$

Hence we obtain

$g_{11}=r^2\kappa_\alpha^2+r^2\tau_\alpha^2\cos^2\theta+\kappa_\alpha^2-2r\,\kappa_\alpha\tau_\alpha\cos\theta,$

$g_{12}=r^2\kappa_\alpha\,,$

$g_{12}=r^2,$

$g=g_{11}g_{22}-g_{12}^2=r^2(\kappa_\alpha-r\tau_\alpha\cos\theta)^2\neq 0.$

The normal vector field the unit U is provided by (equation 4)

$$U(s)=-\sin\theta T_\alpha+\frac{1}{\kappa_\alpha}\cos\theta\,N_\alpha \tag{4}$$

as well as the second fundamental form's coefficients, which are as follows:

$h_{11}=r\kappa_\alpha^2-\tau_\alpha\cos\theta\,(\kappa_\alpha-r\tau_\alpha\cos\theta),$

$h_{12}=r\kappa_\alpha,$

$h_{22}=r,$

from equation 1, the Gaussian and the mean curvatures are provided by (equation 5 and 6)

$$K=-\frac{\tau_\alpha\cos\theta}{r(\kappa_\alpha-r\tau_\alpha\cos\theta)} \tag{5}$$

$$H=\frac{\kappa_\alpha-2r\tau_\alpha\cos\theta}{2r(\kappa_\alpha-r\tau_\alpha\cos\theta)}. \tag{6}$$

The mean curvature H and the Gaussian curvature K of tubular surface $\psi(s,\theta)$ satisfy the following relationship

$$H=\tfrac{1}{2}\left(Kr+\tfrac{1}{r}\right) \tag{7}$$

it follows that, the principal curvatures of $\psi(s,\theta)$ are obtained as follows:

$k_1=\frac{1}{r}\,,k_2=Kr.$

Proposition 1; In Euclidean 3-space, let $\psi(s,\theta)$ be a tubular surface. $\psi(s,\theta)$ is not a flat surface in such case.

Proof; We presume that $\psi(s,\theta)$ is flat. In this case, $\kappa=0$ from equation 5, we get

$-\tau_\alpha\cos\theta=0$, since $\tau_\alpha\neq 0$ from the definition of adjoint curve $\psi(s,\theta)$ the tube surface is not flat.

Proposition 2; In Euclidean 3-space, let $\psi(s,\theta)$ be a tubular surface. Then $\psi(s,\theta)$ is minimal if and only if $r=\frac{\kappa_\alpha}{2\tau_\alpha\cos\theta}$.

Proof; Equation 6 provides the result immediately.

Theorem 1; Let $\psi(s,\theta)$ be a tubular surface with a modified orthogonal frame in E^3. Then

i) Asymptotic curves are s-parameter curves of $\psi(s,\theta)$ if and only if $r=\frac{\tau_\alpha\kappa_\alpha\cos\theta}{\tau_\alpha^2\cos^2\theta+\kappa_\alpha^2}$.

ii) Asymptotic curves cannot be θ-parameter curves of $\psi(s,\theta)$.

Proof; We can infer from the definition of asymptotic curves that

$\langle\psi_{ss},U\rangle=0\,,\langle\psi_{\theta\theta},U\rangle=0$

i) From equations 3 and 4, we can get

$h_{11}=r\kappa_\alpha^2-\tau_\alpha\cos\theta\,(\kappa_\alpha-r\tau_\alpha\cos\theta)=0$

$r=\frac{\tau_\alpha\kappa_\alpha\cos\theta}{\tau_\alpha^2\cos^2\theta+\kappa_\alpha^2}$.

ii) Since $h_{22}\neq 0$, θ-parameter curves of the $\psi(s,\theta)$ cannot be asymptotic.

Theorem 2; In $\mathbb{E}^3$, let $\psi(s,\theta)$ be a tubular surface with a modified orthogonal frame. Then

i) Geodesic curves cannot be s-parameter curves of $\psi(s,\theta)$.

ii) The curves for the θ -parameter of $\psi(s,\theta)$ are geodesic.

Proof; In order to define the parameter curves s and, it is necessary to provide the values of $\psi_{ss}\times U=0$ and $\psi_{\theta\theta}\times U=0$ for the geodesic curves.

i) According to equations 3 and 4, we obtain

$$\psi_{ss}\times U=(r\tau_\alpha'\cos^2\theta-r\cos\theta\sin\theta\,\kappa_\alpha\tau_\alpha-\kappa_\alpha'\cos\theta)T_\alpha+\left(r\cos\theta\sin\theta\tfrac{\tau_\alpha'}{\kappa_\alpha}-r\sin^2\theta\tau_\alpha-\tfrac{\kappa_\alpha'}{\kappa_\alpha}\sin\theta\right)N_\alpha+\left(r\tfrac{\kappa_\alpha'}{\kappa_\alpha}-\tau_\alpha\sin\theta+r\sin\theta\cos\theta\tfrac{\tau_\alpha^2}{\kappa_\alpha}\right)B_\alpha\,.$$

Since T_α , N_α and B_α are linearly independent then, $\psi_{ss}\times U=0$ if and only if $\kappa_\alpha=const.\tau_\alpha=0$. Nevertheless, given $\tau_\alpha\neq 0$, $\psi(s,\theta)$ cannot be a geodesic curve.

ii) Furthermore, we obtain $\psi_{\theta\theta}\times U=0$ from equations 3 and 4. θ-parameter curves are geodesic curves.

We now define the mean curvature and the partial derivative of the Gaussian curvature of the tubular surface as follows:

$$K_s=-\frac{\cos\theta(\kappa_\alpha\tau_\alpha'-\tau_\alpha\kappa_\alpha')}{r(\kappa_\alpha-r\tau_\alpha\cos\theta)^2}\,,$$

$$H_s=-\frac{\cos\theta(\kappa_\alpha\tau_\alpha'-\tau_\alpha\kappa_\alpha')}{2(\kappa_\alpha-r\tau_\alpha\cos\theta)^2},$$

$K_{\theta} = \frac{\kappa_{\alpha}\tau_{\alpha}\sin\theta}{r(\kappa_{\alpha}-r\tau_{\alpha}\cos\theta)^2}$,
$H_{\theta} = \frac{\kappa_{\alpha}\tau_{\alpha}\sin\theta}{2(\kappa_{\alpha}-r\tau_{\alpha}\cos\theta)^2}$. (8)

Theorem 3; Let $\psi(s,\theta)$ be a tubular surface with a modified orthogonal frame in $\mathbb{E}^3$. The surface $\psi(s,\theta)$ is a (K,H) −Weingarten surface.
Proof; If the derivatives of the mean curvature and the Gaussian curvature of the surface $\psi(s,\theta)$, given by equation 8, are substituted into the Jacobi equation
$H_s K_{\theta} - H_{\theta} K_s = 0$
is obtained. Therefore, the surface $\psi(s,\theta)$ is a (K,H) −Weingarten surface.
Theorem 4; Let $\psi(s,\theta)$ be a tubular surface in $\mathbb{E}^3$ equipped with a modified orthogonal frame. If ψ is (K,H) -linear Weingarten surface, then for $a_3 = 1$, the following relations hold $a_1 = -3r^2$ and $a_2 = 2r$, hold.
Proof; Let us assume that ψ is a (K,H) -linear Weingarten surface. In this case, the following equation is satisfied
$a_1 K + a_2 H = 1$
where $a_1, a_2 \in \mathbb{R}$ and $(a_1,a_2) \neq 0$. From equation 7, the following relations are derived

$$2r - a_2 = \frac{-\tau_{\alpha}\cos\theta}{\kappa_{\alpha} - r\tau_{\alpha}\cos\theta}(a_2 r + 2a_1)$$

and
$2\tau_{\alpha}\cos\theta\,(r^2 + a_2 r + a_1) + \kappa_{\alpha}(2r - a_2) = 0$.
Consequently, $a_1 = -3r^2$ is obtained when $a_2 = 2r$.
Example 1; The parametric equation of curve α_1 is provided by

$$\alpha_1(s) = \left(\cos\left(\frac{\sqrt{7}s}{4}\right), \sin\left(\frac{\sqrt{7}s}{4}\right), \frac{3s}{4}\right).$$

The parametric equation of the adjoint curve of α is given by

$$\beta_1(s) = \left(-\frac{3\sqrt{7}}{16}\cos\left(\frac{\sqrt{7}}{4}s\right), -\frac{3\sqrt{7}}{16}\sin\left(\frac{\sqrt{7}}{4}s\right), -\frac{7\sqrt{7}}{64}s\right).$$

The tube surface whose center curve is α_1 according to the modified orthogonal frame with curvature is given by

$$\psi_{\alpha_1}(s,\theta) = \left(\frac{3}{4}\sin\theta\sin\left(\frac{\sqrt{7}}{4}s\right) - \cos\theta\cos\left(\frac{\sqrt{7}}{4}s\right) + \right.$$
$$+\cos\left(\frac{\sqrt{7}}{4}s\right), -\cos\theta\sin\left(\frac{\sqrt{7}}{4}s\right) - \frac{3}{4}\sin\theta\cos\left(\frac{\sqrt{7}}{4}s\right)$$
$$\left. +\sin\left(\frac{\sqrt{7}}{4}s\right), \frac{\sqrt{7}}{4}\sin\theta + \frac{3s}{4}\right).$$

The graph of the tube surface, whose center curve is according to the modified orthogonal frame with curvature, is illustrated in Figure 1.

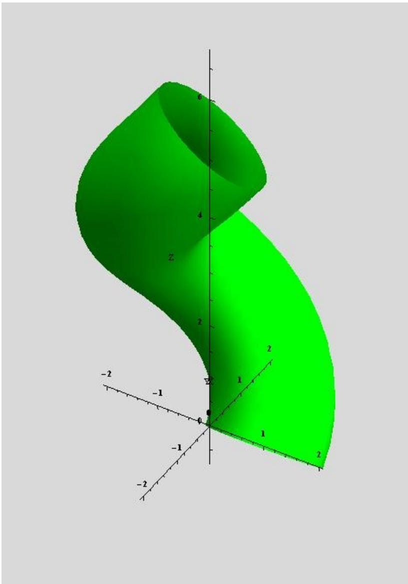

Figure 1 The tube surface, whose center curve is α_1 according to the modified orthogonal frame with curvature.

The tube surface whose center curve is $\beta_1(s)$ according to the modified orthogonal frame with curvature is given by

$$\psi_{\beta_1(s)}(s,\theta) = \left(-\frac{3\sqrt{7}}{16}\cos\left(\frac{\sqrt{7}s}{4}\right) + \cos\theta\cos\left(\frac{\sqrt{7}s}{4}\right)\right.$$
$$-\frac{\sqrt{7}}{4}\sin\theta\sin\left(\frac{\sqrt{7}s}{4}\right), -\frac{3\sqrt{7}}{16}\sin\left(\frac{\sqrt{7}s}{4}\right) + \cos\theta\sin\left(\frac{\sqrt{7}s}{4}\right)$$
$$\left. +\frac{\sqrt{7}}{4}\sin\theta\cos\left(\frac{\sqrt{7}s}{4}\right), \frac{7\sqrt{7}}{64}s + \frac{3}{4}\sin\theta\right)$$

The graph of the tube surface $\psi_{\beta_1(s)}(s,\theta)$ whose center curve is $\beta_1(s)$ according to the modified orthogonal frame with curvature is illustrated in Figure 2.

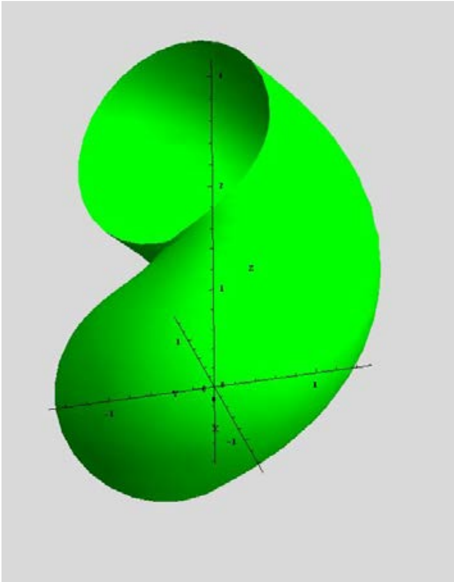

Figure 2. The tube surface whose center curve is β_1 according to the modified orthogonal frame with curvature. (Figure 2. Nin yeri 3.2 bölümünden önce olmalı)

3.2 Tubular Surfaces Whose Center Curve is Adjoint Curve in Modified Orthogonal Frame with Torsion

This section is dedicated to the analysis of the tubular surface whose center curve is the adjoint curve β of the α, according to the modified orthogonal frame with torsion in E^3. The parameterization of the tubular surface exists with respect to the modified orthogonal frame.

$$W(s,\theta)=\alpha(s)+\frac{r}{\tau(s)}(\cos\theta N(s)+\sin\theta B(s)) \tag{9}$$

where $r=const.$ and $\tau\neq 0$. The adjoint curve of this surface is defined as the center curve β of the curve α. Thus, we derive equation 9 and

$$W(s,\theta)=\beta(s)+\frac{r}{\kappa_\beta(s)}\big(\cos\theta\, N_\beta(s)+\sin\theta B_\beta(s)\big),$$

$$W(s,\theta)=\int_{s_0}^{s}B_\alpha(s)ds+r\left(-\frac{\cos\theta}{\tau_\alpha(s)}N_\alpha(s)+\sin\theta T_\alpha(s)\right),$$

$$W_s=(r\kappa_\alpha\cos\theta)T_\alpha+\left(\frac{r\kappa_\alpha}{\tau_\alpha}\sin\theta\right)N_\alpha+(1-r\cos\theta)B_\alpha,$$

$$W_\theta=(r\cos\theta)T_\alpha+\left(\frac{r}{\tau_\alpha}\sin\theta\right)N_\alpha,$$

$$W_{ss}=(r\kappa_\alpha'\cos\theta-r\kappa_\alpha^2\sin\theta)T_\alpha$$
$$+\left(\frac{r\kappa_\alpha^2\cos\theta}{\tau_\alpha}+\frac{r\kappa_\alpha'\sin\theta}{\tau_\alpha}-\tau_\alpha(1-r\cos\theta)\right)N_\alpha$$
$$+\left(r\kappa_\alpha^2\sin\theta+\frac{\tau_\alpha'(1-r\cos\theta)}{\tau_\alpha}\right)B_\alpha,$$

$$W_{\theta\theta}=(-r\sin\theta)T_\alpha+\left(\frac{r\cos\theta}{\tau_\alpha}\right)B_\alpha. \tag{10}$$

Thus, we get at

$$g_{11}=r^2\kappa_\alpha^2+r^2\tau_\alpha^2\cos^2\theta+\tau_\alpha^2-2r\tau_\alpha^2\cos\theta,$$
$$g_{12}=r^2\kappa_\alpha,$$
$$g_{22}=r^2,$$
$$g=g_{11}g_{22}-g_{12}^2=r^2\tau_\alpha^2(1-r\cos\theta)^2\neq 0.$$

The unit normal vector field U^* is given by,

$$U^*=-\sin\theta\, T_\alpha+\frac{\cos\theta}{\tau_\alpha}N_\alpha, \tag{11}$$

$$h_{11}=r\kappa_\alpha^2-\tau_\alpha^2\cos\theta(1-r\cos\theta),$$
$$h_{12}=r\kappa_\alpha,$$
$$h_{22}=r.$$

According to equations 5 and 6, the mean and the Gaussian curvatures are given, respectively, by

$$K=-\frac{\cos\theta}{(1-r\cos\theta)},\ H=\frac{1-2r\cos\theta}{2r(1-r\cos\theta)}. \tag{12}$$

Proposition 3; In Euclidean 3-space, let $W(s,\theta)$be a tubular surface. $W(s,\theta)$ is not a flat surface in such case.

Proof; It is similar to Proof 1.

Proposition 4; In Euclidean 3-space, let $W(s,\theta)$ be a tubular surface. Then $W(s,\theta)$ is minimal if and only if $r=\frac{1}{2\sec\theta}$.

Proof; Equation 12 provides the result explicitly.

Theorem 4; Let $W(s,\theta)$ be a tubular surface with a modified orthogonal frame in E^3. Then

i) Asymptotic curves are s-parameter curves of $W(s,\theta)$ if and only if $r=\frac{\tau_\alpha^2\cos\theta}{\kappa_\alpha^2+\tau_\alpha^2\cos\theta}$.

ii) Asymptotic curves cannot be θ -parameter curves of $W(s,\theta)$.

Proof; We can infer from the definition of asymptotic curves that $\langle W_{ss},U^*\rangle=0$, $\langle W_{\theta\theta},U^*\rangle=0$.

i) From equations 10 and 11 we obtain results

$$r\kappa_\alpha^2-\tau_\alpha^2\cos\theta\,(1-r\cos\theta)=0$$

and

$$r=\frac{\tau_\alpha^2\cos\theta}{\tau_\alpha^2\cos^2\theta+\kappa_\alpha^2}.$$

ii) Since $r\neq 0$, θ-parameter curves of the $W(s,\theta)$ cannot be asymptotic.

Theorem 5; Let $W(s,\theta)$ be a tubular surface with modified orthogonal frame in E^3 . Then

i) s-parameter curves of $W\ (s,\theta)$ cannot be geodesic curves.

ii) θ- parameter curves of $W(s,\theta)$ are geodesic.

Proof; According to the definition of the geodesic curve, the conditions $W_{SS}\times U=0$ and $W_{\theta\theta}\times U=0$ must be satisfied for the s - and θ - parameter curves.

i) According to equations 10 and 11 we obtain

$$W_{SS}\times U=\big(-\tau_\alpha r\kappa_\alpha\sin\theta-\tau_\alpha'(1-r\cos\theta)\big)T_\alpha$$
$$+\left(\frac{\tau_\alpha'(1-r\cos\theta)}{\tau_\alpha}-r\kappa_\alpha\sin^2\theta\right)N_\alpha$$
$$+\left(\frac{r\kappa_\alpha'}{\tau_\alpha}-\tau_\alpha(1-r\cos\theta)\sin\theta\right)B_\alpha$$

Since T_α, N_α and B_α are linearly independent then, $W_{SS}\times U=0$ if and only if $\kappa_\alpha=0$, $\tau_\alpha=0$. Nevertheless, given $\kappa_\alpha\neq 0$, $\tau_\alpha\neq 0$, $W(s,\theta)$ cannot be a geodesic curve.

ii) Furthermore, we obtain $W_{\theta\theta}\times U=0$ from equations 10 and 11. θ-parameter curves are hence geodesic curves.

We now define the mean curvature H and the partial derivative of the Gaussian curvature K of the tubular surface $W(s,\theta)$ as follows

$$K_s=0, \tag{13}$$
$$K_\theta=\frac{-\sin\theta}{r(1-r\cos\theta)^2},$$
$$H_s=0,$$
$$H_\theta=\frac{-\sin\theta}{2(1-r\cos\theta)}$$

Theorem 6 ; Let $W(s,\theta)$ be a tubular surface with a modified orthogonal frame in E^3. The surface $W(s,\theta)$ is a (K,H) −Weingarten surface.

Proof; If the derivatives of the mean curvature and the Gaussian curvature of the surface $W(s,\theta)$, given by equation 13, are substituted into the Jacobi equation

$$H_s K_\theta - H_\theta K_s = 0$$

is obtained. Therefore, the surface $W(s,\theta)$ is a (K,H) −Weingarten surface.

Theorem 7; Let $W(s,\theta)$ be a tubular surface in $\mathbb{E}^3$ equipped with a modified orthogonal frame If W is (K,H) -linear Weingarten surface, then for $a_3 = 1$, the following relations hold $a_1 = -r^2$ and $a_2 = 2r$, hold.

Proof; Let us assume that W is a (K,H) -linear Weingarten surface. In this case, the following equation is satisfied

$$a_1 K + a_2 H = 1$$

where $a_1, a_2 \in \mathbb{R}$ and $(a_1, a_2) \neq 0$. From equation 12, we have the following relation

$$2\cos\theta\,(-r^2 + a_r r + a_1) + (2r - a_2) = 0.$$

Consequently, $a_1 = -r^2$ is obtained when $a_2 = 2r$.

Example 2; The parametric equation of the curve α_2 is given by

$$\alpha_2(s) = \left(2\cos\left(\frac{s}{3}\right), 2\sin\left(\frac{s}{3}\right), \frac{\sqrt{5}s}{3}\right)$$

The parametric equation of the adjoint curve of α_2 is given by

$$\beta_2(s) = \left(\frac{5}{27}\cos\left(\frac{s}{3}\right), -\frac{5}{27}\sin\left(\frac{s}{3}\right), \frac{2\sqrt{5}}{27}s\right)$$

The tube surface whose center curve is α_2 according to the modified orthogonal frame with torsion is given by

$$W_{\alpha_2}(s,\theta) = \Big(\frac{\sqrt{5}}{3}\sin\theta\sin\left(\frac{s}{3}\right) - \cos\theta\cos\left(\frac{s}{3}\right) + 2\cos\left(\frac{s}{3}\right),$$
$$-\cos\theta\sin\left(\frac{s}{3}\right) - \frac{\sqrt{5}}{3}\sin\theta\cos\left(\frac{s}{3}\right) + 2\sin\frac{s}{3},$$
$$\frac{2}{3}\sin\theta + \frac{\sqrt{5}s}{3}\Big)$$

The graph of the tube surface that $W_{\alpha_2}(s,\theta)$ whose center curve is α_2 according to the modified orthogonal frame with torsion is illustrated in Figure 3.

The tube surface whose center curve is β_2 according to the modified orthogonal frame with torsion is given by

$$W_{\beta_2}(s,\theta) = \Big(-\frac{5}{9}\cos\left(\frac{s}{3}\right) + \cos\theta\cos\left(\frac{s}{3}\right) - \frac{2}{3}\sin\theta\sin\left(\frac{s}{3}\right),$$
$$-\frac{5}{9}\sin\left(\frac{s}{3}\right) + \cos\theta\sin\left(\frac{s}{3}\right) + \frac{2}{3}\sin\theta\cos\left(\frac{s}{3}\right),$$
$$\frac{2\sqrt{5}}{27}s + \frac{\sqrt{5}}{3}\sin\theta\Big)$$

The graph of the tube surface that $W_{\beta_2}(s,\theta)$ whose center curve is β_2 according to the modified orthogonal frame with torsion is illustrated in Figure 4.

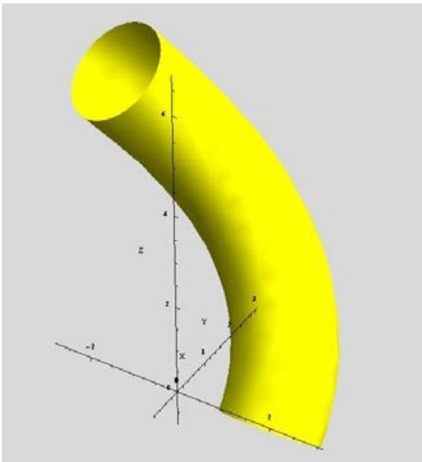

Figure 3. The tube surface, whose center curve is α_2 according to the modified orthogonal frame with torsion.

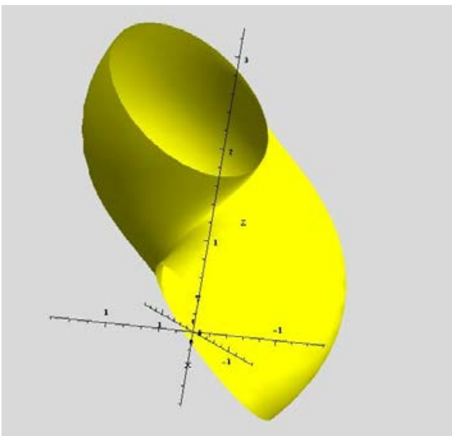

Figure 4. The tube surface, whose center curve is β_2 according to the modified orthogonal frame with torsion.

4. Discussion

The geometry of tubular surfaces in a modified orthogonal frame was investigated in this work, with particular attention to situations in which the center curve in Euclidean 3-space is the adjoint curve of another curve. By using this method, we were able to determine important properties such as Gaussian and mean curvature, which showed that the geometric qualities of these surfaces, such as minimality and flatness, are dependent on certain center curve-related requirements. We discovered that the non-zero Gaussian curvature of these surfaces prevents them from being flat, and minimality is only achievable when the mean curvature is zero. Furthermore, we examined the behavior of asymptotic and geodesic curves, demonstrating that depending on how they relate to curvature and torsion, some parameter curves display asymptotic or geodesic characteristics. Ultimately, we expanded our knowledge of the structural and geometric behavior of these tubular surfaces by classifying them as Weingarten surfaces under particular circumstances.

5. Conclusion

In the modified orthogonal frame, tubular surfaces with center curves that are adjoint curves are thoroughly examined in this work. The importance of the modified orthogonal frame in comprehending the specifics of tubular surface geometry was highlighted when we discovered crucial geometric criteria for the flatness, minimality, asymptotic behavior, and geodesicity of these surfaces by exact mathematical deductions. Our findings show the usefulness of adjoint curves in creating a more comprehensive framework for describing tubular surfaces by exposing unique characteristics not seen in traditional frames. These results provide a more thorough understanding of tubular surface geometry, and the techniques discussed here could be used for additional research in differential geometry, especially in situations where the interplay between curvature and frame structures is essential.

Author Contributions

The percentages of the authors' contributions are presented below. All authors reviewed and approved the final version of the manuscript.

	E.D.	B.S.B.	N.Y.	N.O.
C	25	25	25	25
D	40	20	20	20
S	40	20	20	20
DCP	40	20	20	20
DAI	40	20	20	20
L	40	20	20	20
W	40	20	20	20
CR	40	20	20	20
SR	40	20	20	20
PM	40	20	20	20
FA	0	0	0	0

C=Concept, D= design, S= supervision, DCP= data collection and/or processing, DAI= data analysis and/or interpretation, L= literature search, W= writing, CR= critical review, SR= submission and revision, PM= project management, FA= funding acquisition.

Conflict of Interest

The authors declared that there is no conflict of interest.

Ethical Consideration

Ethics committee approval was not required for this study because of there was no study on animals or humans.

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