BİTLİS EREN ÜNİVERSİTESİ
FEN BİLİMLERİ DERGİSİ

ISSN: 2147-3129 / e-ISSN: 2147-3188

Article Type : Research Article
Received : November 11, 2024
Revised : March 22, 2025
Accepted : June 24, 2025
DOI : 10.17798/bitlisfen.1583030

Year : 2025
Volume : 14
Issue : 2
Pages : 697-712

HANKEL DETERMINANTS OF LOGARITHMIC COEFFICIENTS FOR THE CLASS OF BOUNDED TURNING FUNCTIONS ASSOCIATED WITH CARDIOID DOMAIN

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ABSTRACT

In this study, we initially established bounds for the logarithmic coefficients concerning a specific subclass of bounded turning functions $\mathcal{R}_{\wp}$ that are linked to the cardioid domain. For the functions belonging to this class, we identified sharp bounds for the second Hankel determinant of logarithmic coefficients, denoted as $H_{2,1}(F_f/2)$. In conclusion, we computed the bounds of the third Hankel determinant of logarithmic coefficients $H_{3,1}(F_f/2)$.

Keywords: Logarithmic coefficients, Hankel determinant, Bounded Turning Functions.

1 INTRODUCTION

Denote by $\mathcal{A}$ the class of analytic functions in

$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

It is easy to see that $f(0) = 1 - f'(0) = 0$. We will denote by $\mathcal{S}$ the class of a univalent function in $\mathbb{U}$ from the class $\mathcal{A}$. This class is referred to as the class of univalent functions. Univalent function theory is considered one of the crucial topics within the realm of geometric function theory.

Let $f(z)$ be an analytic and univalent function defined in the domain $\mathbb{U}$ of the class $\mathcal{A}$, and denote that $\mathbb{D}:= F(\mathbb{U})$. If $f(z)$ is analytic in $\mathbb{U}$, $f(0) = F(0)$, and $f(\mathbb{U}) \subset \mathbb{D}$, then we claim that $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$, and we express $f(z) \prec F(z)$. Especially, $f(z)$ and $F(z)$ are analytic in $\mathbb{U}$, and consider the case where $F(z)$ is univalent in $\mathbb{U}$. Then $f(z) \prec F(z)$ if and only if there is a function $b(z)$ that satisfies the conditions of Schwarz's Lemma, $b(z)$ is an analytic function with $b(0) = 0$ and $|b(z)| < 1$ for $z \in \mathbb{U}$, and $f(z) = F\big(b(z)\big)$.

Known as the Caratheodory class, or $\mathcal{P}$-class, it is the class of functions of the following form $p(z) \in \mathcal{P}$ that comply with the conditions $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ for $z \in \mathbb{U}$.

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad (z \in \mathbb{U}) \tag{2}$$

In the field of geometric function theory, the most fundamental and significant subclasses of the set $\mathcal{S}$ are the family $\mathcal{S}^*$ of starlike functions, the subclass $\mathcal{C}$ of convex functions, and the subclass $\mathcal{R}$ of bounded turning functions. In 1994, Ma and Minda [11] proposed a class of analytic univalent functions $\varphi(z)$, which map the unit disk onto the starlike domain with respect to $\varphi(0) = 1$ in the right half plane, which is symmetric about the real axis. For a given analytic in the $\mathbb{U}$ function φ, using the concept of subordination, the Ma-Minda type classes $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, and $\mathcal{R}(\varphi)$ can be characterised as follows:

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$
$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : \frac{\big(zf'(z)\big)'}{f'(z)} \prec \varphi(z) \right\},$$
$$\mathcal{R}(\varphi) := \{ f \in \mathcal{A} : f'(z) \prec \varphi(z) \} \cdot$$

Consequently, functions belonging to $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, and $\mathcal{R}(\varphi)$ are categorized as starlike functions, convex functions, and bounded turning of the Ma-Minda type, respectively. If $\varphi \in \mathcal{P}$, then the inclusions $\mathcal{S}^*(\varphi) \subset \mathcal{S}$, $\mathcal{C}(\varphi) \subset \mathcal{S}$, and $\mathcal{R}(\varphi) \subset \mathcal{S}$ are clearly valid.

In recent years, the consideration of different image domains, $\varphi(\mathbb{U})$, has led to the investigation of various classes of univalent functions, including $\mathcal{S}^*(\varphi)$, $\mathcal{C}(\varphi)$, and $\mathcal{R}(\varphi)$. For example, Kumar et al. [21] introduced and investigated a class of the Ma-Minda type starlike functions, where $\varphi(z)$ maps the unit disk onto a cardioid domain.

The class of functions that are starlike $\mathcal{S}_{C}^{*}\left(1+\frac{4z}{3}+\frac{2z^2}{3}\right)$ was presented by Sharma et al. [18] and consists of functions that map the open unit disc into a cardioid shape. Convex functions associated with a symmetric cardioid domain were presented by Malik et al. [22]. Sharma et al. [25] investigated specific classes of analytic functions that fulfill a Ma–Minda type subordination condition and are related to the crescent-shaped domain. Wani and Swaminathan [32] introduced Ma–Minda-type starlike and convex functions, which are connected to the nephroid domain. Mendiratta et al. [13] further expanded upon this topic by studying strongly starlike functions related to exponential function.

The *logarithmic coefficients* γ_k of each function $f \in \mathcal{S}$ are defined in the following form.

$$\begin{aligned} F_f(z) &:= log\frac{f(z)}{z} \\ &= a_2 z + \left(a_3 - \frac{1}{2}a_2^2\right) z^2 \left(a_4 - a_3 a_2 + \frac{1}{3}a_2^3\right) z^3 + \cdots \\ &= 2\sum_{k=1}^{\infty} \gamma_k z^k, \quad (z \in \mathbb{U}) \end{aligned} \tag{3}$$

where γ_k are referred to as logarithmic coefficients. Logarithmic coefficients play a crucial role in problems involving the coefficients of univalent functions. The logarithmic coefficients $k(z) = z(1-z)^{-2}$ of the Koebe function are defined as $\gamma_k = \frac{1}{k}$. Given the extremal property of the Koebe function in the theory of univalent functions, it is reasonable to expect that $\gamma_k \leq \frac{1}{k}$ for each $f \in \mathcal{S}$. However, this prediction is unfortunately incorrect even for $k = 2$. In regard to the exact bounds of the logarithmic coefficients for the entire class $\mathcal{S}$, it is evident that there is the following inequality:

$$|\gamma_1| \leq 1 \qquad and \qquad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0.6353\cdots$$

In the case of $k > 3$, the exact bounds of the logarithmic coefficients remain uncertain. Over the past few years, numerous researchers (e.g., [5, 6, 14, 15, 16, 23, 24, 33]) have sought to establish upper bounds for the logarithmic coefficients associated with select subclasses within the class of univalent functions.

The Hankel determinant of $f \in \mathcal{A}$, the function for $q, n \in \mathbb{N}$, denoted by $H_{q,n}(f)$, is defined as follows:

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}. \tag{4}$$

The Hankel determinant

$$H_{2,1}(f) = a_3 - a_2^2$$

is also referred to as the Fekete-Szegö functional. Nevertheless, the functional $H_{2,2}(f)$, defined as the second Hankel determinant, is expressed as

$$H_{2,2}(f) = a_2 a_4 - a_3^2 \cdot$$

Determining the upper bound of $|H_{q,n}(f)|$ for various subclasses of $\mathcal{A}$ is a fascinating and well-studied problem in the area of Geometric Function Theory. Many writers have successfully derived sharp upper bounds for $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$ within specific subclasses of analytic functions, as referenced in [3, 8, 20, 26, 28, 29].

The subject of determinants of matrices given by analytic functions has been the focus of extensive research in the field of mathematics. The aforementioned determinants have significant applications in a number of fields, including signal processing, quantum mechanics, and numerical analysis. In particular, the establishment of sharp bounds for the logarithmic coefficients of analytic univalent functions and the determinants of Hankel matrices has attracted significant interest within the theory of complex functions.

The logarithmic coefficients of analytic univalent functions are crucial for understanding the behaviour of a function in the boundary domain where it is defined. The logarithmic coefficients of analytic univalent functions are employed in the investigation of a range of properties, including the sharp bounds of the determinants of Hankel matrices, including the sharp bounds of the determinants of Hankel matrices, growth estimates for the moduli of these functions and their derivatives.

In recent years, there has been a significant focus on the study of Hankel determinants derived from the coefficients of functions within specific subclasses of $\mathcal{S}$. Additionally, Kowalczyk and Lecko [9] recently recommend the definition of the Hankel determinant $H_{q,n}(F_f/2)$, which is generated by the logarithmic coefficients of f.

In a recent publication, Kowalczyk and Lecko [9] improved a new approach to the study of the logarithmic coefficients of f, defining the Hankel determinant in which the coefficients a_n are replaced by the coefficients γ_n to yield the form (4). In that same paper, Kowalczyk and Lecko attained sharp bounds for $|H_{2,1}(F_f/2)|$ for specific classes of starlike and convex functions. In further work, Kowalczyk and Lecko [10] attained sharp bounds on $|H_{2,1}(F_f/2)|$ for classes of starlike and convex functions of order α. The question of computing sharp bounds for strongly starlike and strongly convex functions was addressed by Eker et al. [30]. Additionally, upper bounds for the second Hankel determinant of logarithmic coefficients for various subclasses of $\mathcal{S}$ were attained by et al. [27], Eker et al. [31], Shi et al. [19], and Mandal et al. [12].

For a function $f \in \mathcal{S}$ as defined in equation (1), differentiating equation (3) allows the logarithmic coefficients to be determined.

$$\gamma_1 = \frac{1}{2}a_2, \tag{5}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{6}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{7}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \tag{8}$$

$$\gamma_5 = \frac{1}{2}(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5)\cdot \tag{9}$$

In light of the aforementioned ideas, it is intriguing to consider the Hankel determinant, whose terms are the logarithmic coefficients of the function f.

$$H_{q,n}(f) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}. \tag{10}$$

It is straightforward to demonstrate that the qth-order Hankel determinant $H_{q,n}(F_f/2)$, whose terms are the logarithmic coefficients of f, can be stated as:

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2, \tag{11}$$

$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2, \tag{12}$$

and

$$H_{3,1}(F_f/2) = \gamma_3(\gamma_2\gamma_4 - \gamma_3^2) - \gamma_4(\gamma_1\gamma_4 - \gamma_2\gamma_3) + \gamma_5(\gamma_1\gamma_3 - \gamma_2^2) \tag{13}$$

Moreover, if $f \in \mathcal{S}$, then for $f_\theta \in \mathcal{S}$, $\theta \in \mathbb{R}$, defined as

$$f_\theta(z) \coloneqq e^{-i\theta} f(e^{i\theta} z) \qquad (z \in \mathbb{U}),$$

we find that

$$H_{2,1}(F_{f_\theta}/2) = e^{4i\theta} H_{2,1}(F_f/2)$$

and

$$H_{2,2}(F_{f_\theta}/2) = e^{6i\theta} H_{2,2}(F_f/2) \cdot$$

In light of the aforementioned works, we now turn our attention to a subclass $\mathcal{R}_{\wp}$ of bounded turning functions, identified by the following equation:

$$\mathcal{R}_{\wp} \coloneqq \{f \in \mathcal{A} : f'(z) \prec 1 + ze^z =: \wp(z), \quad z \in \mathbb{U}\} \cdot \tag{14}$$

The aim of this paper is to give the sharp bounds for $|H_{2,1}(F_f/2)|$ for bounded turning functions related to a cardioid domain. In addition, we intended to find the sharp bound of $|H_{3,1}(F_f/2)|$ for this new class. For our consideration, we need the next lemmas.

Lemma 1.1 ([4])**.** If $p \in \mathcal{P}$ is of the form (2) with $c_1 \geq 0$, then

$$\begin{gathered} c_1 = 2d_1, \\ c_2 = 2d_1^2 + 2(1 - d_1^2)d_2, \\ c_3 = 2d_1^3 + 4(1 - d_1^2)d_1d_2 - 2(1 - d_1^2)d_1d_2^2 + 2(1 - d_1^2)(1 - |d_2|^2)d_3 \end{gathered} \tag{15}$$

for some $d_1 \in [0,1]$ and $d_2, d_3 \in \overline{\mathbb{U}} \coloneqq \{z \in \mathbb{C} : |z| \leq 1\} \cdot$

For $d_1 \in \mathbb{U}$ and $d_2 \in \partial\mathbb{U} \coloneqq \{z \in \mathbb{C} : |z| = 1\}$ there exists a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (15), namely

$$p(z) = \frac{1 + (\overline{d_1}d_2 + d_1)z + d_2z^2}{1 + (\overline{d_1}d_2 - d_1)z + d_2z^2}, \qquad (z \in \mathbb{U}) \cdot$$

Lemma 1.2. If $p \in \mathcal{P}$ takes the form (2) then the following inequalities are hold

$$|c_n| \leq 2 \quad for \quad n \geq 1, \tag{16}$$

$$|c_{n+k} - \mu c_n c_k| \leq 2 \quad for \quad 0 \leq \mu \leq 1, \tag{17}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad for \quad m+n=k+l, \tag{18}$$

$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1+2\mu) \quad for \quad \mu \in \mathbb{R}, \tag{19}$$

and for complex number λ, we have

$$|c_2 - \lambda c_1^2| \leq 2\, max(1, |\lambda - 1|) \cdot \tag{20}$$

For the inequalities in (16), (17), (18) and (19), we refer to [16]. Also, see [7] for the inequality (20).

Lemma 1.3 ([1])**.** Let $p \in \mathcal{P}$ and has form (2), then

$$|Kc_1^3 - Lc_1c_2 + Mc_3| \leq 2|K| + 2|L-2K| + 2|K-L+M| \cdot$$

Lemma 1.4 ([2])**.** Given real numbers A, B, C, let

$$Y(A,B,C) := \max\{|A+Bz+Cz^2| + 1 - |z|^2 : z \in \overline{\mathbb{U}}\}.$$

If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A|+|B|+|C|, & |B| \geq 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|) \cdot \end{cases}$$

If $AC < 0$, then

$$Y(A,B,C) = \begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC(C^{-2}-1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(C^{-2}-1)\}, \\ R(A,B,C), & otherwise. \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leq |AB|, \\ -|A|+|B|+|C|, & |AB| \leq |C|(|B|-4|A|) \\ (|A|+|C|)\sqrt{1-\dfrac{B^2}{4AC}}, & otherwise. \end{cases}$$

2 LOGARITHMIC COEFFICIENTS FOR THE CLASS OF BOUNDED TURNING FUNCTIONS ASSOCIATED WITH CARDIOID DOMAIN

Theorem 2.1. If $f \in \mathcal{R}_{\wp}$ and it has the form given in (1), then

$$|\gamma_1| \leq \frac{1}{4}, \tag{21}$$

$$|\gamma_2| \leq \frac{1}{6}, \tag{22}$$

$$|\gamma_3| \leq \frac{7}{48}, \tag{23}$$

$$|\gamma_4| \leq \frac{289}{1280}, \tag{24}$$

$$|\gamma_5| \leq \frac{917}{2880}. \tag{25}$$

The first two inequalities are sharp for $f_1(z)$ and $f_2(z)$ given in the following, respectively.

$$\begin{aligned} f_1(z) &= \int_0^z (1 + te^t)dt, \\ f_2(z) &= \int_0^z \left(1 + t^2 e^{t^2}\right)dt \end{aligned} \tag{26}$$

Proof. Let us consider the function f, which belons to the class of functions $\mathcal{R}_{\wp}$. In accordance with the definition of subordination, there exists a Schwarz function, designated as $w(z)$, which possesses the following features:

$$w(0) = 0 \quad and \quad |w(z)| < 1$$

such that

$$f'(z) = 1 + w(z)e^{w(z)}. \tag{27}$$

Define the function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

It is straightforward that $p(z) \in \mathcal{P}$. This suggests that

$$\begin{aligned} w(z) = \frac{p(z) - 1}{p(z) + 1} &= \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots} \\ &= \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right)z^3 + \cdots. \end{aligned}$$

Since

$$f'(z) = 1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \cdots. \tag{28}$$

and

$$1 + w(z)e^{w(z)} = 1 + \frac{1}{2}c_1 z + \frac{1}{2}c_2 z^2 + \frac{1}{16}(8c_3 - c_1^3)z^3 + \frac{1}{48}(24c_4 - 9c_1^2 c_2 + 2c_1^4)z^4 + \cdots. \tag{29}$$

From equation (27), if we assess the relevant coefficients of (28) and (29), we have

$$a_2 = \frac{1}{4}c_1\,, \tag{30}$$

$$a_3 = \frac{1}{6}c_2\,, \tag{31}$$

$$a_4 = \frac{1}{64}(8c_3 - c_1^3)\,, \tag{32}$$

$$a_5 = \frac{1}{240}(24c_4 - 9c_1^2 c_2 + 2c_1^4)\,, \tag{33}$$

$$a_6 = \frac{1}{4608}(384c_5 - 144c_1 c_2^2 - 144c_1^2 c_3 + 128c_1^3 c_2 - 15c_1^5)\cdot \tag{34}$$

Now, from (5) to (9) and (30) to (34), we obtain

$$\gamma_1 = \frac{1}{8}c_1\,, \tag{35}$$

$$\gamma_2 = \frac{1}{192}(16c_2 - 3c_1^2)\,, \tag{36}$$

$$\gamma_3 = \frac{1}{192}(12c_3 - c_1^3 - 4c_1 c_2)\,, \tag{37}$$

$$\gamma_4 = \frac{1}{92160}(4608c_4 - 1248c_1^2 c_2 - 640c_2^2 + 519c_1^4 - 1440c_1 c_3), \tag{38}$$

$$\gamma_5 = \frac{1}{46080}(856c_1^3 c_2 - 141c_1^5 - 540c_1^2 c_3 - 576c_1 c_4 - 560c_1 c_2^2 - 480c_2 c_3 + 1920c_5)\cdot \tag{39}$$

Applying (16) to (35), we get

$$|\gamma_1| \le \frac{1}{4}\cdot$$

From (36) and using (20), we have

$$|\gamma_2| = \frac{1}{12}\left|c_2 - \frac{3}{16}c_1^2\right| \le \frac{1}{6}\max\left\{1, \left|\frac{3}{16} - 1\right|\right\} = \frac{1}{6}\cdot$$

By using Lemma 1.3 to the equation (37), we get

$$|\gamma_3| \leq \frac{7}{48}.$$

From (38), it follows that

$$\gamma_4 = \frac{1}{20}\left(c_4 - \frac{5}{36}c_2^2\right) + \frac{c_1}{92160}(519c_1^3 - 1248c_1c_2 - 1440c_3).$$

By utilizing (17) and Lemma 1.3, along with the triangle inequality, we get

$$|\gamma_4| \leq \frac{289}{1280}.$$

If we rearrange the equation given in (39), we get

$$\gamma_5 = \frac{1}{46080}\Bigg(-540c_1^2\left(c_3 + \frac{141}{540}c_1c_2^2\right) \\ +1920\left(c_5 - \frac{576}{1920}c_1c_4\right) + c_2(856c_1^3 - 560c_1c_2 - 480c_3)\Bigg).$$

Using triangle inequality along with (16), (17), (19) and Lemma 1.3, we get

$$|\gamma_5| \leq \frac{917}{2880}.$$

Since

$$f_1(z) = \int_0^z (1 + te^t)dt = z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots$$

and

$$f_2(z) = \int_0^z (1 + t^2e^{t^2})dt = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots,$$

from the equations (5) and (6), it is easily obtained that the first two results given in the theorem are sharp.

3 SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR BOUNDED TURNING WITH A CARDIOID DOMAIN

Theorem 3.1. If $f \in \mathcal{R}_{\wp}$, then

$$\left|H_{2,1}\left(\frac{F_f}{2}\right)\right| \leq \frac{1}{36}. \tag{40}$$

The inequality in (40) is sharp.

Proof. Let $f \in \mathcal{R}_{\wp}$ be of the form (1). Then, according to (14), we obtain

$$f'(z) = 1 + w(z)e^{w(z)}, \quad (z \in \mathbb{U}) \tag{41}$$

for certain function $p \in \mathcal{P}$ of the form (2). By equating the corresponding coefficients, we obtain

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{6}c_2, \\ a_4 &= \frac{1}{64}(8c_3 - c_1^3)\cdot \end{aligned} \tag{42}$$

Since the class $\mathcal{R}_{\wp}$ and $|H_{2,1}(F_f/2)|$ are rotationally invariant, without any loss of generality we take it that $a_2 \geq 0$, so $c = c_1 \in [0, 2]$ (i.e., in view of (15) that $d_1 \in [0, 1]$). By using (6)-(8) and (11) we get

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= -\frac{1}{2304}[25d_1^4 - 16(1-d_1^2)d_1^2d_2 \\ &+8(8+d_1^2)(1-d_1^2)d_2^2 - 72d_1(1-d_1^2)(1-|d_2|^2)d_3] \end{aligned} \tag{43}$$

The following cases for d_1 are possible.

Case 1. Assume that $d_1 = 1$. From (43), we get

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{25}{2304}\cdot$$

Case 2. Assume that $d_1 = 0$. From (43), we get

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{36}|d_2|^2 \leq \frac{1}{36}\cdot$$

Case 3. Assume that $d_1 \in (0, 1)$. By the fact that $|d_3| \leq 1$, applying the triangle inequality to (43) we can write

$$|\gamma_1\gamma_3 - \gamma_2^2| \leq \frac{1}{32}(1-d_1^2)d_1(|A + Bd_2 + Cd_2^2| + 1 - |d_2|^2)$$

where

$$A := \frac{25d_1^3}{72(1-d_1^2)} > 0,$$

$$B := -\frac{2d_1}{9} < 0 \quad and \quad C := \frac{8+d_1^2}{9d_1} > 0.$$

Since $AC > 0$, we apply the part I of Lemma 1.3.

We consider the following sub-case. Note that

$$\begin{aligned}|B| - 2(1 - |C|) &= \frac{2d_1}{9} - 2\left(1 - \frac{8 + d_1^2}{9d_1}\right) \\ &= \frac{2(2d_1^2 - 9d_1 + 8)}{9d_1} \\ &\geq \frac{2(d_1 - 8)(d_1 - 1)}{9d_1} > 0 \cdot\end{aligned}$$

Applying Lemma 1.3, we obtain

$$\begin{aligned}|\gamma_1\gamma_3 - \gamma_2^2| &\leq \frac{1}{32}(1 - d_1^2)d_1(|A| + |B| + |C|) \\ &= \frac{1}{32}(1 - d_1^2)d_1\left(\frac{25d_1^3}{72(1 - d_1^2)} + \frac{2d_1}{9} + \frac{d_1^2 + 8}{9d_1}\right) \\ &= \frac{1}{2304}\Psi_1(d_1)\end{aligned}$$

where

$$\Psi_1(x) = x^4 - 40x^2 + 64 \cdot$$

Since $\Psi'_1(x) < 0$ for $x \in (0, 1)$, we deduce that Ψ_1 is a decreasing function. This implies that

$$\begin{aligned}|\gamma_1\gamma_3 - \gamma_2^2| &\leq \frac{1}{2304}\Psi_1(d_1) \\ &\leq \frac{1}{2304}\Psi_1(0) = \frac{1}{36} \cdot\end{aligned}$$

Summarizing parts from Case 1-3, it follows that the inequality (40) is true.

To show the sharpness for the case, consider the function

$$p(z) := \frac{1 + z^2}{1 - z^2} \cdot$$

It is obvious that the function p is in $\mathcal{R}_{\wp}$ with $c_1 = c_3 = 0$ and $c_2 = 2$. The corresponding function $f \in \mathcal{R}_{\wp}$ is described by (41). Hence by (42) it follows that $a_2 = a_4 = 0$ and $a_3 = \frac{1}{3} \cdot$ From (43) we obtain

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{36} \cdot$$

This completes the proof.

4 THIRD HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR BOUNDED TURNING WITH A CARDIOID DOMAIN

Theorem 4.1. If $f \in \mathcal{R}_{\wp}$, then

$$|\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{4877}{69120} \cdot \tag{44}$$

Proof. From (36), (37) and (38), we can write

$$\gamma_2\gamma_4 - \gamma_3^2 = \frac{1}{17694720}(-8208c_1^4c_2 + 2037c_1^6 + 10240c_2^3 + 25728c_1^2c_2^2 \\ -73728c_2c_4 + 13824c_1^2c_4 - 15840c_1^3c_3 - 23040c_1c_2c_3 + 69120c_3^2) \cdot$$

If we rearrange this equation and apply the triangle inequality, we get

$$|\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{1}{17694720}\left(8208|c_1|^4\left|c_2 - \frac{2037}{8208}c_1^2\right| + 10240|c_2|^2\left|c_2 + \frac{25728}{10240}c_1^2\right| \right. \\ \left. + 73728|c_4|\left|c_2 + \frac{13824}{73728}c_1^2\right| + |c_3||-15840c_1^3 - 23040c_1c_2 + 69120c_3|\right) \cdot$$

Using (16), (20), and Lemma 1.3, we get the required result.

Theorem 4.2. If $f \in \mathcal{R}_{\wp}$, then

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| \leq \frac{41}{576} \cdot \tag{45}$$

Proof. From (36), (37), and (38), we obtain

$$\gamma_1\gamma_4 - \gamma_2\gamma_3 = \frac{1}{737280}(-1168c_1^3c_2 + 459c_1^5 \\ +4608c_1c_4 - 720c_1^2c_3 - 3840c_2c_3 + 640c_1c_2^2) \cdot$$

Rearranging the above and applying triangle inequality, we get

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| \leq \frac{1}{737280}\left(1168|c_1|^3\left|c_2 - \frac{459}{1168}c_1^2\right| \right. \\ \left. +4608|c_1|\left|c_4 + \frac{720}{4608}c_1c_3\right| + 3840|c_2|\left|c_3 + \frac{640}{3840}c_1c_2\right|\right) \cdot$$

Using (16), (17), (20), and Lemma 1.3, we get the required result.

Theorem 4.3. If $f \in \mathcal{R}_{\wp}$, then

$$|H_{3,1}(f)| \leq \frac{77869}{2211840} \approx 0.035 \cdot \tag{46}$$

Proof. Since (13), it follows that

$$\left|H_{3,1}(f)\right| \leq |\gamma_3||\gamma_2\gamma_4 - \gamma_3^2| \\ +|\gamma_4||\gamma_1\gamma_4 - \gamma_2\gamma_3| + |\gamma_5||\gamma_1\gamma_3 - \gamma_2^2| \cdot$$

From the (23)-(25), (40), (44), and (45), we achieve the required result.

5 CONCLUSIONS

In this study, we have found the sharp upper bounds for the second Hankel determinant of logarithmic coefficients and computed the bounds of the third Hankel determinant of logarithmic coefficients. The subject of our study is current, and the level of interest of researchers in this subject is increasing. Our prediction is that new logarithmic coefficient classes will emerge, attracting much research.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is compiled with research and publication ethics

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the authors.

Contributions of the Authors

Conceptualization, B.Ş. and O.A.; methodology, B.Ş. and O.A.; writing—original draft preparation, B.Ş. and O.A.; investigation, B.Ş. and O.A. All authors have read and agreed to the published version of the manuscript.

REFERENCES

[1] M. Arif, M. Raza, H. Tang, S. Hussain and H. Khan, "Hankel determinant of order three for familiar subsets of analytic functions related with sine function," *Open Math.*, vol. 17, pp. 1615-1630. 2019.

[2] J. H. Choi, Y. C. Kim and T. Sugawa, "A general approach to the Fekete-Szegö problem," *J. Math. Soc. Japan*, vol. 59, pp. 707-727. 2007.

[3] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, "Some Coefficient Inequalities Related to the Hankel Determinant for Strongly Starlike Functions of Order Alpha," *J. Math. Inequal.*, vol. 11, pp. 429-439. 2017.

[4] N. E. Cho, B. Kowalczyk and A. Lecko, "Sharp Bounds of Some Coefficient Functionals Over The Class of Functions Convex in The Direction of the Imaginary Axis," *Bull. Aust. Math. Soc.*, vol. 100, pp. 86-96. 2019.

[5] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko and Y. J. Sim, "On the third logarithmic coefficient in some subclasses of close-to-convex functions," *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, vol. 114, pp. 1-14. 2020.

[6] D. Girela, "Logarithmic coefficients of univalent functions," *Ann. Acad. Sci. Fenn. Math.*, vol. 25, pp. 337-350. 2000.

[7] F.R. Keogh, E. P. Merkes, "A coefficient inequality for certain classes of analytic functions," *Proc. Am. Math. Soc.*, vol. 20, pp. 8-12. 1969.

[8] B. Kowalczyk, A. Lecko and Y. J. Sim, The sharp bound for the Hankel determinant of the third kind for convex functions, *Bull. Aust. Math. Soc.*, vol. 97, pp. 435-445. 2018.

[9] B. Kowalczyk, A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions," *Bull. Aust. Math. Soc.*, vol. 105, pp. 458-467. 2022.

[10] B. Kowalczyk, A. Lecko, "Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha," *Bull. Malays. Math. Sci. Soc.*, vol. 45, pp. 727-740. 2022.

[11] W. C. Ma, D. Minda, "A unified treatment of some special classes of univalent functions," in: *Proceedings of the Proceedings of the Conference on Complex Analysis*, Tianjin(China): International Press, 1992. pp. 157-169.

[12] S. Mandal, M.B. Ahamed, "Second Hankel determinant of Logarithmic coefficients for Starlike and Convex functions associated with lune," *arXiv.org*, [Online]. Available: https://arxiv.org/pdf/2307.02741. [Accessed: Jul. 6, 2023].

[13] R. Mendiratta, S. Nagpal and V. Ravichandran, "On a subclass of strongly starlike functions associated exponential function," *Bull. Malays. Math. Sci. Soc.*, vol. 38, pp. 365-386. 2015.

[14] N. H. Mohammed, "Sharp bounds of logarithmic coefficient problems for functions with respect to symmetric points", *Mat. Stud*., vol. 59, no. 1, pp. 68-75. 2023.

[15] M. Obradovi´c, S. Ponnusamy and K. J. Wirths, "Logarithmic coefficients and a coefficient conjecture for univalent functions," *Mon. Hefte Math.*, vol. 185, pp. 489-501. 2018.

[16] C. Pommerenke, *Univalent Functions*. Vanderhoeck and Ruprecht, Gottingen, Germany: Springer Science and Business Media, 1975.

[17] S. Ponnusamy, N. L. Sharma and K. J. Wirths, "Logarithmic coefficients problems in families related to starlike and convex functions," *J. Aust. Math. Soc.*, vol. 109, pp. 230-249. 2019.

[18] K. Sharma, N. K. Jain and V. Ravichandran, "Starlike functions associated with cardioid," *Afr. Mat.*, vol. 27, pp. 923-939. 2016.

[19] L. Shi, M. Arif, J. Iqbal, K. Ullah and S.M. Ghufran, "Sharp Bounds of Hankel Determinant on Logarithmic Coefficients for Functions Starlike with Exponential Function," *Fractal Fract.*, vol. 6, pp. 645. 2022.

[20] Y. J. Sim, A. Lecko and D. K. Thomas, "The second Hankel determinant for strongly convex and Ozaki close-to-convex functions," *Ann. Mat. Pura Appl.*, vol. 200, pp. 2515-2533. 2021.

[21] S. S. Kumar, G. Kamaljeet, "A cardioid domain and starlike functions," *Anal. Math. Phys.*, vol. 11, pp. 54. 2021.

[22] S. N. Malik, M. Raza, Q. Xin, J. Sokol, R. Manzoor, et al, "On Convex Functions Associated with Symmetric Cardioid Domain," *Symmetry*, vol. 13, pp. 2321. 2021.

[23] B. Rath, D. V. Krishna, K. S. Kumar, and G. K. S. Viswanadh, "The sharp bound of the third Hankel determinants for inverse of starlike functions with respect to symmetric points", *Mat. Stud*., vol. 58, no. 1, pp. 45-50. 2022.

[24] B. Rath, K. S. Kumar, and D. V. Krishna, “An exact estimate of the third Hankel determinants for functions inverse to convex functions”, *Mat. Stud*., vol. 60, no. 1, pp. 34-39. 2023.

[25] P. Sharma, R. K. Raina and J. Sokół, “Certain Ma–Minda type classes of analytic functions associated with the crescentshaped region,” *Anal. Math. Phys.*, vol. 9, pp. 1887-1903. 2019.

[26] J. Sokół, D. K. Thomas, “The second Hankel determinant for alpha-convex functions,” *Lith. Math. J.*, vol. 58, pp. 212-218. 2018.

[27] H. M. Srivastava, S. Sümer Eker, B. Seker and B. Çekiç, “Second Hankel Determinant of Logarithmic Coefficients for a subclass of univalent functions,” *Miskolc Math. Notes.*, vol. 25, pp. 479-488. 2024.

[28] H. M. Srivastava, Q. Z. Ahmad, M. Darus, N. Khan, B. Khan, et al, “Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli,” *Mathematics*, vol. 7, pp. 1-10. 2019.

[29] L. Shi, H. M. Srivastava, M. Arif, S. Hussain and H. Khan, “An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential unction*,*” *Symmetry*, vol. 11, pp. 1-14. 2019.

[30] S. Sümer Eker, B. Seker, B. Çekiç and M. Acu, “Sharp Bounds for the Second Hankel Determinant of Logarithmic Coefficients for Strongly Starlike and Strongly Convex functions,” *Axioms*, vol. 11, pp. 1-14. 2022.

[31] S. Sümer Eker, A. Lecko, B. Çekiç and B. Seker, “The Second Hankel Determinant of Logarithmic Coefficients for Strongly Ozaki Close-to-Convex Functions,” *Bull. Malays. Math. Sci. Soc.*, vol. 46, pp. 1-23. 2023.

[32] L. A. Wani, A. Swaminathan, “Starlike and convex functions associated with a Nephroid domain,” *Bull. Malays. Math. Sci. Soc.*, vol. 44, pp. 79–104. 2021.

[33] P. Zaprawa, “Initial logarithmic coefficients for functions starlike with respect to symmetric points,” *Bol. Soc. Mat. Mex.*, vol. 27, pp. 1-13. 2021.