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# An Innovative Technique for Solving Singularly Perturbed Problems with Integral Boundary Conditions on the Non-Uniform Mesh

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**Abstract**

We investigate an innovative numerical method for solving linear singularly perturbed problems with integral boundary conditions. Firstly, proving a uniformly convergent numerical method for solving the singularly perturbed problem is the main goal of this problem. Secondly, a piecewise equidistant mesh is used to generate the finite difference scheme. Next, the difference technique's stability and convergence analysis are covered. Lastly, two test instances' numerical results are shown.

***Keywords:*** *Linear Singularly Perturbed Problem, Non-Uniform Mesh, Numerical Method, Integral Boundary.*
***2010 Mathematics Subject Classification:*** *65N12; 65N30; 65N06*

## 1. Introduction

In recent times, many authors have concentrated on solving the singularly perturbed differential equations (SPDEs) [1, 4-9, 12-15]. In general, these equations occur in the simulation of natural processes, which includes high Reynolds number flow, electrical networks, control theory, chemical reactions, quantum control, flow equations, drift-diffusion equations with semiconductors, and other physical models. In these modeling processes, many important results have been obtained in both numerical methods and theoretical methods for computing these equations [10-11, 21-28].
In this study, we consider the following linear SPDEs:

$$Lu \equiv \varepsilon u^{''}(x) + a(x)u^{'}(x) - b(x)u(x) = f(x), \; x\varepsilon(0,l), \tag{1}$$

$$u(0) = A + \int_0^l g_1(x)u(x)dx, \tag{2}$$

$$u(l) = B + \int_0^l g_2(x)u(x)dx, \tag{3}$$

where the functions $a(x) \geq \alpha > 0$, $b(x) \geq \beta > 0$, $g(x)$, and $f(x)$ are supposed to be sufficiently smooth on $[0,l]$. $A$ and $B$ are given constants, and $0 < \varepsilon << 1$ is the perturbation parameter. There is an initial layer of the function $u(x)$ around $x = 0$. Under these conditions, there is only one solution to the problem (1)-(3).
SPDEs are differential equations in which the parameter "$\varepsilon$" multiplies with the highest-order terms. The presence of this parameter causes the behavior of the solutions to be irregular in such problems. Moreover, the parameter $\varepsilon$ has an important limiting value for such differential equations. Therefore, it often creates complex situations for the solutions of SPDEs.
SPDEs generally produce oscillatory numerical solutions, and it is widely recognized that typical numerical methods are not appropriate for solving them. Because of this, researchers frequently use Bakhvalov-type and Shishkin-type meshes ( see [19-20]) for these kinds of problems. Shishkin-type meshes are known for their straightforward structure and ease of analysis. SPDEs with Shishkin-type meshes have

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been extensively studied, as evidenced by the works [3, 7, 16-18, 20, 24, 29]. However, the convergence order deteriorates when employing a Shishkin-type mesh since the convergence typically contains a logarithmic component. Bakhvalov-type meshes typically perform better numerically when compared to Shishkin-type meshes. Nevertheless, convergence analysis faces significant challenges because of the transition points of Bakhvalov-type meshes. As a result, the number of articles on Bakhvalov-type meshes has been limited thus far [30].
In general, with a perturbation parameter, numerical analyses of both linear and nonlinear SPDEs are quite difficult and complex. In addition, due to the parameter $\varepsilon$ of these equations, standard numerical methods do not provide reliable results for the solutions. Thus, this study focuses on establishing innovative and reliable numerical techniques. The project's goal is to establish an innovative numerical method for SPDEs with the integral boundary case on Bakhvalov-type meshes. The foundation of this approach is the use of quadrature interpolation with integral form for the weight and remainder terms, using exponentially base functions. It also contains a new, and suitable difference operator. These results are checked against classical methods in a local truncation error that includes very small derivatives of the exact solution, making convergence analysis possible. Because our approach requires fewer smooth conditions for the problem estimate, it offers several advantages. Thus, the approach is more beneficial and produces quality outcomes. It is also evident from these results that a large class of differential equations can be solved using our method.
The structure below outlines the order in which this article operates: In Section 2, the characteristics of the precise solution are presented. In Section 3, the difference method is shown. In Section 4, the finite difference technique presents the stability analysis, the approximation solution's error evaluation, the first-order uniformity convergence, and error assessments of SPVIDEs with a first-order uniformly convergent perturbation parameter. In Section 5, two illustrative examples are given that confirm the numerical results. Finally, the numerical results are confirmed by two tables.
**Notations:** $C$, $C_0$, $C_1$,... denote positive constants that are independent of the mesh parameter and the perturbation parameter $\varepsilon$. With a continuous function $v(x)$ described in a region, the maximum norm of $v$ is $\|v\|_\infty = \max_{x\in[0,l]} |v(x)|$ given as.

## 2. Properties of The Exact Solution

Here, we state a lemma that will be used later and give some properties of the solutions for the problem (1)-(3). These characteristics will be required in the following sections for the evaluation of suitable numerical solutions.

**Lemma 2.1.** *We presume that $a(x)$, $b(x)$, $g(x)$ and $f(x) \in C^1[0,l]$. Moreover $\gamma = \int_0^l (|g_1(x)| + |g_2(x)|)dx < 1$. In these circumstances, the following inequalities are satisfied by the solution $u$ of the problem (1)-(3):*

$$\|u\|_\infty \le C_0, \quad 0 \le x \le l, \tag{4}$$

*and*

$$\left|u'(x)\right| \le C\left\{1 + \frac{1}{\varepsilon} e^{-\frac{\alpha x}{\varepsilon}}\right\}, \tag{5}$$

*here*

$$C_0 = (|A| + |B| + \alpha^{-1}\|f\|_\infty)(1-\gamma)^{-1}.$$

*Proof.* The maximum principle will be used for problem (1)-(3) [29-30]. We take the following inequality and use the maximum principle:

$$|u(x)| \le \alpha^{-1}\|f\|_\infty + |u(l)| + |u(0)|, \quad x\varepsilon\,[0,l]. \tag{6}$$

Once boundary conditions (2) and (3) are applied, we have

$$|u(0)| \le |A| + \int_0^l |g_1(x)|\,|u(x)|\,dx, \tag{7}$$

and

$$|u(l)| \le |B| + \int_0^l |g_2(x)|\,|u(x)|\,dx. \tag{8}$$

We obtain the following inequality by substituting the inequalities (7) and (8) in inequality (6):

$$|u(x)| \le |A| + \|u\|_\infty \int_0^l |g_1(x)|\,dx + |B| + \|u\|_\infty \int_0^l |g_2(x)|\,dx + \alpha^{-1}\|f\|_\infty.$$

Hence, we get

$$\|u\|_\infty \le \left(|A| + |B| + \alpha^{-1}\|f\|_\infty\right)\left(1 - \left(\int_0^l |g_1(x)|\,dx + \int_0^l |g_2(x)|\,dx\right)\right)^{-1}$$

$$\le \left(|A| + |B| + \alpha^{-1}\|f\|_\infty\right)(1-\gamma)^{-1}. \tag{9}$$

Thus, inequality (4) has been proved.
Now, let's present the proof of (5). We examine the following using equation (1):

$$\varepsilon u''(x) + a(x)u'(x) = F(x), \tag{10}$$

where

$F(x) = f(x) + b(x)u(x).$

We utilize the relation shown below for $u'(x)$ based on equation (10),

$$u'(x) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi. \tag{11}$$

We need to obtain an evaluation for $u'(0)$ based on (11). When (11) is integrated throughout $[0,l]$, we get

$$B - A - \int_0^l g_1(x)u(x) + \int_0^l g_2(x)u(x)$$

$$= \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx + u'(0)\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx. \tag{12}$$

From (12), we have

$$u'(0) = \frac{\int_0^l g_1(x)u(x)dx + B - A + \int_0^l g_2(x)u(x)dx - \frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx}{\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx}. \tag{13}$$

To estimate the integral in the denominator of (13), we utilize

$$\int_0^l e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau}dx \geq \int_0^l e^{-\frac{1}{\varepsilon}\int_0^x \overline{a}d\tau}dx$$

$$= \delta\varepsilon \ \ (\delta \neq \delta\varepsilon > 0, \quad \overline{a} = \max_{[0,l]}|a(x)|). \tag{14}$$

After analyzing the integral in (13), we get the following by applying the mean value theorem

$$\left|\frac{1}{\varepsilon}\int_0^l\int_0^x F(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi dx\right| \leq \frac{\|F\|_\infty}{\varepsilon}\int_0^l\int_0^x e^{-\frac{1}{\varepsilon}\int_\xi^x \alpha d\tau}d\xi dx,$$

$$\leq \alpha^{-1}l\,\|F\|_\infty$$

$$\leq C_1, \tag{15}$$

and

$$\left|\int_0^l g_1(x)u(x)dx + \int_0^l g_2(x)u(x)dx\right| \leq C_2. \tag{16}$$

We give by replacing inequalities (14) and (15) in (13) as follows:

$$|u'(0)| \leq \frac{|A| + |B| + C_2 + \|F\|_\infty C_1}{\delta\varepsilon} = \frac{C}{\varepsilon}. \tag{17}$$

By considering (17) in (11), we get

$$|u'(x)| = |u'(0)|\,e^{-\frac{1}{\varepsilon}\int_0^x a(\tau)d\tau} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|\,e^{-\frac{1}{\varepsilon}\int_\xi^x a(\tau)d\tau}d\xi$$

$$= \frac{C}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}} + C. \tag{18}$$

Thus, we prove inequality (5).
So, Lemma 2.1 has been proved. □

## 3. Difference Scheme Generating Using Finite Difference Method

For (1)-(3), we give the well-known finite difference method over the Bakhvalov-type mesh in this part of the paper. Assume $\omega_h$ is a uniform mesh over $[0,l]$, in which case we can write the following expression:

$$\omega_h = \{0 < x_1 < x_2 < ... < x_{N-1} < x_N = l,\ h_i = x_i - x_{i-1},\ \ i = 1,2,...,N\},$$

and

$$\overline{\omega}_N = \omega_h \cup \{x = 0\}.$$

A non-uniform mesh is used to compute the approximation to the solution $u(x)$ of problem (1)-(3). When a positive integer $N$ is divisible by two, the range $[0,l]$ gives rise to two sub-intervals, $[0,\sigma]$ and $[0,l]$. In practical application, $\sigma << l$ is commonly utilized as a transitional point and is represented in the following format:

$$\sigma = \min\{\frac{1}{2},\ \alpha^{-1}\varepsilon\,|\ln\varepsilon|\}.$$

A collection of the mesh points is defined as:

$$x_i \in [0,\sigma] : x_i = -\alpha^{-1}\varepsilon\ln[1-(1-\varepsilon)\frac{2i}{N}],\ \ i = 0,...,\frac{N}{2},$$

if $\sigma < \frac{l}{2}$,

$$x_i = -\alpha^{-1}\varepsilon\ln[1-(1-e^{-\frac{\alpha l}{2\varepsilon}})\frac{2i}{N}],$$

if $\sigma = \frac{l}{2}$,

$$x_i \in (\sigma,l] : x_i = \sigma + (i-\frac{N}{2})h.$$

We provide a collection of $\overline{\omega}_h = \omega_h \cup \{t = 0\}$ mesh points. To simplify the form of the formula, we take $v_i = v(x_i)$ for a function $v(x)$, where $v_i$ signifies the approximate value of $v(t)$ at $v_i$,

$$v_{\bar{x},i} = \frac{v_i - v_{i-1}}{h_i}, \quad v_{\overset{o}{x},i} = \frac{v_{x,i} - v_{\bar{x},i}}{2},$$

$$v_{\overset{\wedge}{x},i} = \frac{v_{i+1} - v_i}{\hbar_i}, \qquad v_{\bar{x}\overset{\wedge}{x},i} = \frac{v_{x,i} + v_{\bar{x},i}}{\hbar_i}, \qquad \hbar_i = \frac{h_i + h_{i+1}}{2}.$$

From the following identity, we demonstrate how to construct the difference scheme and how to integrate problem (1) over $(x_{i-1}, x_{i+1})$:

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(t)Lu(t)dt = \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(t)f(t)dt, \tag{19}$$

where the basis functions are denoted by $\psi_i(x)$, defined as in

$$\psi_i(x) = \begin{cases} \psi_i^{(1)}(x) \equiv \frac{x-x_{i-1}}{\hbar_i}, & x_{i-1} < x < x_i, \\ \psi_i^{(2)}(x) \equiv \frac{x_{i+1}-x}{\hbar_{i+1}}, & x_i < x < x_{i+1}, \\ 0, \ \ x \notin (x_{i-1}, x_{i+1}), \end{cases}$$

It is evident that the following problems have basis functions $\psi_i^{(1)}(x)$ and $\psi_i^{(2)}(x)$ as their respective solutions:

$$\psi^{''}(x) = 0, \quad x_{i-1} < x < x_i,$$

$$\psi(x_{i-1}) = 0, \quad \psi(x_i) = 1,$$

and

$$\psi^{''}(x) = 0, \quad x_i < x < x_{i+1},$$

$$\psi(x_i) = 1, \quad \psi(x_{i+1}) = 0.$$

Moreover, we offer

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i^{(1)}(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i^{(2)}(x)dx = 1.$$

When the required steps are completed and partial integration is used in (19), we have

$$\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)\varepsilon u^{''}(x)dx + \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)a(x)u'(x)dx - \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)b(x)u(x)dx$$

$$= \hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}} \psi_i(x)f(x)dx,$$

and

$$-\varepsilon\hbar_i^{-1}\int_{x_{i-1}}^{x_i}\psi_i^{(1)'}(x)u'(x)dx-\varepsilon\hbar_i^{-1}\int_{x_i}^{x_{i+1}}\psi_i^{(2)^1}(x)u'(x)dx$$
$$+a_i\hbar_i^{-1}\int_{x_{i-1}}^{x_i}\psi_i^{(1)}(x)u'(x)dx+a_i\hbar_i^{-1}\int_{x_i}^{x_{i+1}}\psi_i^{(2)}(x)u'(x)dx$$
$$-b_i\hbar_i^{-1}\int_{x_{i-1}}^{x_i}\psi_i^{(1)}(x)u(x)dx+b_i\hbar_i^{-1}\int_{x_i}^{x_{i+1}}\psi_i^{(2)}(x)u(x)dx$$

$$=f_i-R_i,\quad i=1,2,...,N-1, \tag{20}$$

where

$$R_i=R_i^{(1)}+R_i^{(2)}+R_i^{(3)}. \tag{21}$$

The following definitions apply to the remainder terms $R_i^{(1)}$, $R_i^{(2)}$, and $R_i^{(3)}$, respectively:

$$R_i^{(1)}=\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\psi_i(x)\left[a(x)-a(x_i)\right]u'(x)dx, \tag{22}$$

$$R_i^{(2)}=\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}\psi_i(x)\left[b(x)u(x)-b(x_i)u(x_i)\right]dx,$$

$$\equiv\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}dx\psi_i(x)\int_{x_{i-1}}^{x_{i+1}}\frac{d}{dx}(bu+f)-T_0(x_i-\xi)d\xi, \tag{23}$$

and

$$R_i^{(3)}=-\hbar_i^{-1}\int_{x_{i-1}}^{x_{i+1}}dx\psi_i^{'}(x)\int_{x_{i-1}}^{x_{i+1}}u'(\xi)K_0(x,\xi). \tag{24}$$

When we apply the quadrature formulas in (2.1) and (2.2) from [2] for (20) on both the intervals $(x_{i-1},x_i)$ and $(x_i,x_{i+1})$, we obtain the precise connection that follows:

$$-\varepsilon\hbar_i^{-1}u_{\bar{x},i}+a_iu_{\bar{x},i}\hbar_i^{-1}\int_{x_{i-1}}^{x_i}\psi_i^{(1)}(x)dx+a_iu_{x,i}\hbar_i^{-1}\int_{x_i}^{x_{i+!}}\psi_i^{(2)}(x)dx+\varepsilon\hbar^{-1}u_{x,i}-b_iu_i$$
$$=f_i-R_i^{(1)}-R_i^{(2)}-R_i^{(3)},$$

and

$$\varepsilon u_{\bar{x}\hat{x},i}+a_iu_{\overset{\circ}{x},i}-b_iu_i=f_i-R_i^{(1)}-R_i^{(2)}-R_i^{(3)},\quad i=\overline{1,N-1}. \tag{25}$$

For (1), we use the difference technique that follows from (20) and (25)

$$\varepsilon u_{\bar{x}\hat{x},i}+a_iu_{\overset{\circ}{x},i}-b_iu_i=f_i-R_i. \tag{26}$$

To get a suitable approximation to (3), we apply the rule of the right-sided rectangle. Then, we get

$$u_0=A+\sum_{i=1}^{N}h_ig_1u_i-r_1, \tag{27}$$

and

$$u_N=B+\sum_{i=1}^{N}h_ig_2u_i-r_2, \tag{28}$$

where

$$r_1=\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\frac{d}{dx}(g_1(x)u(x))dx, \tag{29}$$

and

$$r_2=\sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\frac{d}{dx}(g_2(x)u(x))dx. \tag{30}$$

For the approximating problem (1)-(3), we suggest the kind of difference technique that follows, disregarding the remaining terms $r_1$, $r_2$, and $R_i$ in (26), (27), and (28):

$$ly_i\equiv\varepsilon y_{\bar{x}\hat{x},i}+a_iy_{\overset{\circ}{x},i}-b_iy_i=f_i, \tag{31}$$

$$y_0=A+\sum_{i=1}^{N}h_ig_1y_i, \tag{32}$$

$$y_N=B+\sum_{i=1}^{N}h_ig_2y_i. \tag{33}$$

# 4. Stability and uniform convergence

Firstly, we indicate the uniform convergence of our method. Due to the difference schemes (31)-(33), we get

$$lz_i \equiv \varepsilon z_{\bar{x}\hat{x},i} + a_i z_{\overset{o}{x},i} - b_i z_i = R_i, \tag{34}$$

$$z_0 = \sum_{i=1}^{N} h_i g_1 z_i - r_1, \tag{35}$$

$$z_N = \sum_{i=1}^{N} h_i g_2 z_i - r_2, \tag{36}$$

in which the remaining terms in (21), (29), and (30), respectively, are $r_1$, $r_2$, and $R_i$.

**Lemma 4.1.** *With the situation of Lemma 2.1, within the remainder terms $r_1$, $r_2$, and $R_i$ of the scheme (31)-(33), the following estimates are satisfied on the Bakhvalov mesh:*

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}, \tag{37}$$

$$|r_1| \leq CN^{-1}, \tag{38}$$

*and*

$$|r_2| \leq CN^{-1}. \tag{39}$$

*Proof.* We evaluate the residual terms $R_i$, $r_1$, and $r_2$ for $[0,\sigma]$ and $[\sigma,l]$, in that order. With the sum of the error functions being $|R_i| = R_i^{(1)} + R_i^{(2)} + R_i^{(3)}$ :

$$|R_i| \leq \left| \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x)[a(x_i) - a(x)]u'(x)dx + \hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x)[g(x) - g(x_i)]dx \right|, \tag{40}$$

where

$$|a(x_i) - a(x)| \leq Ch_i, \tag{41}$$

$$|g(x) - g(x_i)| \leq Ch_i, \tag{42}$$

and

$$\hbar_i^{-1} \int_{x_{i-1}}^{x_{i+1}} \psi_i(x)dx \leq Ch_i. \tag{43}$$

Once the equations (41)-(43) are written in (40), we obtain

$$|R_i| \leq Ch_i. \tag{44}$$

Let $x_i \in [0,\sigma]$ :
**(1)** For $\sigma < \frac{1}{2}$,

$$x_{i-1} = \ln[1 - \frac{2(i-1)}{N}(1-\varepsilon)]\alpha^{-1}\varepsilon, \tag{45}$$

and

$$h_i = \ln[1 - \frac{2i}{N}(1-\varepsilon)]\alpha^{-1}\varepsilon - \ln[1 - \frac{2(i-1)}{N}(1-\varepsilon)]\alpha^{-1}\varepsilon. \tag{46}$$

After applying the mean value theorem to the expression (46) the result is

$$h_i \leq CN^{-1}. \tag{47}$$

The following expression is so constructed from (44) and (47):

$$|R_i| \leq CN^{-1}, \ i = \overline{0, \frac{N}{2}}.$$

**(2)** For $\sigma = \frac{1}{2}$,

$$x_{i-1} = \alpha^{-1}\varepsilon \ln[1 - (1 - e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}], \tag{48}$$

and

$$h_i = \ln[1-(1-e^{\frac{\alpha}{2\varepsilon}})\frac{2i}{N}]\alpha^{-1}\varepsilon - \ln[1-(1-e^{\frac{\alpha}{2\varepsilon}})\frac{2(i-1)}{N}]\alpha^{-1}\varepsilon. \tag{49}$$

Utilizing the mean value theorem in (49), we find

$$h_i \leq CN^{-1}. \tag{50}$$

As a result, from (44) and (50), we give the following expression:

$$|R_i| \leq CN^{-1},\ \ i=\overline{0,\frac{N}{2}}.$$

Let $x_i \in [\sigma, l]$ :

$$x_i = \sigma + h(i-\frac{N}{2}),\ \ i=\overline{\frac{N}{2}+1,N}. \tag{51}$$

By utilizing expressions (44) and (51), we obtain

$$|R_i| \leq Ch \leq CN^{-1}.$$

Thus, we have

$$|R_i| \leq CN^{-1}.$$

As a result, we reach (37). Next, we calculate (38). Using the clearly expressed expressions (29) and (30), we arrive at

$$|r_1| \leq \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\frac{d}{dx}(g_1(x)u(x))\,dx,$$

and

$$|r_2| \leq \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i}(x-x_{i-1})\frac{d}{dx}(g_2(x)u(x))\,dx.$$

We formulate the following expression using the inequality (5):

$$|r_1| \leq \|g_1\|_\infty \sum_{i=1}^{N} h_i \int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx. \tag{52}$$

We obtain the following inequality from (29) and (52)

$$|r_1| \leq CN^{-1}.$$

We create the following phrase by using (5)

$$|r_2| \leq \|g_2\|_\infty \sum_{i=1}^{N} h_i \int_{x_{i-1}}^{x_i}(1+\frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}})dx. \tag{53}$$

We find the following inequality with (53) and (30)

$$|r_2| \leq CN^{-1}.$$

The convergence outcomes of problems (1)-(3) can therefore be expressed. Thus, this demonstrates the accuracy of (37)–(39). □

**Lemma 4.2.** *Let $z_i$, $0 \leq i \leq N$ be a solution of (34)-(36). Furthermore*

$$\overline{\gamma} = h\sum_{i=1}^{N-1}|g_i| < 1.$$

*Then the prediction is given as follows:*

$$\|z\|_{\infty,\overline{\omega}_h} \leq C(\alpha^{-1}\|R\|_{\infty,\omega_h} + |r_1| + |r_2|). \tag{54}$$

*Proof.* The following inequality is written according to the discrete maximum rule [29]:

$$\|z\|_{\infty,\overline{\omega}_N} \leq |z_0| + |z_N| + \alpha^{-1}\|R\|_{\infty,\overline{\omega}_N}. \tag{55}$$

By the boundary conditions (2)-(3), we obtain inequality as follows:

$$|z_0| \leq |r_1| + \sum_{i=1}^{N}\left(\int_{x_{i-1}}^{x_i}|g_1(x)|\,dx\right)|z_i|, \tag{56}$$

and

$$|z_N| \leq |r_2| + \sum_{i=1}^{N} \left( \int_{x_{i-1}}^{x_i} |g_2(x)|\, dx \right) |z_i| . \tag{57}$$

We write the following expression if we substitute inequality (56)-(57) into (55):

$$\|z\|_{\infty,\overline{\omega}_N} \leq \alpha^{-1} \|R\|_{\infty} + |r_1| + |r_2| . \tag{58}$$

From (58), we have

$$\|z\|_{\infty} \leq (1-\overline{\gamma})^{-1} (\alpha^{-1} \|R\|_{\infty,\overline{\omega}_N} + |r_1| + |r_2|).$$

This proves inequality (54).
Consequently, we state the primary findings of this work □

We now present this paper's primary finding.

**Theorem 4.3.** *Let u and y represent the respective solutions to (1)-(3) and (31)-(33). Afterward, with the assumptions of the prior lemma, the ε-uniform estimate is satisfied by the next inequality:*

$$\|y-u\|_{\infty,\overline{\omega}_h} \leq CN^{-1}.$$

*Proof.* The results of the previous two lemmas are used to illustrate this theorem. □

## 5. Test Examples and Numerical Results

Here, we provide the numerical findings derived with the difference scheme (31)-(33) to corroborate with the theoretical findings.

**Example 5.1.** *Take into consideration the following problem*

$$\varepsilon u''(x) + (1+\frac{x}{2})u'(x) - 4u(x) + \arctan h(x+u) = 0, \ \ 0 < x < 1,$$

$$u(0) = \int_0^1 2x^2 \sin x u(x) dx + 1,$$

*and*

$$u(1) = \int_0^1 3 \tanh x u(x) dx + 2.$$

*The following formula is used for calculated absolute errors:*

$$e_{\varepsilon}^N = \max_{\omega_N} |u_{\varepsilon}(x_i) - y_i| .$$

*Here is the convergence rate:*

$$p_{\varepsilon}^N = \frac{\ln(e_{\varepsilon}^N / e_{\varepsilon}^{2N})}{\ln 2}.$$

*Table 1. Calculated the convergence process rates and errors for Example 5.1.*

| $\varepsilon$ | $N=2^6$ | $N=2^7$ | $N=2^8$ | $N=2^9$ | $N=2^{10}$ |
|---|---|---|---|---|---|
| $2^{-2}$ | 0.03346378 | 0.01674393 | 0.00837347 | 0.00418692 | 0.00209348 |
| | 0.99896 | 0.99974 | 0.99993 | 1.00212 | |
| $2^{-4}$ | 0.06696914 | 0.03353289 | 0.01677248 | 0.00838699 | 0.00419359 |
| | 0.99791 | 0.99948 | 0.99987 | 0.99996 | |
| $2^{-6}$ | 0.08641245 | 0.04328675 | 0.02165344 | 0.01082798 | 0.00541414 |
| | 0.99731 | 0.99932 | 0.99983 | 0.99995 | |
| $2^{-8}$ | 0.09412933 | 0.04716026 | 0.023559208 | 0.01179753 | 0.00589895 |
| | 0.99707 | 0.99926 | 0.99981 | 0.99995 | |
| $2^{-10}$ | 0.09673932 | 0.04847064 | 0.02424795 | 0.01212555 | 0.00606297 |
| | 0.99699 | 0.99924 | 0.99982 | 1.00001 | |
| $e^N$ | 0.09673932 | 0.04847064 | 0.02424795 | 0.01179753 | 0.00606297 |
| $p^N$ | 0.99707 | 0.99924 | 0.99981 | 0.99995 | |

**Example 5.2.** *Take into consideration the following problem*

$$\varepsilon u''(x) + (1+x)u'(x) - 2u(x) = \arctan h(x), \ \ 0 < x < 1,$$

$$u(0) = \int_0^1 \cos(\pi x) u(x) dx + 2,$$

*and*

$$u(1) = \int_0^1 \sin(\pi x) u(x) dx + 3.$$

*We implement the double-mesh design to analyze the suggested solutions' errors and calculate their rate of convergence because the solution to the testing problem is unknowable. Thus, we verify a computed solution twice with a solution on the mesh divided into two [11, 19, 24]. Error evaluations made in this process are given by:*

$$e_{\varepsilon}^{N}=\max_{\omega_N}\left|y_i^{\varepsilon,N}-\widetilde{y}_i^{\varepsilon,2N}\right|,$$

*where* $\widetilde{y}_i^{\varepsilon,2N}$ *is an approximation of the associated technique in the mesh* $\overline{\omega}_{2N}=\left\{x_{i/2}:i=\overline{0,2N}\right\}$ *and* $x_{i+1/2}=\frac{x_i+x_{i+1}}{2}$ *for* $i=\overline{1,N-1}$*. The following expression is the appropriate convergence rate:*

$$p_{\varepsilon}^{N}=\frac{\ln(e_{\varepsilon}^{N}/e_{\varepsilon}^{2N})}{\ln 2}.$$

*Table 2. Calculated the convergence process rates and errors for Example 5.2.*

| $\varepsilon$ | $N=2^4$ | $N=2^5$ | $N=2^6$ | $N=2^7$ | $N=2^8$ | $N=2^9$ |
|---|---|---|---|---|---|---|
| $2^{-2}$ | 0.056423 | 0.030111 | 0.001554 | 0.007962 | 0.004023 | 0.002067 |
| | 0.83 | 0.88 | 0.94 | 0.96 | 0.97 | |
| $2^{-4}$ | 0.058451 | 0.030320 | 0.001553 | 0.007950 | 0.004015 | 0.002034 |
| | 0.84 | 0.86 | 0.95 | 0.96 | 0.98 | |
| $2^{-6}$ | 0.059014 | 0.030212 | 0.001550 | 0.007863 | 0.004012 | 0.001995 |
| | 0.84 | 0.86 | 0.93 | 0.95 | 0.96 | |
| $2^{-8}$ | 0.059014 | 0.030125 | 0.001550 | 0.007833 | 0.004007 | 0.001995 |
| | 0.84 | 0.86 | 0.93 | 0.95 | 0.99 | |
| $2^{-10}$ | 0.059012 | 0.030984 | 0.001546 | 0.007828 | 0.004004 | 0.001994 |
| | . | . | . | . | . | . |
| | . | . | . | . | . | . |
| | . | . | . | . | . | . |
| $e^N$ | 0.059024 | 0.030988 | 0.001555 | 0.007963 | 0.004024 | 0.002035 |
| $p^N$ | 0.85 | 0.89 | 0.95 | 0.97 | 0.99 | |

In the example above, we use finite difference technique. Based on Tables 1-2, as $N$ increases, it becomes apparent that the smooth rapid convergence $p^N$ exhibits a first-order convergence rate. In other words, numerical findings demonstrate that the suggested approach works very successfully.

**Conclusion**

In this work, we give numerical method for solving linear singularly perturbed problems with integral boundary conditions. The numerical method of convergence has been the primary focus. Thus, the uniform convergence of a finite difference method with a perturbation parameter has been shown. That method is very useful for solving the linear SPDEs with integral boundary conditions and boundary layer, as it gives the first-order uniformly correct solutions. By using weight and remainder terms, exponential basis functions, asymptotic estimation, and interpolating quadrature forms, the method supports the boundary layer. As a result, appropriate techniques are used to solve the test problems. Our method's greatest advantage is that it yields excellent and appropriate solutions. After all, the numerical outcomes show the accuracy and robustness of our suggested analytical method for reaction-diffusion-type singularly perturbed situations.

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