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AN EXACT ANALYSIS OF FINE RESOLUTION FREQUENCY ESTIMATION METHOD FROM THREE DFT SAMPLES

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Abstract: An exact closed-form expression for the variance of frequency estimators proposed by Jacobsen and Kootsookos (2007) and Candan (2011) has been derived. Additionally, an approximate variance formula applicable for sufficiently large data lengths is provided. Theoretical results have been validated through computer simulations.

Keywords: Frequency estimation, Complex sinusoid, Variance analysis

Üç DFT Örneğinden İnce Çözünürlüklü Frekans Tahmin Yönteminin Tam Analizi

Öz: Jacobsen ve Kootsookos (2007) ve Candan (2011) makalelerinde önerilen frekans kestiricilerinin değişintisi için bir kesin kapalı biçim ifadesi türetilmiştir. Ek olarak, yeterince büyük veri uzunlukları için geçerli olan yaklaşık bir değişinti formülü verilmiştir. Teorik sonuçlar bilgisayar simülasyonları ile doğrulanmıştır.

Anahtar Kelimeler: Frekans kestirimi, Kompleks sinüzoid, Değişinti analizi

1. INTRODUCTION

The problem of sinusoidal frequency estimation has been frequently studied in the signal processing literature due to its wide range of applications. Consider the following data model consisting of a complex sinusoid in noise:

$$x_n = \alpha e^{j\phi} \exp(j2\pi f n/N) + e_n, \qquad n = 0, 1, \ldots, N-1 \tag{1}$$

where α, ϕ and f are real-valued unknown deterministic parameters, which are amplitude, phase and frequency of the complex sinusoid, e_n is assumed to be a zero-mean complex white noise process with unknown variance σ^2. The frequency is divided into an integer bin index k_0 and a fractional term δ as $f = k_0 + \delta$ with $|\delta| \leq 0.5$.

A suboptimal but simple approach to estimate f from the N samples of x_n, which is first suggested in Jacobsen and Kootsookos (2007) and later modified in Candan (2011), uses the three (the peak and its two neighbors) samples of the Discrete Fourier Transform (DFT) $\{X_k\}_{k=0}^{N-1}$ of the data $\{x_n\}_{n=0}^{N-1}$. Let k_0 be the peak index. Provided the peak is correctly identified, i.e., the parameter k_0 is known, the fractional part δ of the frequency is estimated via

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$$\hat{\delta} = c_N \operatorname{Re}\left(\frac{X_{k_0-1} - X_{k_0+1}}{2X_{k_0} - X_{k_0-1} - X_{k_0+1}}\right) \tag{2}$$

where

$$c_N = \frac{\tan(\pi/N)}{\pi/N} \tag{3}$$

is the so-called bias correction factor, which is introduced in Candan (2011) to reduce the bias of the estimator in Jacobsen and Kootsookos (2007).

For small values of δ, the variance of the estimator $\hat{\delta}$, denoted by $\operatorname{var}(\hat{\delta})$, has been derived in Candan (2013) as

$$\operatorname{var}(\hat{\delta}) \approx \frac{1}{4N\mathrm{SNR}}\left[\left(\frac{\tan(\pi/N)}{\pi/N}\right)^2 + 3\left(\frac{\tan(\pi\delta/N)}{\pi/N}\right)^2\right] \tag{4}$$

where $\mathrm{SNR} = \alpha^2/\sigma^2$ is the signal-to-noise ratio. However, the expression in (4), which is derived under small δ assumption, gives inaccurate results when $|\delta|$ is relatively large. In this paper, we develop an exact expression for the variance of $\hat{\delta}$. By using a Taylor series expansion technique (see, e.g., Papoulis (1991)), a novel variance formula is presented in Section 2. A simple and yet accurate variance expression for sufficiently large N is also given. Computer simulations are presented in Section 3 to validate the theoretical results. Finally, in Section 4, conclusions are drawn.

2. EXACT VARIANCE DERIVATION

Let

$$\hat{d}(\theta) = \operatorname{Re}\left(\frac{X_{k_0-1} - X_{k_0+1}}{2X_{k_0} - X_{k_0-1} - X_{k_0+1}}\right) \tag{5}$$

with

$$\theta = \left[\bar{X}_{k_0-1} \;\; \tilde{X}_{k_0-1} \;\; \bar{X}_{k_0} \;\; \tilde{X}_{k_0} \;\; \bar{X}_{k_0+1} \;\; \tilde{X}_{k_0+1}\right]^T \tag{6}$$

where superscript T denotes the transpose. For notational convenience, here and onwards, an over bar denotes the real part of the quantity beneath it, and an over tilde denotes the imaginary part, i.e., $\bar{X} = \operatorname{Re}(X)$ and $\tilde{X}=\operatorname{Im}(X)$ for a complex quantity X.

Let $\theta_0 = E\{\theta\}$, i.e., the mean of θ. It is calculated as (see Appendix A)

$$\theta_0 = [\bar{g}(\delta+1) \;\; \tilde{g}(\delta+1) \;\; \bar{g}(\delta) \;\; \tilde{g}(\delta) \;\; \bar{g}(\delta-1) \;\; \tilde{g}(\delta-1)]^T \tag{7}$$

where

$$g(\delta + l) = \alpha e^{j\phi} \sum_{n=0}^{N-1} \exp\left(j\frac{2\pi}{N}(\delta + l)n\right), \quad \text{for } l = 0, 1, \text{and } -1. \tag{8}$$

For sufficiently large N and/or SNR, θ will be in the vicinity of θ_0, and a first order Taylor expansion of $\hat{d}(\theta)$ around θ_0 yields

$$\hat{d}(\theta) \approx \hat{d}(\theta_0) + v^T(\theta - \theta_0) \tag{9}$$

where $v \triangleq \nabla\hat{d}(\theta)\big|_{\theta=\theta_0}$ is the gradient of $\hat{d}(\theta)$ evaluated at $\theta = \theta_0$, which is a function of N, α, ϕ and δ. From (9) it follows that

$$\mathrm{var}\left(\hat{d}(\theta)\right) \approx v^T C_\theta v \tag{10}$$

where $C_\theta = E\{(\theta - \theta_0)(\theta - \theta_0)^T\}$, which is the covariance matrix of θ with size 6×6. C_θ is derived as a diagonal matrix with diagonal elements are given by (see Appendix A)

$$[C_\theta]_{k,k} = \frac{N\sigma^2}{2} \quad \text{for } k = 1, \ldots, 6. \tag{11}$$

Thus

$$\mathrm{var}(\hat{d}) \approx \frac{N\sigma^2\|v\|^2}{2} \tag{12}$$

and

$$\mathrm{var}(\hat{\delta}) = c_N^2 \mathrm{var}(\hat{d}) \approx \frac{c_N^2 N\sigma^2\|v\|^2}{2}. \tag{13}$$

It is shown in Appendix B that $\|v\|^2$ does not depend on ϕ, so we can take $\phi = 0$ for the computation of v in (13). With this convenience, the elements of v are calculated as

$$[v|_{\phi=0}] = \frac{\alpha^{-1}}{2}\begin{bmatrix} -\cot(\pi\delta)\tan\left(\frac{\pi\delta}{N}\right)A(N,\delta) \\ -\tan\left(\frac{\pi\delta}{N}\right)A(N,\delta) \\ 2\cot(\pi\delta)\tan^2\left(\frac{\pi\delta}{N}\right)B(N,\delta) \\ 2\tan^2\left(\frac{\pi\delta}{N}\right)B(N,\delta) \\ -\cot(\pi\delta)\tan\left(\frac{\pi\delta}{N}\right)C(N,\delta) \\ -\tan\left(\frac{\pi\delta}{N}\right)C(N,\delta) \end{bmatrix} \tag{14}$$

where

$$A(N,\delta) = \frac{\sin\left(\frac{\pi(\delta-1)}{N}\right)\sin^2\left(\frac{\pi(\delta+1)}{N}\right)}{\sin^3\left(\frac{\pi}{N}\right)\cos\left(\frac{\pi\delta}{N}\right)} \tag{15}$$

$$B(N,\delta)=\frac{\cos\left(\frac{\pi}{N}\right)\sin\left(\frac{\pi(\delta-1)}{N}\right)\sin\left(\frac{\pi(\delta+1)}{N}\right)}{\sin^3\left(\frac{\pi}{N}\right)} \tag{16}$$

$$C(N,\delta)=\frac{\sin^2\left(\frac{\pi(\delta-1)}{N}\right)\sin\left(\frac{\pi(\delta+1)}{N}\right)}{\sin^3\left(\frac{\pi}{N}\right)\cos\left(\frac{\pi\delta}{N}\right)}. \tag{17}$$

Inserting (14) into (13), the variance of $\hat{\delta}$ is expressed as

$$\operatorname{var}(\hat{\delta})\approx\frac{c_N^2}{4N\mathrm{SNR}} \times\left\{\frac{\sin^2\left(\frac{\pi(\delta-1)}{N}\right)\sin^2\left(\frac{\pi(\delta+1)}{N}\right)\left[3\sin^2\left(\frac{\pi\delta}{N}\right)\cos^2\left(\frac{\pi}{N}\right)+\cos^2\left(\frac{\pi\delta}{N}\right)\sin^2\left(\frac{\pi}{N}\right)\right]}{\sin^6\left(\frac{\pi}{N}\right)\cos^4\left(\frac{\pi\delta}{N}\right)D^2(N,\delta)}\right\} \tag{18}$$

where

$$D(N,\delta)=\frac{\sin(\pi\delta)}{N\sin\left(\frac{\pi\delta}{N}\right)}. \tag{19}$$

The variance expression in (18) is independent of ϕ and is a function of N, SNR and δ. When comparing the expressions (18) and (4), we see that they are identical for $\delta=0$.

It can be observed that, for a fixed value of δ, the term in curly brackets in (18) is $O(1)$ as $N\to\infty$. Replacing this term with its limit as $N\to\infty$ in (18), an approximate expression of $\operatorname{var}(\hat{\delta})$ for sufficiently large N is obtained as

$$\operatorname{var}(\hat{\delta})\approx\frac{c_N^2(\delta^2-1)^2(3\delta^2+1)}{4N\mathrm{SNR}\,\mathrm{sinc}^2(\delta)}. \tag{20}$$

Note that (18) and (20) are also applicable for the variance of the estimator $\hat{\delta}$ in Jacobsen and Kootsookos (2007) with $c_N=1$.

3. NUMERICAL EXAMPLES

Computer simulations have been conducted to verify our theoretical variance expressions for estimating the frequency of a complex sinusoid in complex white Gaussian noise. The phase of the sinusoid was set to zero. The amplitude of the sinusoid was fixed to 1 and the noise samples were scaled appropriately to obtain different SNRs. The frequency was $f=k_0+\delta$. The integer part k_0 of the frequency was treated as a known parameter and set to one fourth of the data length N. The bias correction factor was $c_N=\tan(\pi/N)/(\pi/N)$. The normalized variance of $\hat{\delta}$, i.e., $\operatorname{var}(\hat{\delta})\times N\times\mathrm{SNR}$, was evaluated as a function of δ and SNR. To obtain all simulation results, we averaged the outcomes of 100,000 independent runs.

Figure 1 shows normalized variances versus δ at SNR $=20$ dB, $N=16$ and $k_0=4$. The expressions of (4), (18) and (20) and the corresponding Cramér-Rao lower bound (CRLB) (see, e.g., Kay (1993)) are also shown. We noted that the observed variances aligned closely with the theoretical calculations from equation (18), and equation (20) provided a good approximation across the entire range of δ. On the other hand, (4), whose derivation assumed small δ, could predict the estimator variance only when, say, $|\delta| \leq 0.1$. We also observe that the estimator variance does not attain the CRLB for the whole range of δ. Figure 2 shows the results of the above experiment with $N = 128$ and $k_0 = 32$. Same observations were obtained; moreover, comparing normalized variance values in Figure 1 and Figure 2, we see that increasing N by a factor of eight only slightly changed the results, indicating the weak dependence of the normalized variances on N. Figure 3 shows normalized variances as a function of SNR at $\delta = 0.35$, $N = 128$ and $k_0 = 32$. One can notice that (18) and (20) showed good agreement with the simulation results across all considered SNRs, while (4) incorrectly predicted the estimator variances at each SNR.

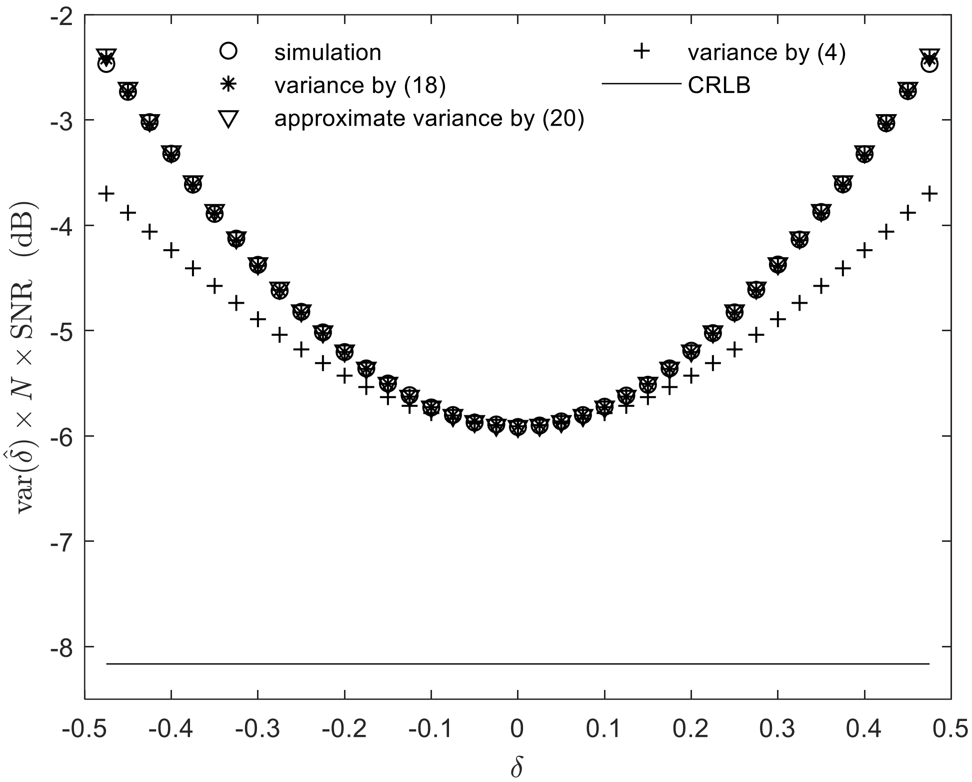

Figure 1:
Normalized variances versus δ at SNR $= 20$ dB, $N = 16$ and $k_0 = 4$

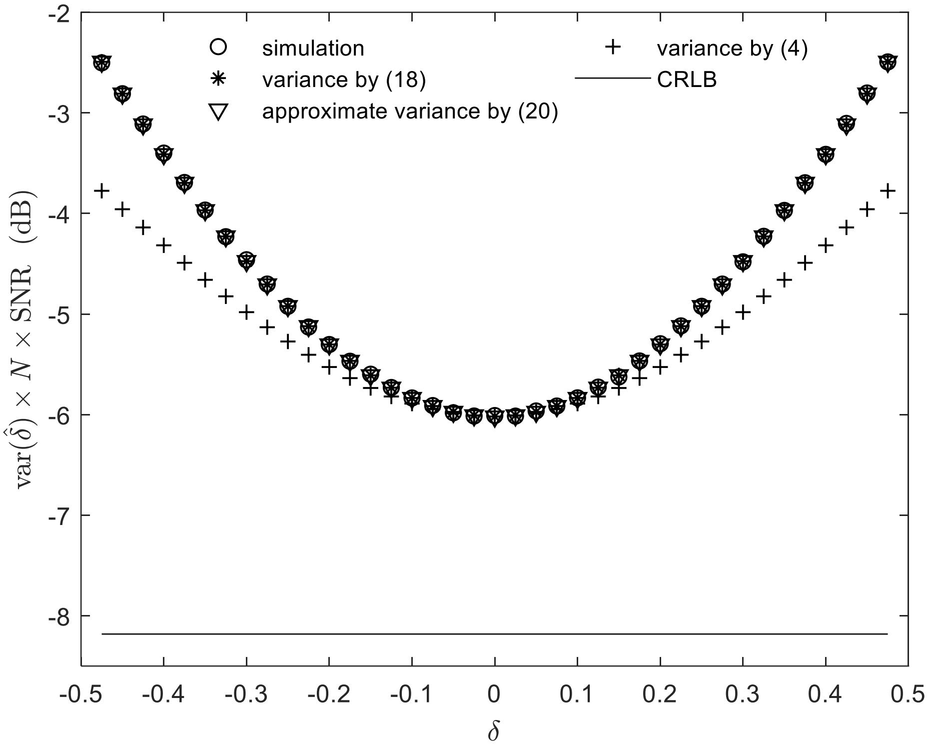

Figure 2:
As in Figure 1, but for $N = 128$ *and* $k_0 = 32$

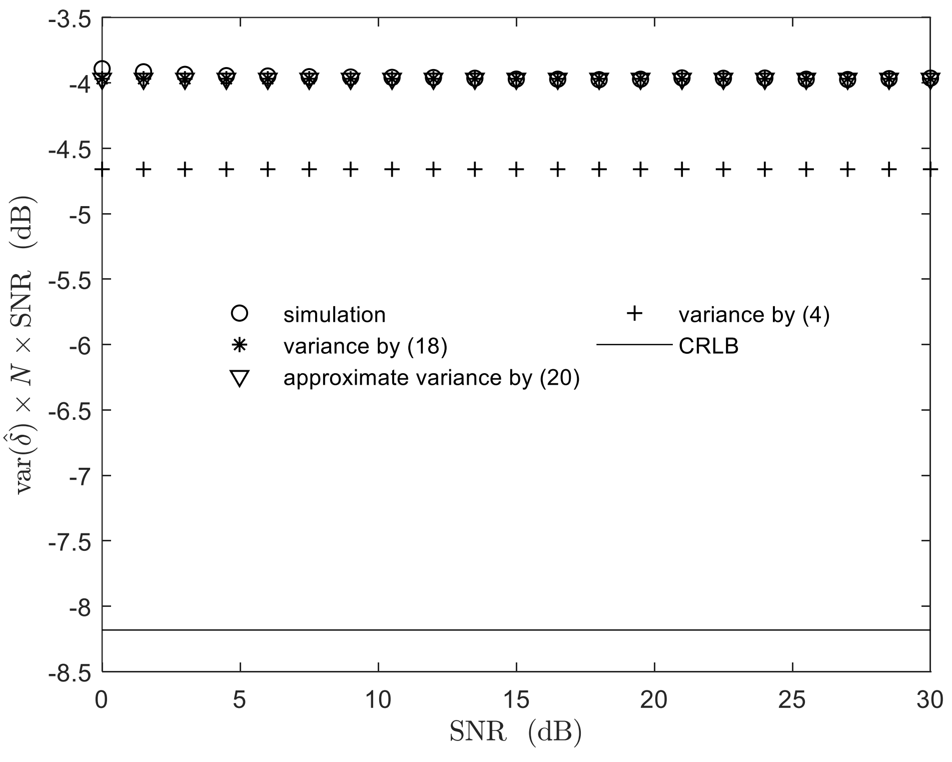

Figure 3:
Normalized variances versus SNR at $\delta = 0.35$, $N = 128$ *and* $k_0 = 32$

4. CONCLUSIONS

An exact closed-form frequency variance expression of the frequency estimators suggested in Jacobsen and Kootsookos (2007) and Candan (2011) for a single complex sinusoid in complex white noise has been derived. A simple approximate variance formula for sufficiently large data lengths and signal-to-noise ratios is also developed. Computer simulations have been provided to validate the theoretical results.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest regarding the publication of this article.

AUTHOR CONTRIBUTION

Betofe Mboyo KEYTA: Responsible for data collection, analysis and interpretation, drafting the initial version of the article, giving final approval, and taking full responsibility.

Erdoğan DİLAVEROĞLU: Oversaw and managed the conceptual and design processes of the study, critically reviewed the intellectual content, gave final approval, and assumed full responsibility.

APPENDIX A

Computation of θ_0:

The noise samples satisfy the following conditions:

$$E\{e_n\} = 0 \tag{A.1}$$

$$E\{e_n e_m^*\} = \begin{cases} \sigma^2, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{A.2}$$

$$E\{e_n e_m\} = 0, \quad \text{for all } n, m \tag{A.3}$$

where E denotes the expectation operation and superscript * denotes the complex conjugate.

Let $W_N = \exp\left(j\frac{2\pi}{N}\right)$. Then

$$\begin{aligned} E\{X_{k_0-1}\} &= E\left\{\sum_{n=0}^{N-1} x_n W_N^{-(k_0-1)n}\right\} = E\left\{\sum_{n=0}^{N-1}\left(\alpha e^{j\phi} W_N^{(k_0+\delta)n} + e_n\right) W_N^{-(k_0-1)n}\right\} \\ &= \alpha e^{j\phi}\sum_{n=0}^{N-1} W_N^{(\delta+1)n} + \sum_{n=0}^{N-1} E\{e_n\} W_N^{-(k_0-1)n} = \alpha e^{j\phi}\sum_{n=0}^{N-1} W_N^{(\delta+1)n} = g(\delta+1) \end{aligned} \tag{A.4}$$

using (A.1). Similarly, $E\{X_{k_0}\} = g(\delta)$ and $E\{X_{k_0+1}\} = g(\delta-1)$.

Computation of C_θ:

The elements of C_θ are either $\text{cov}(\bar{X}_k, \bar{X}_l)$, or $\text{cov}(\tilde{X}_k, \tilde{X}_l)$, or $\text{cov}(\bar{X}_k, \tilde{X}_l)$ for $k, l = k_0 - 1$, k_0, and $k_0 + 1$, where $\text{cov}(X, Y) = E\{(X - E\{X\})(Y - E\{Y\})\}$, i.e., the covariance between the real random variables X and Y. We have, using (A.1) – (A.3)

$$E\{(X_k - E\{X_k\})(X_l - E\{X_l\})^*\} = E\left\{\left(\sum_{n=0}^{N-1} e_n W_N^{-kn}\right)\left(\sum_{m=0}^{N-1} e_m^* W_N^{lm}\right)\right\}$$
$$= \sum_{n,m=0}^{N-1} E\{e_n e_m^*\} W_N^{-kn+lm} = \sigma^2 \sum_{n=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N\sigma^2, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases} \tag{A.5}$$

and

$$E\{(X_k - E\{X_k\})(X_l - E\{X_l\})\} = E\left\{\left(\sum_{n=0}^{N-1} e_n W_N^{-kn}\right)\left(\sum_{m=0}^{N-1} e_m W_N^{-lm}\right)\right\}$$
$$= \sum_{n,m=0}^{N-1} E\{e_n e_m\} W_N^{-kn-lm} = 0, \quad \text{for all } k, l. \tag{A.6}$$

On the other hand

$$E\{(X_k - E\{X_k\})(X_l - E\{X_l\})^*\}$$
$$= \left[\mathrm{cov}(\bar{X}_k, \bar{X}_l) + \mathrm{cov}(\tilde{X}_k, \tilde{X}_l)\right] + j\left[\mathrm{cov}(\tilde{X}_k, \bar{X}_l) - \mathrm{cov}(\bar{X}_k, \tilde{X}_l)\right] \tag{A.7}$$

and

$$E\{(X_k - E\{X_k\})(X_l - E\{X_l\})\}$$
$$= \left[\mathrm{cov}(\bar{X}_k, \bar{X}_l) - \mathrm{cov}(\tilde{X}_k, \tilde{X}_l)\right] + j\left[\mathrm{cov}(\tilde{X}_k, \bar{X}_l) + \mathrm{cov}(\bar{X}_k, \tilde{X}_l)\right]. \tag{A.8}$$

It follows from (A.5) – (A.8) that

$$\mathrm{cov}(\bar{X}_k, \bar{X}_l) = \mathrm{cov}(\tilde{X}_k, \tilde{X}_l) = \begin{cases} N\sigma^2/2, & \text{if } k = l \\ 0, & \text{if } k \neq l \end{cases} \tag{A.9}$$

and

$$\mathrm{cov}(\bar{X}_k, \tilde{X}_l) = 0, \quad \text{for all } k, l. \tag{A.10}$$

Hence

$$C_\theta = N\sigma^2 I/2 \tag{A.11}$$

where I is the identity matrix of size 6×6.

APPENDIX B

Decompose $g(\delta + l)$ as

$$g(\delta + l) = e^{j\phi} h(\delta + l) \tag{B.1}$$

where

$$h(\delta + l) = \alpha \sum_{n=0}^{N-1} \exp\left(j \frac{2\pi}{N}(\delta + l)n\right). \tag{B.2}$$

From (B.1), we have

$$\bar{g}(\delta + l) = \bar{h}(\delta + l) \cdot \cos\phi - \tilde{h}(\delta + l) \cdot \sin\phi \tag{B.3}$$

and

$$\tilde{g}(\delta + l) = \bar{h}(\delta + l) \cdot \sin\phi + \tilde{h}(\delta + l) \cdot \cos\phi. \qquad \text{(B.4)}$$

Putting (B.3) and (B.4) into (7), we can express θ_0 as

$$\theta_0 = M(\phi)\omega_0 \qquad \text{(B.5)}$$

where

$$M(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & & & & \\ \sin\phi & \cos\phi & & & & \\ & & \cos\phi & -\sin\phi & & \\ & & \sin\phi & \cos\phi & & \\ & & & & \cos\phi & -\sin\phi \\ & & & & \sin\phi & \cos\phi \end{bmatrix} \qquad \text{(B.6)}$$

and

$$\omega_0 = \left[\bar{h}(\delta + 1)\ \ \tilde{h}(\delta + 1)\ \ \bar{h}(\delta)\ \ \tilde{h}(\delta)\ \ \bar{h}(\delta - 1)\ \ \tilde{h}(\delta - 1)\right]^T. \qquad \text{(B.7)}$$

Let variables θ and ω be related as $\theta = M(\phi)\omega$. Then we have

$$v = \nabla\hat{d}(\theta)\big|_{\theta=\theta_0=M(\phi)\omega_0} = M(\phi)\nabla\hat{d}(\omega)\big|_{\omega=\omega_0} \qquad \text{(B.8)}$$

using the chain rule and the fact that $\hat{d}(M(\phi)\omega) = \hat{d}(\omega)$. Since $M(\phi)$ is an orthogonal matrix, it follows from (B.8) that

$$\|v\|^2 = \left\|\nabla\hat{d}(\theta)\big|_{\theta=\theta_0=M(\phi)\omega_0}\right\|^2 = \left\|\nabla\hat{d}(\omega)\big|_{\omega=\omega_0}\right\|^2 \qquad \text{(B.9)}$$

where the last term does not depend on ϕ.

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