Konuralp Journal of Mathematics
Volume 4 No. 2 pp. 34–41 (2016) ©KJM

GRAPHS WHICH ARE DETERMINED BY THEIR SPECTRUM

ALI ZEYDI ABDIAN

Abstract. It is well-known that the problem of spectral characterization is related to the Hückel theory from Chemistry. E. R. van Dam and W. H. Haemers [11] conjectured almost all graphs are determined by their spectra. Nevertheless, the set of graphs which are known to be determined by their spectra is small. Hence discovering infinite classes of graphs that are determined by their spectra can be an interesting problem and helps reinforce this conjecture. The main aim of this work is to characterize new classes of graphs that are known as multicone graphs. In this work, it is shown that any graph cospectral with multicone graphs $K_w \bigtriangledown GQ(2,1)$ or $K_w \bigtriangledown GQ(2,2)$ is determined by its adjacency spectra, where $GQ(2,1)$ and $GQ(2,2)$ denote the strongly regular graphs that are known as the generalized quadrangle graphs. Also, we prove that these graphs are determined by their Laplacian spectrum. Moreover, we propose four conjectures for further reseache in this topic.

1. Introduction

All graph considered here are simple and undirected. All notions on graph that are not defined here can be found in [3, 4, 6, 15]. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, ..., v_n\}$ and edge set $E = E(G) = \{e_1, ..., e_m\}$. Denote by $d(v)$ the degree of vertex v. Let $A(G)$ be the $(0, 1)$-adjacency matrix of graph G. The characteristic polynomial of G is $\det(\lambda I - A(G))$, and it is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjaceny eigenvalues of G and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If G has n vertices, then it has n eigenvalues in descending order as $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Let λ_1, λ_2, ..., λ_n be the distinct eigenvalues of G with multiplicity m_1, m_2, ... , m_n, respectively. The multi-set of eigenvalues of $A(G)$ is called the adjacency spectrum of G. The matrices $L(G) = D(G) - A(G)$ and $SL(G) = D(G) + A(G)$ are called the Laplacian matrix and signless Laplacian matrix of G, respectively, where $D(G)$ is the diagonal matrix $diag\{d(v_1), ..., d(v_n)\}$ and $A(G)$ is the $(0, 1)$ adjacency matrix of G. Two graph with the same spectrum are called cospectral. A graph G is determined by its spectrum (DS for short) if every graph cospectral to it is in fact isomorphic to it. About the

2010 *Mathematics Subject Classification.* 05C50.

Key words and phrases. Adjacency spectrum, Laplacian spectrum, Determined by their spectra, generalized quadrangle .

background of the guestion "which graph are determined by their spectrums?", we refer to [11, 12]. A spectral characterization of multicone graph is studied in [13]. In [13], Wang, Zhao and Huang investigated on the spectral characterization of multicone graph and also they claimed that friendship graph F_n(that are special classes of multicone graph) are DS with respect to their adjacency spectra. In addition, Wang, Belardo, Huang and Borovićanin [14] proposed such conjecture on the adjacency spectrum of F_n. This conjecture caused some activity on the spectral characterization of F_n. Das [5] claims to have a proof, but Abdollahi, Janbaz and Oboudi [2] found a mistake. In addition, these authors give correct proofs in some special cases. Abdian and Mirafzal [1] characterized new classes of multicone graph that were DS with respect to their spectra. In this paper, we present new classes of multicone graph that are DS with respect to their spectra.
This paper is organized as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$ must be bidegreed (Lemma 3.1). In Subsection 3.2, we prove that any graph cospectral with $K_1 \bigtriangledown GQ(2,1)$ is determined by its adjacency spectra (Lemma 3.2). In Subsection 3.3, we prove that complement of $K_w \bigtriangledown GQ(2,1)$ is DS with respect to their adjacency spectra (Theorem 3.1). In Subsection 3.4, we show that graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectra (Theorem 3.2). In Section 4, we characterize multicone graph $K_w \bigtriangledown GQ(2,2)$ and we show that these graph are DS with respect to their spectra. Subsections 4.1, 4.2 and 4.3 are the similar of Subsections 3.2, 3.3 and 3.4, respectively. We conclude with final remarks and open problems in Section 5.

2. Some definitions and preliminaries

Lemma 2.1. [1, 9] *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*

(*i*) *The number of vertices,*
(*ii*) *The number of edges.*
For the adjacency matrix, the following follows from the spectrum:
(*iii*) *The number of closed walks of any length,*
(*iv*) *Being regular or not and the degree of regularity,*
(*v*) *Being bipartite or not.*
For the Laplacian matrix, the following follows from the spectrum:
(*vi*) *The number of spanning trees,*
(*vii*) *The number of components,*
(*viii*) *The sum of squares of degrees of vertices.*

Theorem 2.1. [4] *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by:*

$$P_{G_1 \bigtriangledown G_2(y)} = \frac{P_{G_1}(y)P_{G_2}(y)}{(y-r_1)(y-r_2)}((y-r_1)(y-r_2)-n_1n_2).$$

Proposition 2.1. [12, *Proposition* 4] *Let G be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over G, the graph H ; that is, obtained from G by adding one vertex that is adjacent to all vertices of G, is also determined by its Laplacian spectrum.*

Theorem 2.2. [1] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 2.3. [8] *Let G and H be two graphs with the Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then the Laplacian spectrum of $\overline{G}$ and $G \bigtriangledown H$ are $n-\lambda_1, n-\lambda_2, ..., n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., n+\mu_{m-1}, 0$, respectively.*

Theorem 2.4. [8] *Let G be a graph on n vertices. Then n is one of the Laplacian eigenvalue of G if and only if G is the join of two graph.*

Theorem 2.5. [7, p.163] *For a graph G, the following statements are equivalent:*
(i) G is d-regular.
(ii) $\varrho(G) = d_G$, the average vertex degree.
(iii) G has $v = (1,1,...,1)^t$ as an eigenvector for $\varrho(G)$.

Proposition 2.2. [4] *Let $G - j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j}(y) = P_G(y)\sum_{i=1}^{m}\frac{\alpha_{ij}^2}{y-\mu_i}$, where m is the number of distinct eigenvalues of graph G.*

3. Main Results

In this subsection, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$ must be bidegreed.

3.1. Connected graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$.

Proposition 3.1. *Let G be a graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$. Then* $Spec(G) = \left\{ [-1]^{w-1}, [-2]^4, [1]^4, \left[\frac{\Omega + \sqrt{\Omega^2 + 4\Gamma}}{2}\right]^1, \left[\frac{\Omega - \sqrt{\Omega^2 + 4\Gamma}}{2}\right]^1 \right\}$, *where $\Omega = w+3$ and $\Gamma = 5w+4$.*

Proof. It is well-known that $Spec(GQ(2,1)) = \left\{[-2]^4, [1]^4, [4]^1\right\}$. Now, by Theorem 2.1 the proof is clear. □

Lemma 3.1. *Let G be cospectral with a multicone graph $K_w \bigtriangledown GQ(2,1)$. Then G is bidegreed in which any vertex of G is of degree $w+4$ or $w+8$.*

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the degrees sequence of graph G consists of at least three number. Hence the equality in Theorem 2.2 cannot happen for any δ. But, if we put $\delta = w+4$, then the equality in Theorem 2.2 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w+8$. By contrary, we suppose that $\Delta < w+8$. Therefore, the equality in Theorem 2.2 cannot hold for any δ. But, if we put $\delta = w+4$, then this equality holds . This is a contradiction and so $\Delta = w+8$. Now, $\delta = w+4$, since G is bidegreed and G has

$w+9$, $\Delta = w+8$ and $w(w+8)+9(w+4) = w\Delta + 9(w+4) = \sum_{i=1}^{w+9} \deg v_i$.
This completes the proof. $\square$

In the following subsection, we prove that the cone of the generalized quadrangle graph $GQ(2,1)$ is DS with respect to its adjacency spectra.

3.2. Connected graph cospectral with the multicone graph $K_1 \bigtriangledown GQ(2,1)$.

Lemma 3.2. *Any graph cospectral with the multicone graph $K_1 \bigtriangledown GQ(2,1)$ is DS with respect to its adjacency spectrum.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown GQ(2,1)$. By Lemma 3.1, it is easy to see that G has one vertex of degree 9, say j. Now, Proposition 2.2 implies that $P_{G-j}(y) = (y-\mu_3)^3(y-\mu_4)^3[\alpha_{1j}^2 F_1 + \alpha_{2j}^2 F_2 + \alpha_{3j}^2 F_3 + \alpha_{4j}^2 F_4]$, where

$\mu_1 = \dfrac{4+\sqrt{52}}{2}$, $\mu_2 = \dfrac{4-\sqrt{52}}{2}$, $\mu_3 = 1$ and $\mu_4 = -2$.
$F_1 = (y-\mu_2)(y-\mu_3)(y-\mu_4)$,
$F_2 = (y-\mu_1)(y-\mu_3)(y-\mu_4)$,
$F_3 = (y-\mu_1)(y-\mu_2)(y-\mu_4)$,
$F_4 = (y-\mu_1)(y-\mu_2)(y-\mu_3)$.
Now, we have:
$a+b+4 = -(3\mu_3 + 3\mu_4)$,
$a^2+b^2+16 = 36 - (3\mu_3^2 + 3\mu_4^2)$,

where a and b are the eigenvalues of graph $G-j$. If we solve the above equations, then $a=1$ and $b=-2$. Hence $Spec(G-j) = Spec(GQ(2,1))$ and so $G - j \cong GQ(2,1)$.
This follows the result. $\square$

Until now, we have shown the cone of generalized quadrangle graph $K_1 \bigtriangledown GQ(2,1)$ is DS. The natural question is; what happens for multicone graph $K_w \bigtriangledown GQ(2,1)$? we will respond to this question in the following theorem.

3.3. Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$.

Theorem 3.1. *Multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their adjacency spectrums.*

Proof. We solve the problem by induction on w. If $w=1$, by Lemma 3.3 there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown GQ(2,1))$, then $G_1 \cong K_w \bigtriangledown GQ(2,1)$, where G_1 is an arbitrary graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown GQ(2,1))$, then $G \cong K_{w+1} \bigtriangledown GQ(2,1)$, where G is an arbitrary graph cospectral with multicone graph $K_{w+1} \bigtriangledown GQ(2,1)$. It is clear that G has one vertex and 9 edges more than G_1. Also, By Lemma 3.1 and the spectrums of G and G_1, we can conclude that $G \cong K_1 \bigtriangledown G_1$.
Now, induction hypothesis follows the result. $\square$

In the following subsection, we prove that multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectrum.

3.4. **Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,1)$ with respect to Laplacian spectrum.**

Theorem 3.2. *Multicone graph $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(L(G_1)) = Spec(L(K_w \bigtriangledown GQ(2,1))) = \left\{[w+9]^w,\ [w+3]^4,\ [w+6]^{21},\ [0]^1\right\}$, then $G_1 \cong K_w \bigtriangledown GQ(2,1)$. We show that the problem is true for $w+1$; that is, we show that $Spec(L(G)) = Spec(L(K_{w+1} \bigtriangledown GQ(2,1))) = \left\{[w+10]^{w+1},\ [w+4]^4,\ [w+7]^{21},\ [0]^1\right\}$ follows that $G \cong K_w \bigtriangledown GQ(2,1)$, where G is a graph. Theorem 2.4 implies that G_1 and G are the join of two graph. On the other hand, $Spec(L(K_1 \bigtriangledown G_1)) = Spec(L(G)) = spec(L(K_{w+1} \bigtriangledown GQ(2,1)))$ and also G has one vertex and $w+9$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Because, G is the join of two graph and also according to the spectrum of G, must K_1 be joined to G_1 and this is only possibility. □

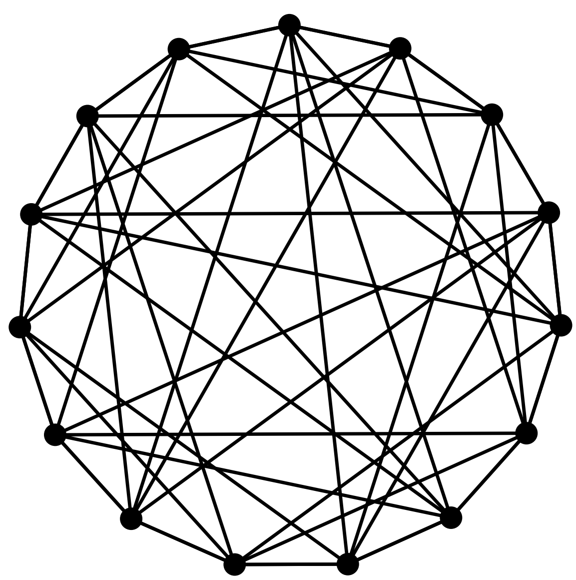

FIGURE 1. Generalized quadrangle $GQ(2,2)$

Hereafter, we characterize another new classes of multicone graph that are DS with respect to their spectra. Our arguments are the similar of the above subsection. So, we will avoid bringing description before each subsection.

4. Connected graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$

Proposition 4.1. *Let G be a graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$. Then*

$$Spec(G) = \left\{ [-1]^{w-1}, [-3]^5, [1]^9, \left[\frac{\vartheta + \sqrt{\vartheta^2 + 4\Upsilon}}{2}\right]^1, \left[\frac{\vartheta - \sqrt{\vartheta^2 + 4\Upsilon}}{2}\right]^1 \right\}, \textit{ where}$$

$\vartheta = 5 + w$ *and* $\Upsilon = 9w + 6$.

Proof. It is well-known that $Spec(GQ(2,2)) = \left\{[-3]^5, [1]^9, [6]^1\right\}$. Now, by Theorem 2.1 the proof is clear. □

In the following lemma, we show that any graph cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$ must be bidegreed.

Lemma 4.1. *Let G be cospectral with multicone graph $K_w \bigtriangledown GQ(2,2)$. Then G is bidegreed in which any vertex of G is of degree $w + 6$ or $w + 14$.*

Proof. It is obvious that G cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph G consists of at least three number. Hence the equality in Theorem 2.2 cannot happen for any δ. But, if we put $\delta = w+6$, then the equality in Theorem 2.2 holds. So, G must be bidegreed. Now, we show that $\Delta = \Delta(G) = w + 14$. By contrary, we suppose that $\Delta < w + 14$. Therefore, the equality in Theorem 2.2 cannot hold for any δ. But, if we put $\delta = w + 6$, then this equality holds. This is a contradiction and so $\Delta = w + 14$. Now, $\delta = w + 6$, since G is bidegreed and G has $w+15$ vertices, $\Delta = w+14$ and $w(w+14)+15(w+6) = w\Delta+15(w+6) = \sum_{i=1}^{w+15} \deg v_i$. Therefore, the assertion holds. □

4.1. **Connected graph cospectral with multicone graph $K_1 \bigtriangledown GQ(2,2)$.**

Lemma 4.2. *Any graph cospectral with a multicone graph $K_1 \bigtriangledown GQ(2,2)$ is isomorphic to $K_1 \bigtriangledown GQ(2,2)$.*

Proof. Let G be cospectral with multicone graph $K_1 \bigtriangledown GQ(2,2)$. By Lemma 4.1, it is easy to see that G has one vertex of degree 15, say j. Now, Proposition 2.2 implies that $P_{G-j}(y) = (y - \mu_3)^4(y - \mu_4)^8[\alpha_{1j}^2 N_1 + \alpha_{2j}^2 N_2 + \alpha_{3j}^2 N_3 + \alpha_{4j}^2 N_4]$, where

$\mu_1 = \dfrac{6 + \sqrt{96}}{2}$, $\mu_2 = \dfrac{6 - \sqrt{96}}{2}$, $\mu_3 = -3$ and $\mu_4 = 1$.
$N_1 = (y - \mu_2)(y - \mu_3)(y - \mu_4)$,
$N_2 = (y - \mu_1)(y - \mu_3)(y - \mu_4)$,
$N_3 = (y - \mu_1)(y - \mu_2)(y - \mu_4)$,
$N_4 = (y - \mu_1)(y - \mu_2)(y - \mu_3)$.

Now, we have:

$\eta + \xi + 6 = -(3\mu_3 + 3\mu_4)$,
$\eta^2 + \xi^2 + 36 = 90 - (3\mu_3^2 + 3\mu_4^2)$,

where η and ξ are the eigenvalues of graph $G - j$. If we solve above equation, then $\eta = 1$ and $\xi = -3$. Hence $Spec(G - j) = Spec(GQ(2,2))$ and so $G - j \cong GQ(2,2)$. Therefore, the assertion holds. □

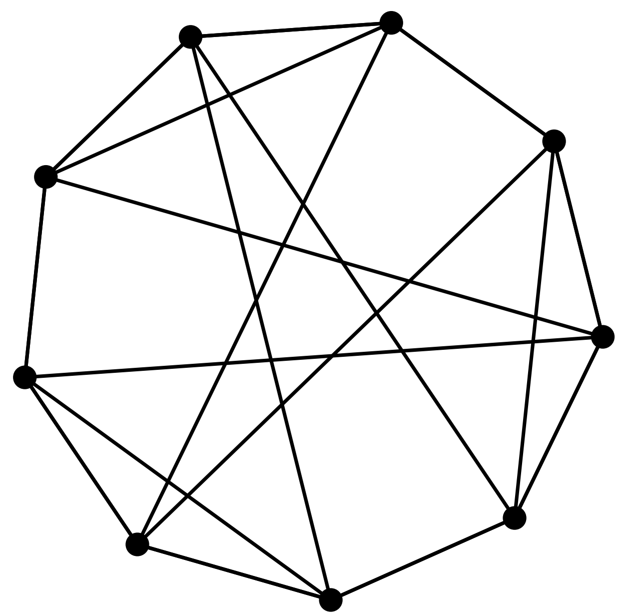

FIGURE 2. Generalized quadrangle $GQ(2,1)$

4.2. **Connected graph cospectral with a multicone graph $K_w \bigtriangledown GQ(2,2)$.**

Theorem 4.1. *Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their adjacency spectra.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown GQ(2,2))$, then $G_1 \cong K_w \bigtriangledown GQ(2,2)$, where G_1 is a graph. We show that the claim is true for $w+1$; that is, if $Spec(G) = Spec(K_{w+1} \bigtriangledown GQ(2,2))$, then $G \cong K_{w+1} \bigtriangledown GQ(2,2)$, where G is a graph. By Lemma 4.2, G has one vertex, 15 edges and 280 triangle more than G_1. Hence $G \cong K_1 \bigtriangledown G_1$.
This follows the result. □

4.3. **Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their Laplacian spectrum.**

Theorem 4.2. *Multicone graph $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their Laplacian spectrums.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, $Spec(L(G_1)) = Spec(L(K_w \bigtriangledown GQ(2,2))) = \Big\{[w+15]^w,\ [w+5]^9,\ [w+9]^5,\ [0]^1\Big\}$
follows that $G_1 \cong K_w \bigtriangledown GQ(2,2)$. We show that the claim is true for $w+1$; that is, we show that $Spec(L(G)) = Spec(L(K_{w+1} \bigtriangledown GQ(2,2))) = \Big\{[w+16]^{w+1},\ [w+6]^9,\ [w+10]^5,\ [0]^1\Big\}$ follows that $G \cong K_{w+1} \bigtriangledown GQ(2,2)$, where G is a graph. Theorem 2.4 implies that

G_1 and G are the join of two graph. On the other hand, $Spec(L(K_1 \bigtriangledown G_1)) = Spec(L(G)) = spec(L(K_{w+1} \bigtriangledown GQ(2,2)))$ and also G has one vertex and $w+15$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Because, G is the join of two graph and also according to spectrum of G, must K_1 be joined to G_1 and this is only available state. □

5. Conclusion remarks and open problems

In this paper, we have shown multicone graph $K_w \bigtriangledown GQ(2,1)$ and $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their adjacency spectra as well as their Laplacian spectra. Now, in the following, we pose these conjectures.

Conjecture 1. *Graphs $\overline{K_w \bigtriangledown GQ(2,1)}$ are DS with respect to their adjacency spectra.*

Conjecture 2. *Multicone graphs $K_w \bigtriangledown GQ(2,1)$ are DS with respect to their signless Laplacian spectra.*

Conjecture 3. *Graphs $\overline{K_w \bigtriangledown GQ(2,2)}$ are DS with respect to their adjacency spectra.*

Conjecture 4. *Multicone graphs $K_w \bigtriangledown GQ(2,2)$ are DS with respect to their signless Laplacian spectra.*

References

[1] Abdian A.Z. and Mirafzal S.M., On new classes of multicone graphs determined by their spectrums, Alg. Struc. Appl, 2 (2015), no. 1, 21-32.
[2] Abdollahi A., Janbaz S. and Oubodi M., Graphs cospectral with a friendship graph or its complement, Trans. Comb., 2 (2013), no. 4, 37–52.
[3] Biggs N. L., Algebraic Graph Theory, Cambridge university press, 1993.
[4] Cvetković D., Rowlinson P. and Simić S. , An introduction to the theory of graph spectra, London Mathematical Society Student Texts, 75, Cambridge University Press, 2010.
[5] Das. K. C., Proof of conjectures on adjacency eigenvalues of graphs, Disceret Math, 313 (2013) , no. 1, 19–25.
[6] Godsil C. D. and Royle G., Algebraic graph theory, Graduate Texts in Mathematics 207, 2001.
[7] Knauer U., Algebraic graph theory: morphisms, monoids and matrices, 41, Walter de Gruyter, 2011.
[8] Mohammadian A., Tayfeh-Rezaie B., Graphs with four distinct Laplacian eigenvalues, J. Algebraic Combin., 34 (2011), no. 4, 671–682.
[9] Omidi G. R., On graphs with largest Laplacian eignnvalues at most 4, Australas. J. Combin., 44 (2009) 163–170.
[10] Rowlinson P., The main eigenvalues of a graph: a survey, Appl. Anal. Discrete Math., 1 (2007) 445-471.
[11] van Dam E. R. and Haemers W. H., Which graphs are determined by their spectrum?, Linear Algebra. Appl., 373 (2003) 241–272.
[12] van Dam E. R., Haemers W. H., Developments on spectral characterizations of graphs, Discrete Math., 309 (2009), no.3, 576–586.
[13] Wang J., Zhao H., and Huang Q., Spectral charactrization of multicone graphs, Czech. Math. J., 62 (2012), no.1, 117–126.
[14] Wang J., Belardo F., Huang Q., and Borovićanin B., On the two largest Q-eigenvalues of graphs, Discrete Math., 310 (2010), no. 1, 2858–2866.
[15] West D. B., Introduction to graph theory, Upper Saddle River: Prentice hall; 2001.

Department of mathematics, College of science, Lorestan university, Lorestan, Khorramabad, Iran.
E-mail address: `aabdian67@gmail.com`