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ON THE UNIQUENESS OF PRODUCT OF DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we study the uniqueness of product of difference polynomials $f^n[\prod_{j=1}^{d} f(z+c_j)^{s_j}]^{(k)}$ and $g^n[\prod_{j=1}^{d} g(z+c_j)^{s_j}]^{(k)}$, which are sharing a fixed point z and f, g share ∞ IM. The result extends the previous results of Cao and Zhang[1] into product of difference polynomials.

1. Introduction, Definitions and Results

Let $\mathbb{C}$ denote the complex plane and f be a non-constant meromorphic function in $\mathbb{C}$. We shall use the standard notations in the Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f), N(r,f), \overline{N}(r,f)$ and $m(r,f)$, as explained in Yang and Yi[14], L.Yang[12] and Hayman[8]. The notation $S(r,f)$ is defined to be any quantity satisfying $S(r,f) = o(T(r,f))$, as $r \to \infty$ possibly outside a set r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$, provided that $T(r,a) = S(r,f)$. A point $z_0 \in \mathbb{C}$ is called as a fixed point of $f(z)$ if $f(z_0) = z_0$.

The following definitions are useful in proving the results.

Definition 1.1. We denote $\rho(f)$ for order of $f(z)$.

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

And $\rho_2(f)$ is to denote hyper order of $f(z)$, defined by

$$\rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

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Definition 1.2. Let a be a finite complex number and k be a positive integer. We denote by $N_{k)}(r, 1/(f-a))$ the counting function for the zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\leq k$ and by $\overline{N}_{k)}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Let $N_{(k}(r, 1/(f-a))$ be the counting function for the zeros of $f(z)-a$ in $|z| \leq r$ with multiplicity $\geq k$ and by $\overline{N}_{(k}(r, 1/(f-a))$ the corresponding one for which multiplicity is not counted. Then we have

$$N_k(r, 1/(f-a)) = \overline{N}_{(1}(r, 1/(f-a)) + \overline{N}_{(2}(r, 1/(f-a)) + \ldots + \overline{N}_{(k}(r, 1/(f-a))$$

Definition 1.3. Let $f(z)$ and $g(z)$ be two meromorphic functions in the complex plane $\mathbb{C}$. If $f(z)-a$ and $g(z)-a$ assume the same zeros with the same multiplicities, then we say that $f(z)$ and $g(z)$ share the value $'a'$ CM, where $'a'$ is a complex number.

In 2010, J.F.Xu, F.Lu and H.X.Yi obtained the following result on meromorphic function sharing a fixed point.

Theorem A. *([11]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let n, k be two positive integers with $n > 3k+10$. If $(f^n(z))^{(k)}$ and $(g^n(z))^{(k)}$ share z CM, f and g share ∞ IM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4n^2(c_1c_2)^n c^2 = -1$, or $f \equiv tg$ for a constant t such that $t^n = 1$.*

Further, Fang and Qiu investigated uniqueness for the same functions as in the theorem A, when $k = 1$.

Theorem B. *([7]) Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$, or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.*

In 2012, Cao and Zhang replaced f' with $f^{(k)}$ and obtained the following theorem.

Theorem C. *([1]) Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros are of multiplicities atleast k, where k is a positive integer. Let $n > max\{2k-1, 4+4/k+4\}$ be a positive integer. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.*
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$
(2) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

Recently, X.B.Zhang reduced the lower bond of n and relax the condition on multiplicity of zeros in theorem C and proved the below result.

Theorem D. *([15]) Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and n, k two positive integers with $n > k+6$. If $f^n(z)f^{(k)}(z)$ and $g^n(z)g^{(k)}(z)$ share z CM, and f and g share ∞ IM, then one of the following two conclusions holds.*
(1) $f^n(z)f^{(k)}(z) = g^n(z)g^{(k)}(z)$;
(2) $f(z) = c_1 e^{cz^2}$, $g(z) = c_2 e^{-cz^2}$, where c_1, c_2 and c are constants such that $4(c_1c_2)^{n+1}c^2 = -1$.

We define a difference product of meromorphic function $f(z)$ as follows.

$$F(z) = f(z)^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j} \right]^{(k)} \tag{1.1}$$

$$F_1(z) = f(z)^n \prod_{j=1}^{d} f(z+c_j)^{s_j} \tag{1.2}$$

Where $c_j \in \mathbb{C} \setminus \{0\}(j = 1, 2, \ldots, d)$ are distinct constants. $n, k, d, s_j(j = 1, 2, \ldots, d)$ are positive integers and $\lambda = \sum_{j=1}^{d} s_j$.
For $j = 1, 2, 3 \ldots d$, $\lambda_1 = \sum_{j=1}^{d} \alpha_j s_j$ and $\lambda_2 = \sum_{j=1}^{d} \beta_j s_j$, where $f(z + c_j)$ and $g(z + c_j)$ have zeros with maximum orders α_j and β_j respectively.

In this article, we prove the theorem on product of difference polynomials sharing a fixed point as follows.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions of hyper order $\rho_2(f) < 1$ and $\rho_2(g) < 1$. Let k, n, d, λ be positive integers and $n > \max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. If $F(z)$ and $G(z)$ share z CM and f, g share ∞ IM, then one of the following two conclusions holds.*

(1) $F(z) = G(z)$
(2) $\prod_{j=1}^{d} f(z+c_j)s_j = C_1 e^{Cz^2}$, $\prod_{j=1}^{d} g(z+c_j)s_j = C_2 e^{-Cz^2}$, *where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.*

2. Lemmas

We need following Lemmas to prove our results.

Lemma 2.1. *([13]) Let f and g be two non-constant meromorphic functions, $'a'$ be a finite non-zero constant. If f and g share $'a'$ CM and ∞ IM, then one of the following cases holds.*

(1) $T(r, f) \leq N_2\left(r, \frac{1}{f}\right) + N_2\left(r, \frac{1}{g}\right) + 3\overline{N}(r, f) + S(r, f) + S(r, g)$.
The same inequality holding for $T(r, g)$;
(2) $fg \equiv a^2$;
(3) $f \equiv g$.

Lemma 2.2. *([10]) Let $f(z)$ be a transcendental meromorphic functions of hyper order $\rho_2(f) < 1$, and let c be a non-zero complex constant. Then we have*

$$T(r, f(z+c)) = T(r, f(z)) + S(r, f(z)),$$

$$N(r, f(z+c)) = N(r, f(z)) + S(r, f(z)),$$

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f(z)).$$

Lemma 2.3. *([14]) Let f be a non-constant meromorphic function, let $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4. *([14]) Let f be a non-constant meromorphic function and p, k be positive integers. Then*

$$T\left(r, f^{(k)}\right) \leq T(r,f) + k\overline{N}\left(r,f\right) + S(r,f), \tag{1}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r,f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r,f), \tag{2}$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \tag{3}$$

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f). \tag{4}$$

Lemma 2.5. *([8]) Suppose that f is a non-constant meromorphic function, $k \geq 2$ is an integer. If*

$$N(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}}\right) = S\left(r, \frac{f'}{f}\right),$$

then $f(z) = e^{az+b}$, where $a \neq 0, b$ are constants.

Lemma 2.6. *([14]) Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f'}{f}\right) = S(r,f)$$

Lemma 2.7. *Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$ and $F_1(z)$ be stated as in (1.2). Then*

$$(n-\lambda)T(r,f) + S(r,f) \leq T(r, F_1(z)) \leq (n+\lambda)T(r,f) + S(r,f)$$

Proof: Since f is a meromorphic function with $\rho_2(f) < 1$. From Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} T(r, F_1(z)) &\leq T(r, f(z)^n) + T\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + S(r,f) \\ &\leq (n+\lambda)T(r,f) + S(r,f) \end{aligned}$$

On the other hand, from Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} (n+\lambda)T(r,f) &= T(r, f^n f^\lambda) + S(r,f) \\ &= m(r, f^n f^\lambda) + N(r, f^n f^\lambda) + S(r,f) \\ &\leq m\left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r, \frac{F_1(z) f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) \\ &\quad + S(r,f) \\ &\leq m(r, F_1(z)) + N(r, F_1(z)) + T\left(r, \frac{f^\lambda}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) \\ &\quad + S(r,f) \\ &\leq T(r, F_1(z)) + 2\lambda T(r,f) + S(r,f) \\ (n-\lambda)T(r,f) &\leq T(r, F_1(z)) + S(r,f) \\ \Rightarrow (n-\lambda)T(r,f) + S(r,f) &\leq T(r, F_1(z)) \end{aligned}$$

Hence we get Lemma 2.7.

3. Proof of theorem

Proof of the theorem 1.1

$$Let, \quad F^* = \frac{F}{z} \qquad and \qquad G^* = \frac{G}{z} \tag{3.1}$$

From the hypothesis of the theorem 1.1, we have F and G share z CM and f, g share ∞ IM. It follows that F^* and G^* share 1 CM and ∞ IM.

By Lemma 2.1, we arrive at 3 cases as follows.

Case 1. Suppose that case (1) of Lemma 2.1 holds.

$$T(r, F^*) \leq N_2\left(r, \frac{1}{F^*}\right) + N_2\left(r, \frac{1}{G^*}\right) + 3\overline{N}(r, F^*) + S(r, F^*) + S(r, G^*) \tag{3.2}$$

We deduce from (3.2) and obtained the following

$$T(r, F) \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + S(r, F) + S(r, G) \tag{3.3}$$

From Lemma 2.2 and Lemma 2.7, we have $S(r, F) = S(r, f)$ and $S(r, G) = S(r, g)$. From (3.3), we have

$$\begin{aligned} T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + 3\overline{N}(r, F) + S(r, f) + S(r, g) \\ &\leq N_2\left(r, \frac{1}{f^n}\right) + N_2\left(r, \frac{1}{\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}}\right) + N_2\left(r, \frac{1}{g^n}\right) \\ &\quad + N_2\left(r, \frac{1}{\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)}}\right) + 3\overline{N}(r, f^n) + 3\overline{N}\left(r, \left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\right) \\ &\quad + S(r, f) + S(r, g) \end{aligned} \tag{3.4}$$

Using (2) of Lemma 2.4 in (3.4), we have

$$\begin{aligned} T(r,F) &\leq 2\overline{N}_{(2}\left(r,\frac{1}{f^n}\right)+T\left(r,\left(\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}\right)-T\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)\\ &+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^{d}f(z+c_j)^{s_j}}\right)+2\overline{N}_{(2}\left(r,\frac{1}{g^n}\right)+T\left(r,\left(\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)^{(k)}\right)\\ &-T\left(r,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)+N_{k+2}\left(r,\frac{1}{\prod_{j=1}^{d}g(z+c_j)^{s_j}}\right)+3N(r,f)\\ &+3N\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+S(r,f)+S(r,g)\end{aligned}$$

$$\begin{aligned} T(r,F) &\leq 2T(r,f)+T\left(r,\left(\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}\right)+T(r,f^n)-T(r,f^n)\\ &-T\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+(k+2)\ d\ T(r,f)+2T(r,g)\\ &+T\left(r,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)+k\overline{N}\left(r,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)\\ &-T\left(r,\prod_{j=1}^{d}g(z+c_j)^{s_j}\right)+(k+2)\ d\ T(r,g)\\ &+3T(r,f)+3\lambda T(r,f)+S(r,f)+S(r,g)\\ T(r,F) &\leq 2T(r,f)+T(r,F)-T(r,F_1)+(k+2)\ d\ T(r,f)+2T(r,g)+k\lambda T(r,g)\\ &+(k+2)\ d\ T(r,g)+(3+3\lambda)T(r,f)+S(r,f)+S(r,g)\\ T(r,F_1) &\leq 2[T(r,f)+T(r,g)]+(k+2)\ d\ [T(r,f)+T(r,g)]+k\lambda T(r,g)\\ &+(3+3\lambda)T(r,f)+S(r,f)+S(r,g)\end{aligned}$$

From Lemma 2.7, we have

$$(n-\lambda)T(r,f)\leq((k+2)d+2)[T(r,f)+T(r,g)]+k\lambda T(r,g)+(3+3\lambda)T(r,f)+S(r,f)$$

$$+S(r,g) \tag{3.5}$$

Similarly for $T(r,g)$, we obtain the following

$$(n-\lambda)T(r,g)\leq(2+(k+2)d)[T(r,f)+T(r,g)]+k\lambda T(r,f)+(3+3\lambda)T(r,g)+S(r,f)$$

$$+S(r,g) \tag{3.6}$$

From (3.5) and (3.6), we have

$$(n-\lambda)[T(r,f)+T(r,g)] \leq 2(2+(k+2)d))[T(r,f)+T(r,g)]+(k\lambda+3+3\lambda)[T(r,f)+T(r,g)] +S(r,f)+S(r,g)$$

Which is contradiction to $n > 2d(k+2)+\lambda(k+3)+7$.

Case 2. Suppose that $FG \equiv z^2$ holds.

$$\text{i.e} \quad f^n\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} g^n\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)} \equiv z^2 \tag{3.7}$$

Now, (3.7) can be written as

$$f^n g^n = \frac{z^2}{\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}$$

By using Lemma 2.2, Lemma 2.3 and (4) of Lemma 2.4, we derive

$$n\left[N(r,f)+N(r,g)\right] \leq \lambda\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right] +kd[N(r,f)+N(r,g)]+S(r,f)+S(r,g) \tag{3.8}$$

From (3.7), we can write

$$\frac{1}{f^n g^n} = \frac{\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}}{z^2}$$

Similarly, as (3.8), we obtain

$$n\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right] \leq (\lambda+kd)\left[N(r,f)+N(r,g)\right]+S(r,f)+S(r,g) \tag{3.9}$$

From (3.8) and (3.9), deduce

$$(n-(\lambda+2kd))[N(r,f)+N(r,g)]+(n-\lambda)\left[N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right] \leq S(r,f)+S(r,g)$$

Since $n > 2d(k+2)+\lambda(k+3)+7$, we have

$$N(r,f)+N(r,g)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right) < S(r,f)+S(r,g)$$

Hence, we conclude that f and g have finitely many zeros and poles.

Let z_0 be a pole of f of multiplicity p, then z_0 is pole of f^n of multiplicity np, since f and g share ∞ IM, then z_0 is pole of g of multiplicity q.

If z_0 also zero of $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ then we have from (3.7) that

$$n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k$$

$$\Rightarrow 2n < n(p+q) \leq \sum_{j=1}^{d} \alpha_j s_j + \sum_{j=1}^{d} \beta_j s_j - 2k = \lambda_1 + \lambda_2 - 2k < \lambda_1 + \lambda_2 \leq 2\max\{\lambda_1, \lambda_2\}$$

$\Rightarrow n < \max\{\lambda_1, \lambda_2\}$, which is contradiction to $n > \max\{2d(k+2) + \lambda(k+3) + 7, \lambda_1, \lambda_2\}$. Therefore f has no poles.

Similarly, we can get contradiction for other two cases namely, if z_0 is zero of $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$, but not zero of $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ and other way. Therefore f has no poles. Similarly, we get that g also has no poles. By this we conclude that f and g are entire functions and hence $\left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)}$ and $\left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$ are entire functions.

Then from (3.7), we deduce that f and g have no zeros.
Therefore,

$$f = e^{\alpha(z)},\ g = e^{\beta(z)} \qquad \text{and}$$

$$\prod_{j=1}^{d} f(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j} \quad , \quad \prod_{j=1}^{d} g(z+c_j)^{s_j} = \prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j} \tag{3.10}$$

where α, β are entire functions with $\rho_2(f) < 1$. Substitute f and g into (3.7), we get

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j}\right]^{(k)} e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j}\right]^{(k)} \equiv z^2 \tag{3.11}$$

If $k = 1$, then

$$e^{n\alpha(z)} \left[\prod_{j=1}^{d} (e^{\alpha(z+c_j)})^{s_j}\right]' e^{n\beta(z)} \left[\prod_{j=1}^{d} (e^{\beta(z+c_j)})^{s_j}\right]' \equiv z^2 \tag{3.12}$$

$$\Rightarrow e^{n(\alpha+\beta)} e^{\sum_{j=1}^{d} (\alpha(z+c_j) + \beta(z+c_j)) s_j} \sum_{j=1}^{d} (\alpha'(z+c_j)) s_j \sum_{j=1}^{d} (\beta'(z+c_j)) s_j \equiv z^2 \tag{3.13}$$

Since $\alpha(z)$ and $\beta(z)$ are non-constant entire functions, then we have

$$T\left(r, \frac{\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)'}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) = T\left(r, \frac{\left(\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}\right)'}{\prod_{j=1}^{d} e^{\alpha(z+c_j)s_j}}\right)$$

(3.14)
$$= T\left(r, \frac{\sum_{j=1}^{d}\alpha'(z+c_j)s_j\prod_{j=1}^{d}e^{\alpha(z+c_j)s_j}}{\prod_{j=1}^{d}e^{\alpha(z+c_j)s_j}}\right) = T\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)$$

$$\begin{aligned}\text{Let}\quad nT(r,f)=T(r,f^n) &= T\left(r, \frac{F}{(\prod_{j=1}^{d}f(z+c_j)^{s_j})^{(k)}}\right)\\ &\leq T(r,F)+T\left(r,\left(\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}\right)+S(r,f)\\ &\leq T(r,F)+T\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+k\overline{N}\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)\\ &\quad +S(r,f)\\ nT(r,f) &\leq T(r,F)+(\lambda+kd)T(r,f)+S(r,f)\end{aligned}$$

(3.15) $(n-\lambda-kd)T(r,f)\leq T(r,F)+S(r,f)$

We obtain from (3.15) that

$$T(r,f)=O(T(r,F)) \tag{3.16}$$

as $r\in E$ and $r\to\infty$, where $E\subset(0,+\infty)$ is some subset of finite linear measure.

On the other hand, we have

$$\begin{aligned}T(r,F)=T\left(r,f^n\left[\prod_{j=1}^{d}f(z+c_j)^{s_j}\right]^{(k)}\right) &\leq nT(r,f)+\lambda T(r,f)\\ &\quad +k\overline{N}\left(r,\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)+S(r,f)\\ &\leq (n+kd+\lambda)T(r,f)+S(r,f)\end{aligned}$$

$$\Rightarrow\quad T(r,F)=O(T(r,f)) \tag{3.17}$$

as $r\in E$ and $r\to\infty$, where $E\subset(0,+\infty)$ is some subset of finite linear measure.

Thus from (3.16), (3.17) and the standard reasoning of removing exceptional set we deduce $\rho(f)=\rho(F)$. Similarly, we have $\rho(g)=\rho(G)$. It follows from (3.7) that $\rho(F)=\rho(G)$. Hence we get $\rho(f)=\rho(g)$.

We deduce that either both α and β are polynomials or both α and β are transcendental entire functions. Moreover, we have

$$N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) \leq N\left(r, \frac{1}{z^2}\right) = O(\log r) \tag{3.18}$$

From (3.18) and (3.10), we have

$$N\left(r, \prod_{j=1}^{d} f(z+c_j)^{s_j}\right) + N\left(r, \frac{1}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right)$$
$$+N\left(r, \frac{1}{(\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)}}\right) = O(\log r)$$

If $k \geq 2$, then it follows from (3.14),(3.18) and Lemma 2.5 that $\sum_{j=1}^{d} \alpha'(z+c_j)s_j$ is a polynomial and therefore we have $\alpha(z)$ is a non- constant polynomial.

Similarly, we can deduce that $\beta(z)$ is also a non-constant polynomial. From this, we deduce from (3.10) that

$$\left(\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \alpha(z+c_j)s_j}\left[P_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^k\right]$$

$$\left(\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)} = e^{\sum_{j=1}^{d} \beta(z+c_j)s_j}\left[Q_{k-1}(\alpha'(z+c_j)) + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^k\right]$$

Where P_{k-1} and Q_{k-1} are difference-differential polynomials in $\alpha'(z+c_j)$ with degree at most $k-1$.

Then (3.11) becomes

$$e^{n(\alpha+\beta)}e^{\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left[\sum_{j=1}^{d} \alpha^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \alpha'(z+c_j)s_j\right)^k\right]$$

$$\left[\sum_{j=1}^{d} \beta^{(k)}(z+c_j)s_j + \left(\sum_{j=1}^{d} \beta'(z+c_j)s_j\right)^k\right] = z^2 \tag{3.19}$$

We deduce from (3.19) that $\alpha(z)+\beta(z) \equiv C$ for a constant C.
If $k=1$, from (3.13), we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left[\sum_{j=1}^{d}(\alpha'(z+c_j))s_j \sum_{j=1}^{d}(\beta'(z+c_j))s_j\right] \equiv z^2 \tag{3.20}$$

Next, we let $\alpha+\beta=\gamma$ and suppose that α, β both are transcendental entire functions.

If γ is a constant, then $\alpha'+\beta'=0$ and $\sum_{j=1}^{d} \alpha'(z+c_j) = -\sum_{j=1}^{d} \beta'(z+c_j)$.

From (3.20) we have

$$e^{n(\alpha+\beta)+\sum_{j=1}^{d}(\alpha(z+c_j)+\beta(z+c_j))s_j}\left\{-\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]^2\right\}=z^2$$

$$e^{n\gamma+d\gamma}\left\{-\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]^2\right\}=z^2 \tag{3.21}$$

Which implies that α' is a non-constant polynomial of degree 1. This together with $\alpha'+\beta'=0$ which implies that β' is also non-constant polynomial of degree 1. Which is contradiction to α, β both are transcendental entire functions.

If γ is not a constant, then we have

$$\alpha+\beta=\gamma \qquad \text{and} \qquad \sum_{j=1}^{d}\alpha(z+c_j)s_j+\sum_{j=1}^{d}\beta(z+c_j)s_j=\sum_{j=1}^{d}\gamma(z+c_j)s_j$$

From (3.20) we have

$$\left[\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}=z^2 \tag{3.22}$$

Since $T\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)=m\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)+N\left(r,\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)$

$$\leq m\left(r,\frac{(e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j})'}{e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j}}\right)+O(1)=S\left(r,e^{\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) \tag{3.23}$$

And also we have

$$T\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)=m\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)+N\left(r,n\gamma'+\sum_{j=1}^{d}\gamma'(z+c_j)s_j\right)$$

$$\leq m\left(r,\frac{(e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j})'}{e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}}\right)+O(1)=S\left(r,e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) \tag{3.24}$$

From (3.22), we have

$$T\left(r,e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right)\leq T\left(r,\frac{z^2}{\sum_{j=1}^{d}\alpha'(z+c_j)s_j\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]}\right)$$

$$+O(1)$$

$$\leq T(r,z^2)+T\left(r,\sum_{j=1}^{d}\alpha'(z+c_j)s_j\left[\sum_{j=1}^{d}\gamma'(z+c_j)s_j-\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right]\right)$$

$$+O(1)$$

$$\leq 2\log r + 2T\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right) + O(1)$$

$$\Rightarrow T\left(r, e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) \leq O\left(T\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)\right) \tag{3.25}$$

Similarly, we have

$$T\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right) \leq O\left(T\left(r, e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right)\right) \tag{3.26}$$

Thus, from (3.23)-(3.26) we have
$T\left(r, n\gamma' + \sum_{j=1}^{d}\gamma'(z+c_j)s_j\right) = S\left(r, e^{n\gamma+\sum_{j=1}^{d}\gamma(z+c_j)s_j}\right) = S\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)$

By the second fundamental theorem and (3.22), we have

$$T\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right) \leq \overline{N}\left(r, \frac{1}{\sum_{j=1}^{d}\alpha'(z+c_j)s_j}\right)$$

$$+\overline{N}\left(r, \frac{1}{\sum_{j=1}^{d}\alpha'(z+c_j)s_j - \sum_{j=1}^{d}\gamma'(z+c_j)s_j}\right) + S\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)$$

$$\leq O(\log r) + S\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)$$

This implies $\sum_{j=1}^{d}\alpha'(z+c_j)s_j$ is a polynomial, which leads to $\alpha'(z)$ is a polynomial. Which contradicts that $\alpha(z)$ is a trascendental entire function.
Thus α and β are both polynomials and $\alpha(z)+\beta(z) \equiv C$ for a constant C.
Hence, from (3.19) and using $\alpha+\beta = C$ we get

$$(-1)^k\left(\sum_{j=1}^{d}\alpha'(z+c_j)s_j\right)^{2k} = z^2 + P_{2k-1}(\alpha'(z+c_j)s_j) \quad for \;\; j = 1, 2, \ldots, d. \tag{3.27}$$

Where P_{2k-1} is difference-differential polynomial in $\alpha'(z+c_j)s_j$ of degree at most $2k-1$. From (3.27), we have

$$2kT\left(r, \sum_{j=1}^{d}\alpha'(z+c_j)s_j\right) = 2\log r + S(r, \alpha'(z+c_j)s_j) \tag{3.28}$$

From (3.28), we can see that $\sum_{j=1}^{d}\alpha'(z+c_j)s_j$ is a non-constant polynomial of degree 1 and $k=1$.

Which implies,

$$\sum_{j=1}^{d} \alpha'(z+c_j)s_j = zl_1$$

Since $\alpha' + \beta' = 0$, we get $\sum_{j=1}^{d} \beta'(z+c_j)s_j = -\sum_{j=1}^{d} \alpha'(z+c_j)s_j$. Which implies $\sum_{j=1}^{d} \beta'(z+c_j)s_j$ is also a non-constant polynomial of degree 1. Hence we have

$$\sum_{j=1}^{d} \beta'(z+c_j)s_j = zl_2$$

Hence, we get

$$\prod_{j=1}^{d} f(z+c_j)s_j = C_1 e^{Cz^2}$$

Similarly, we have

$$\prod_{j=1}^{d} g(z+c_j)s_j = C_2 e^{-Cz^2}$$

where C_1, C_2 and C are constants such that $4(C_1C_2)^{n+1}C^2 = -1$.

This proves the conclusion (2) of theorem 1.1.

Case 3. If $F \equiv G$

i.e $f^n \left[\prod_{j=1}^{d} f(z+c_j)^{s_j}\right]^{(k)} \equiv g^n \left[\prod_{j=1}^{d} g(z+c_j)^{s_j}\right]^{(k)}$

This proves the conclusion (1) of theorem 1.1.

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