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Formulas and Finite Sums Covering Beta-type Rational Functions and Euler-Frobenius-type Polynomials

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Abstract

The aim of this article is to derive some novel formulas and finite sums covering Stirling type numbers, the Frobenius Euler numbers and polynomials, the beta-type rational functions, and combinatorial numbers with the help of both generating functions and their functional equations, and also some special identities associated with special numbers and polynomials.

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1. INTRODUCTION

Polynomial classes are one of the indispensable fields in both mathematics and all applied sciences, especially engineering. For this reason, thousands of important studies on polynomial classes have been encountered in recent years. The motivation and subject of this article focus on special classes of polynomials. In this study, we will examine special polynomials, especially those of the Euler-Frobenius type polynomials. This class has very intensive and important applications in spline theory. Consequently, the Frobenius-Euler polynomials are also known as the Euler-Frobenius polynomials. There are those who call these polynomials Eulerian polynomials. These polynomials are known as a generalization of the well-known Euler polynomials of the first kind. These polynomials belong to the class of the Appell polynomials, which is a subclass of the well-known Sheffer polynomials within the polynomial families. Polynomial classes of this type have valuable applications in many areas of mathematics, mathematical and applied physics, theoretical and applied chemistry, and even in engineering. Moreover, novel formulas, finite sums, and relations containing these polynomial classes are given.

Let ζ be a complex number: $\mathbb{C}$. The higher-order Frobenius-Euler numbers and polynomials are respectively described by,

$$\left(\frac{1-\text{ȝ}}{e^t-\text{ȝ}}\right)^{\zeta}=\sum_{\text{ꞑ}=0}^{\infty}\frac{H_{\text{ꞑ}}^{(\zeta)}(\text{ȝ})}{\text{ꞑ}!}t^{\text{ꞑ}} \tag{1}$$

and

$$\left(\frac{1-\text{ȝ}}{e^t-\text{ȝ}}\right)^{\zeta}=\sum_{\text{ꞑ}=0}^{\infty}\frac{H_{\text{ꞑ}}^{(\zeta)}(\chi;\text{ȝ})}{\text{ꞑ}!}t^{\text{ꞑ}} \tag{2}$$

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(Carlitz & Sholander, 1963; Kucukoglu & Simsek, 2019).

Gun and Simsek (2023) defined the following monic polynomials, which are denoted by $u_{ɳ}^{(\zeta)}(\chi;ʒ)$:

$$\left(\frac{1+ʒ}{ʒe^{t}+1}\right)^{\zeta}=e^{-t\chi}\sum_{ɳ=0}^{\infty}\frac{u_{ɳ}^{(\zeta)}(\chi;ʒ)}{ɳ!}t^{ɳ}. \quad (3)$$

From (2) and (3), one has

$$u_{ɳ}^{(\zeta)}(\chi;ʒ)-H_{ɳ}^{(\zeta)}\left(\chi;-\frac{1}{ʒ}\right)=0 \quad (4)$$

(Gun & Simsek, 2023).

For $ʒ\in\mathbb{C}$ (or real number: $\mathbb{R}$), the second kind $ʒ$ -Stirling numbers are described by

$$(ʒe^{t}-1)^{v}=\sum_{ɳ=0}^{\infty}\frac{v!\,S_{2}(ɳ,v;ʒ)}{ɳ!}t^{ɳ}. \quad (5)$$

When $ʒ$ =1 Eq. (5), yields the second kind Stirling numbers:

$$S_{2}(ɳ,v;1)-S_{2}(ɳ,v)=0 \quad (6)$$

(Simsek, 2013a; 2013b; 2018a; 2018b; 2023; Srivastava, 2011; Srivastava & Choi, 2001; 2012).

The Apostol Euler numbers of order ζ are described by

$$(ʒe^{t}+1)^{-\zeta}=\sum_{ɳ=0}^{\infty}\frac{2^{-\zeta}\mathcal{E}_{ɳ}^{(\zeta)}(ʒ)}{ɳ!}t^{ɳ} \quad (7)$$

(Luo, 2006; Gun & Simsek, 2020; Simsek, 2013a; 2013b; 2018a; 2018b; 2023; Srivastava, 2011; Srivastava & Choi, 2001; 2012).
Replacing ζ by $-\zeta$, Eq. (7) reduces to the Apostol Euler numbers of negative order are described by

$$(ʒe^{t}+1)^{\zeta}=\sum_{ɳ=0}^{\infty}2^{\zeta}\frac{\mathcal{E}_{ɳ}^{(-\zeta)}(ʒ)}{ɳ!}t^{ɳ} \quad (8)$$

and also

$$\mathcal{E}_{ɳ}^{(\zeta)}(ʒ)=\sum_{z=1}^{ɳ}(-1)^{z}S_{2}(ɳ,z)ʒ^{z}\frac{(ʒ+1)^{-z-\zeta}(\zeta)^{(z)}}{2^{-\zeta}}, \quad (9)$$

where $(\zeta)^{(ɳ)}$ denotes

$$(\zeta)^{(ɳ)}=\begin{cases}\zeta(\zeta+1)\dots(\zeta+ɳ-1), & ɳ\in\mathbb{N}\\ 1, & ɳ=0\end{cases} \quad (10)$$

(Luo, 2006; Gun & Simsek, 2020; 2023; Srivastava, 2011; Srivastava & Choi, 2001; 2012).

The description of the beta-type rational functions $\mathfrak{M}_{j,ɳ}(ʒ)$:

$$\left(\frac{\mathfrak{z}}{\mathfrak{z}+1}\right)^{\mathfrak{j}} = e^{t(-\mathfrak{z}-1)} \sum_{\mathfrak{n}=0}^{\infty} \mathfrak{M}_{\mathfrak{j},\mathfrak{n}}(\mathfrak{z}) \frac{1}{\mathfrak{n}!} t^{\mathfrak{n}}. \tag{11}$$

From Eq. (11), we have

$$\mathfrak{M}_{\mathfrak{j},\mathfrak{n}}(\mathfrak{z}) - \mathfrak{z}^{\mathfrak{j}}(1+\mathfrak{z})^{\mathfrak{n}-\mathfrak{j}} = 0, \tag{12}$$

where $\mathfrak{n}$ and $\mathfrak{z} \in \mathbb{R}$, (or $\mathbb{C}$) (Simsek, 2015; 2018c).

With the aid of (4), Gun and Simsek (2023) also defined the following polynomials $q_{\mathfrak{n}}(\mathfrak{z};\zeta)$:

$$q_{\mathfrak{n}}(\mathfrak{z};\zeta) - (1+\mathfrak{z})^{\mathfrak{n}} u_{\mathfrak{n}}^{(\zeta)}(\mathfrak{z}) = 0. \tag{13}$$

Gun and Simsek (2023) also gave the following formula:

$$q_{\mathfrak{n}}(\mathfrak{z};\zeta) = \sum_{\mathfrak{j}=1}^{\mathfrak{n}} (-1)^{\mathfrak{j}} S_2(\mathfrak{n},\mathfrak{j}) \mathfrak{M}_{\mathfrak{j},\mathfrak{n}}(\mathfrak{z}) (\zeta)^{(\mathfrak{n})}. \tag{14}$$

Let $\mathbb{N}_0$ correspond to the set of nonnegative integers. For $v \in \mathbb{N}_0$ and $\mathfrak{z} \in \mathbb{C}$, Simsek (2018c) described the combinatorial numbers, $y_1(\mathfrak{n}, v; \mathfrak{z})$:

$$(\mathfrak{z}e^t + 1)^v = \sum_{\mathfrak{n}=0}^{\infty} \frac{v!\, y_1(\mathfrak{n}, v; \mathfrak{z})}{\mathfrak{n}!} t^{\mathfrak{n}} \tag{15}$$

and also

$$\mathfrak{k}^{\mathfrak{n}} \mathfrak{z}^{\mathfrak{k}} - \sum_{\mathfrak{j}=0}^{\mathfrak{k}} (-1)^{\mathfrak{k}-\mathfrak{j}} \binom{\mathfrak{k}}{\mathfrak{j}} y_1(\mathfrak{n}, \mathfrak{j}; \mathfrak{z}) \mathfrak{j}! = 0. \tag{16}$$

Combining Eq. (5) and Eq. (16), we have

$$S_2(\mathfrak{k}, v; \mathfrak{z}) + (-1)^{1+v} y_1(\mathfrak{k}, v; -\mathfrak{z}) = 0 \tag{17}$$

(Simsek, 2018c).

Therefore, this article is motivated to give novel formulas, finite sums, and relations containing special classes of polynomials. (Simsek, 2018a; 2018b; 2023).

2. FORMULAS AND FINITE SUMS COVERING BETA-TYPE RATIONAL FUNCTIONS AND COMBINATORIAL-TYPE NUMBERS AND POLYNOMIALS

We give some novel formulas and finite sums covering Stirling-type polynomials and numbers, the Frobenius Euler numbers and polynomials, the beta-type rational functions, and combinatorial numbers via not only generating functions, but also identities covering special numbers and polynomials.

Theorem 2.1. Let $\zeta \in \mathbb{C}$ and $\mathfrak{n} \in \mathbb{N}_0$.

$$\sum_{\mathfrak{k}=0}^{\infty} \binom{\zeta}{\mathfrak{k}} \mathfrak{z}^{\mathfrak{k}} \sum_{\mathfrak{m}=0}^{\mathfrak{n}} \binom{\mathfrak{n}}{\mathfrak{m}} u_{\mathfrak{m}}^{(\zeta)}(\chi; \mathfrak{z})\, \mathfrak{k}^{\mathfrak{n}-\mathfrak{m}} - (1+\mathfrak{z})^{\zeta} \chi^{\mathfrak{n}} = 0.$$

Proof. Using the umbral calculus convention in Eq. (3) yields, we have

$$\left(\frac{1+ȥ}{ȥe^{t}+1}\right)^{\zeta}=e^{u^{(\zeta)}(ꭓ;ȥ)t-tꭓ}.$$

Assuming that $|ȥe^{t}|<1$. Using binomial series yields, we have

$$\sum_{ɳ=0}^{\infty}(1+ȥ)^{\zeta}ꭓ^{ɳ}\frac{t^{ɳ}}{ɳ!}=\sum_{ƙ=0}^{\infty}\binom{\zeta}{ƙ}ȥ^{ƙ}\sum_{ɳ=0}^{\infty}\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}u_{ɱ}^{(\zeta)}(ꭓ;ȥ)ƙ^{ɳ-ɱ}\frac{t^{ɳ}}{ɳ!}.$$

Aligning the coefficients $\frac{t^{ɳ}}{ɳ!}$ with each side completes the proof.

Substituting $\zeta=v$ into Theorem 2.1. and combining Eq. (12) with Theorem 2.1, after performing several calculations, we get the following corollary:

Corollary 2.2. Let $\zeta\in\mathbb{C}$ and $ɳ\in\mathbb{N}_0$.

$$ꭓ^{ɳ}-\sum_{ƙ=0}^{\infty}\binom{v}{ƙ}\mathfrak{M}_{ƙ,ƙ-v}(ȥ)\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}u_{ɱ}^{(v)}(ꭓ;ȥ)\,ƙ^{ɳ-ɱ}=0.$$

Combining Theorem 2.1. with Eq. (4) yields, we get the following corollary:

Corollary 2.3. Let $\zeta\in\mathbb{C}$ and $ɳ\in\mathbb{N}_0$.

$$(1+ȥ)^{\zeta}ꭓ^{ɳ}-\sum_{ƙ=0}^{\infty}\binom{\zeta}{ƙ}ȥ^{ƙ}\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}ƙ^{ɳ-ɱ}H_{ɱ}^{(\zeta)}\left(ꭓ;-\frac{1}{ȥ}\right)=0.$$

Joining Eq. (16) with Theorem 2.1. after performing several calculations, we get the following corollary:

Corollary 2.4. Let $\zeta\in\mathbb{C}$ and $ɳ\in\mathbb{N}_0$.

$$(1+ȥ)^{\zeta}ꭓ^{ɳ}+\sum_{ƙ=0}^{\infty}\binom{\zeta}{ƙ}\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}u_{ɱ}^{(\zeta)}(ꭓ;ȥ)\frac{1}{ƙ^{ɱ}}\sum_{ʝ=0}^{ƙ}(-1)^{ƙ+1-ʝ}y_1(ɳ,ʝ;ȥ)\binom{ƙ}{ʝ}ʝ!=0\,.$$

Replacing $ȥ$ by $-ȥ$, Corollary 2.4. reduces to

Corollary 2.5. Let $\zeta,ȥ\in\mathbb{C}$, $ɳ\in\mathbb{N}_0$.

$$(1-ȥ)^{\zeta}ꭓ^{ɳ}+\sum_{ƙ=0}^{\infty}\binom{\zeta}{ƙ}\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}u_{ɱ}^{(\zeta)}(ꭓ;-ȥ)\frac{1}{ƙ^{ɱ}}\sum_{ʝ=0}^{ƙ}(-1)^{1+ƙ}S_2(ɳ,ʝ;ȥ)\binom{ƙ}{ʝ}ʝ!=0.$$

Theorem 2.6. Let $\zeta\in\mathbb{C}$ and $ɳ\in\mathbb{N}_0$.

$$(1+ȥ)^{\zeta}ꭓ^{ɳ}-\sum_{ƙ=0}^{\infty}ȥ^{ƙ}\binom{\zeta}{ƙ}\sum_{v=0}^{ƙ}v!\binom{ƙ}{v}\sum_{ɱ=0}^{ɳ}\binom{ɳ}{ɱ}S_2(ɱ,v)u_{ɳ-ɱ}^{(\zeta)}(ꭓ;ȥ)=0.$$

Proof. From Eq. (3) yields, we have

$$(1+ȥ)^{\zeta}\sum_{ɳ=0}^{\infty}ꭓ^{ɳ}\frac{t^{ɳ}}{ɳ!}=\sum_{ƙ=0}^{\infty}ȥ^{ƙ}\binom{\zeta}{ƙ}e^{tƙ}\sum_{ɳ=0}^{\infty}u_{ɳ}^{(\zeta)}(ꭓ;ȥ)\frac{t^{ɳ}}{ɳ!}.$$

Thus

$$(1+\mathfrak{z})^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=\sum_{\mathfrak{k}=0}^{\infty}\binom{\zeta}{\mathfrak{k}}\mathfrak{z}^{\mathfrak{k}}\sum_{v=0}^{\mathfrak{k}}\binom{\mathfrak{k}}{v}v!\,\frac{(e^t-1)^v}{v!}\sum_{\mathfrak{n}=0}^{\infty}u_{\mathfrak{n}}^{(\zeta)}(\chi;\mathfrak{z})\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}.$$

Combining Eq. (6) with above equation yields, we get

$$(1+\mathfrak{z})^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=\sum_{\mathfrak{k}=0}^{\infty}\mathfrak{z}^{\mathfrak{k}}\binom{\zeta}{\mathfrak{k}}\sum_{v=0}^{\mathfrak{k}}\binom{\mathfrak{k}}{v}v!\sum_{\mathfrak{n}=0}^{\infty}S_2(\mathfrak{n},v)\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}\sum_{\mathfrak{n}=0}^{\infty}u_{\mathfrak{n}}^{(\zeta)}(\chi;\mathfrak{z})\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}.$$

Hence

$$(1+\mathfrak{z})^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=\sum_{\mathfrak{k}=0}^{\infty}\mathfrak{z}^{\mathfrak{k}}\binom{\zeta}{\mathfrak{k}}\sum_{v=0}^{\mathfrak{k}}v!\binom{\mathfrak{k}}{v}\sum_{\mathfrak{n}=0}^{\infty}\sum_{\mathfrak{m}=0}^{\mathfrak{n}}u_{\mathfrak{n}-\mathfrak{m}}^{(\zeta)}\binom{\mathfrak{n}}{\mathfrak{m}}S_2(\mathfrak{m},v)(\chi;\mathfrak{z})\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}.$$

Aligning the coefficients $\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}$ with each side completes the proof.

Replacing $\mathfrak{z}$ by $-\frac{1}{\mathfrak{z}}$ respectively, Theorem 2.6., and also combining Eq. (4) yields, we get the following corollary:

Corollary 2.7. Let $\zeta,\mathfrak{z}\in\mathbb{C}$ with $\mathfrak{z}\neq 0$ and $\mathfrak{n}\in\mathbb{N}_0$.

$$\chi^{\mathfrak{n}}(\mathfrak{z}-1)^{\zeta}=\sum_{\mathfrak{k}=0}^{\infty}(-1)^{\mathfrak{k}}\binom{\zeta}{\mathfrak{k}}\mathfrak{z}^{\zeta-\mathfrak{k}}\sum_{v=0}^{\mathfrak{k}}v!\binom{\mathfrak{k}}{v}\sum_{\mathfrak{m}=0}^{\mathfrak{n}}S_2(\mathfrak{m},v)\binom{\mathfrak{n}}{\mathfrak{m}}H_{\mathfrak{n}-\mathfrak{m}}^{(\zeta)}(\chi;\mathfrak{z})\,.$$

Theorem 2.8. Let $\zeta,\mathfrak{z}\in\mathbb{C}$ and $\mathfrak{n}\in\mathbb{N}_0$.

$$(1+\mathfrak{z})^{\zeta}\chi^{\mathfrak{n}}-2^{\zeta}\sum_{\mathfrak{k}=0}^{\mathfrak{n}}\binom{\mathfrak{n}}{\mathfrak{k}}\mathcal{E}_{\mathfrak{n}-\mathfrak{k}}^{(-\zeta)}(\mathfrak{z})u_{\mathfrak{k}}^{(\zeta)}(\chi;\mathfrak{z})=0.$$

Proof. From Eq. (3) yields, we get

$$\sum_{\mathfrak{n}=0}^{\infty}(1+\mathfrak{z})^{\zeta}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=(\mathfrak{z}e^t+1)^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\frac{u_{\mathfrak{n}}^{(\zeta)}(\chi;\mathfrak{z})}{\mathfrak{n}!}t^{\mathfrak{n}}.$$

Thus

$$\sum_{\mathfrak{n}=0}^{\infty}(1+\mathfrak{z})^{\zeta}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=2^{\zeta}\left(\frac{\mathfrak{z}e^t+1}{2}\right)^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\frac{u_{\mathfrak{n}}^{(\zeta)}(\chi;\mathfrak{z})}{\mathfrak{n}!}t^{\mathfrak{n}}.$$

Joining last equation with (8) yields, we have

$$\sum_{\mathfrak{n}=0}^{\infty}(1+\mathfrak{z})^{\zeta}\chi^{\mathfrak{n}}\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}=2^{\zeta}\sum_{\mathfrak{n}=0}^{\infty}\sum_{\mathfrak{k}=0}^{\mathfrak{n}}\binom{\mathfrak{n}}{\mathfrak{k}}u_{\mathfrak{k}}^{(\zeta)}(\chi;\mathfrak{z})\mathcal{E}_{\mathfrak{n}-\mathfrak{k}}^{(-\zeta)}(\mathfrak{z})\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}.$$

Aligning the coefficients $\frac{t^{\mathfrak{n}}}{\mathfrak{n}!}$ with each side completes the proof.

Combining the Theorem 2.8. with Eq. (4) yields, we get the following corollary:

Corollary 2.9. Let $\zeta,\mathfrak{z}\in\mathbb{C}$ and $\mathfrak{n}\in\mathbb{N}_0$.

$$(1+\mathfrak{z})^{\zeta}\chi^{\mathfrak{n}}-2^{\zeta}\sum_{\mathfrak{k}=0}^{\mathfrak{n}}\binom{\mathfrak{n}}{\mathfrak{k}}\mathcal{E}_{\mathfrak{n}-\mathfrak{k}}^{(-\zeta)}(\mathfrak{z})H_{\mathfrak{k}}^{(\zeta)}\left(\chi;-\frac{1}{\mathfrak{z}}\right)=0.$$

Substituting $\chi = 0$ into Theorem 2.8. and combining Eq. (13) yields, we get the following corollary:

Corollary 2.9. Let $\zeta, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

$$0 = \sum_{k=0}^{n} \frac{\binom{n}{k}\mathcal{E}_{n-k}^{(-\zeta)}(z)q_k(z;\zeta)}{(1+z)^k}.$$

Theorem 2.10. Le $\zeta, z \in \mathbb{C}$. Let $u_0(\chi; z) = 1$.

$$(1+z)\chi^n - u_n(\chi;z) = z\sum_{k=0}^{n} u_k(\chi;z)\binom{n}{k}; \qquad (n > 0).$$

Proof. Substituting $\zeta = 1$ into Eq. (3) yields, we have

$$\left(\frac{1+z}{ze^t+1}\right)e^{t\chi} = e^{u(\chi;z)t}$$

and the umbral calculus convention and some calculation give:

$$(1+z)\sum_{n=0}^{\infty}\chi^n\frac{t^n}{n!} = z\sum_{n=0}^{\infty}(u(\chi;z)+1)^n\frac{t^n}{n!} + \sum_{n=0}^{\infty}u_n(\chi;z)\frac{t^n}{n!}.$$

Aligning the coefficients $\frac{t^n}{n!}$ with each side completes the proof.

Substituting $\chi = 0$ into Theorem 2.10. and using Eq. (13) yields, we get the following corollary:

Corollary 2.11. Let $\zeta, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

$$0 = z\sum_{k=0}^{n}\binom{n}{k}q_k(z)(1+z)^{-k} + q_n(z)(1+z)^{-n}.$$

Theorem 2.12. Let $\zeta, z \in \mathbb{C}$ with $z \neq 0$ and $n \in \mathbb{N}_0$.

$$\sum_{m=0}^{n}\binom{n}{m}H_m^{(\zeta)}\left(\chi; -\frac{1}{z}\right)H_{n-m}^{(\zeta)}\left(\chi; \frac{1}{z}\right) - 2^n u_n^{(\zeta)}(\chi; -z^2) = 0.$$

Proof. Replacing z by $-z^2$ and t by $2t$ respectively, Eq. (3) reduces to

$$\left(\frac{z-1}{ze^t-1}\right)^{\zeta}e^{t\chi}\left(\frac{1+z}{ze^t+1}\right)^{\zeta}e^{t\chi} = \sum_{n=0}^{\infty}\frac{u_n^{(\zeta)}(\chi;-z^2)}{n!}2^n t^n. \tag{18}$$

Hence

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}H_m^{(\zeta)}\left(\chi; -\frac{1}{z}\right)\binom{n}{m}H_{n-m}^{(\zeta)}\left(\chi;\frac{1}{z}\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty}u_n^{(\zeta)}(\chi;-z^2)2^n\frac{t^n}{n!}.$$

Aligning the coefficients $\frac{t^n}{n!}$ with each side completes proof.

Theorem 2.13. Let $\zeta, z \in \mathbb{C}$ with $z \neq 1$ and $n \in \mathbb{N}_0$.

$$2^n u_n^{(-v)}(-z^2) = \frac{(v!)^2}{(z^2-1)^v}\sum_{m=0}^{n}\binom{n}{m}S_2(m,v;z)y_1(n-m,v;z).$$

Proof. Replacing ζ by $-v$ and χ by 0 respectively, Eq. (18) gives

$$\sum_{ꞑ=0}^{\infty} 2^{ꞑ} u_{ꞑ}^{(-v)}(-ʒ^2)\frac{t^{ꞑ}}{ꞑ!} = (v!)^2 \frac{1}{(ʒ^2-1)^v}\frac{(ʒe^t-1)^v}{v!}\frac{(ʒe^t+1)^v}{v!}.$$

Joining Eq. (5) with Eq. (15) yields, we have

$$\sum_{ꞑ=0}^{\infty} 2^{ꞑ} u_{ꞑ}^{(-v)}(-ʒ^2)\frac{t^{ꞑ}}{ꞑ!} = \frac{(v!)^2}{(ʒ^2-1)^v}\sum_{ꞑ=0}^{\infty} S_2(ꞑ,v;ʒ)\frac{t^{ꞑ}}{ꞑ!}\sum_{ꞑ=0}^{\infty} y_1(ꞑ,v;ʒ)\frac{t^{ꞑ}}{ꞑ!}.$$

Hence

$$\sum_{ꞑ=0}^{\infty} 2^{ꞑ} u_{ꞑ}^{(-v)}(-ʒ^2)\frac{t^{ꞑ}}{ꞑ!} = \frac{(v!)^2}{(ʒ^2-1)^v}\sum_{ꞑ=0}^{\infty}\sum_{ɱ=0}^{ꞑ} S_2(ɱ,v;ʒ)\binom{ꞑ}{ɱ} y_1(ꞑ-ɱ,v;ʒ)\frac{t^{ꞑ}}{ꞑ!}.$$

Aligning the coefficients $\frac{t^{ꞑ}}{ꞑ!}$ with each side completes proof.

Theorem 2.14. Let $\zeta, ʒ \in \mathbb{C}$ with $ʒ \neq 0$ and $ꞑ \in \mathbb{N}_0$.

$$\frac{(ʒ^2-1)^\zeta}{2^\zeta}\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ}\mathcal{E}_{ꞑ-ɱ}^{(\zeta)}(ʒ)\chi^{ɱ} = \zeta!\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ} u_{ɱ}^{(\zeta)}(\chi;-ʒ^2)S_2(ꞑ-ɱ,\zeta;ʒ).$$

Proof. By using Eq. (18) yields, we have

$$\frac{(ʒ^2-1)^\zeta}{2^\zeta}\left(\frac{2}{ʒe^t+1}\right)^\zeta e^{2t\chi} = \sum_{ꞑ=0}^{\infty} u_{ꞑ}^{(\zeta)}(\chi;-ʒ^2)2^{ꞑ}\,\frac{(ʒe^t-1)^\zeta}{\zeta!}\zeta!\frac{t^{ꞑ}}{ꞑ!}.$$

Combining Eq. (5) with Eq. (8) yields

$$\frac{(ʒ^2-1)^\zeta}{2^\zeta}\sum_{ꞑ=0}^{\infty}\mathcal{E}_{ꞑ}^{(\zeta)}(ʒ)\frac{t^{ꞑ}}{ꞑ!}\sum_{ꞑ=0}^{\infty}\chi^{ꞑ}2^{ꞑ}\frac{t^{ꞑ}}{ꞑ!} = \sum_{ꞑ=0}^{\infty} u_{ꞑ}^{(\zeta)}(\chi;-ʒ^2)2^{ꞑ}\frac{t^{ꞑ}}{ꞑ!}\sum_{ꞑ=0}^{\infty} S_2(ꞑ,\zeta;ʒ)\zeta!\frac{t^{ꞑ}}{ꞑ!}.$$

Hence

$$\frac{(ʒ^2-1)^\zeta}{2^\zeta}\sum_{ꞑ=0}^{\infty}\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ}\mathcal{E}_{ꞑ-ɱ}^{(\zeta)}(ʒ)2^{ɱ}\frac{\chi^{ɱ}t^{ꞑ}}{ꞑ!} = \zeta!\sum_{ꞑ=0}^{\infty}\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ} u_{ɱ}^{(\zeta)}(\chi;-ʒ^2)2^{ɱ}\,S_2(ꞑ-ɱ,\zeta;ʒ)\frac{t^{ꞑ}}{ꞑ!}.$$

Aligning the coefficients $\frac{t^{ꞑ}}{ꞑ!}$ with each side completes the proof.

Substituting $\chi = 1$ into Theorem 2.14. and using Eq. (13) yields, after performing several calculations, we get the following corollary:

Corollary 2.15. Let $\zeta, ʒ \in \mathbb{C}$ with $ʒ \neq 1$ and $ꞑ \in \mathbb{N}_0$.

$$\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ}\mathcal{E}_{ꞑ-ɱ}^{(\zeta)}(ʒ) = 2^\zeta\zeta!\sum_{ɱ=0}^{ꞑ}\binom{ꞑ}{ɱ}\frac{(-1)^{ɱ}q_{ɱ}(-ʒ^2;\zeta)}{(ʒ^2-1)^{ɱ+\zeta}}S_2(ꞑ-ɱ,\zeta;ʒ).$$

Theorem 2.16. Let $\zeta, ʒ \in \mathbb{C}$ and $ꞑ \in \mathbb{N}_0$.

$$(ʒ-1)^{ɱ}\chi^{ꞑ} = ɱ!\sum_{ƙ=0}^{ꞑ}\binom{ꞑ}{ƙ} u_{ꞑ-ƙ}^{(ɱ)}(\chi;-ʒ)\,S_2(ƙ,ɱ;ʒ).$$

Proof. Replacing ζ by m and z by $-z$ respectively, Eq. (3) yields, we have

$$(z-1)^{m}\sum_{n=0}^{\infty}\chi^{n}\frac{t^{n}}{n!}=m!\sum_{n=0}^{\infty}S_2(n,m;z)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}u_{n}^{(m)}(\chi;-z)\frac{t^{n}}{n!}.$$

Hence

$$(z-1)^{m}\sum_{n=0}^{\infty}\chi^{n}\frac{t^{n}}{n!}=m!\sum_{n=0}^{\infty}\sum_{k=0}^{n}S_2(k,m;z)\binom{n}{k}u_{n-k}^{(m)}(\chi;-z)\frac{t^{n}}{n!}.$$

Aligning the coefficients $\frac{t^{n}}{n!}$ with each side completes the proof.

Combining the above Theorem 2.16. and Eq. (17) yields, we get the following corollary:

Corollary 2.17. Let $\zeta, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

$$(1+z)^{m}\chi^{n}=m!\sum_{k=0}^{n}\binom{n}{k}u_{n-k}^{(m)}(\chi;z)\,y_1(k,m;z).$$

Theorem 2.18. Let $\zeta, z \in \mathbb{C}$ and $n \in \mathbb{N}_0$.

$$\sum_{j=1}^{k}(-1)^{j}S_2(n,j)z^{j}(\zeta)^{(j)}\mathfrak{M}_{j,2j}(z)\sum_{k=0}^{j-1}\frac{1}{\zeta+k}=\frac{d}{d\zeta}\left\{u_{n}^{(\zeta)}(z)\right\}.$$

Proof. Substituting $\chi = 1$ into Eq. (3) yields, we have

$$\left(\frac{1+z}{2}\right)^{\zeta}\sum_{n=0}^{\infty}\frac{\mathcal{E}_{n}^{(\zeta)}(z)}{n!}t^{n}=\sum_{n=0}^{\infty}\frac{u_{n}^{(\zeta)}(z)}{n!}t^{n}.$$

Combining the Eq. (9) and taking derivative wrt ζ yields, we have

$$\frac{d}{d\zeta}\left\{(1+z)^{\zeta}\sum_{n=0}^{\infty}\sum_{j=1}^{n}(-1)^{j}S_2(n,j)z^{j}\frac{(\zeta)^{(j)}}{(z+1)^{j+\zeta}}\frac{t^{n}}{n!}\right\}=\sum_{n=0}^{\infty}\frac{d}{d\zeta}\left\{u_{n}^{(\zeta)}(z)\right\}\frac{t^{n}}{n!}.$$

Aligning the coefficients $\frac{t^{n}}{n!}$ with each side completes the proof.

3. RESULTS AND DISCUSSION

The results in this paper are obtained by blending formulas, finite sums via generating functions and their functional equations methods. These results have the value of being a resource for researchers in applied sciences.

4. CONCLUSION

Generating functions for special numbers and polynomials covering Stirling type numbers, the Frobenius Euler numbers and polynomials, the beta-type rational functions, and also combinatorial numbers have been given. Using these functions, some novel formulas and finite sums, containing combinatorial numbers, Stirling-type numbers, the Frobenius Euler numbers and polynomials, and beta-type rational functions, have been derived.

Since these numbers and polynomials are related to spline curves, our future project is to blend the results of this paper with many different kinds of spline curves and their applications.

AUTHOR CONTRIBUTIONS

The authors jointly prepared this article and made equal contributions.

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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