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ON SOME NEW DIFFERENCE SEQUENCE SPACES DERIVED BY USING RIESZ MEAN AND A MUSIELAK-ORLICZ FUNCTION

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Abstract. In this paper we introduce new difference sequence spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ by using Riesz mean and Musielak-Orlicz function. We also make an effort to study some topological properties and compute $\alpha-, \beta-$ and $\gamma-$ duals of these spaces. Finally, we study matrix transformations on newly formed spaces.

1. Introduction and Preliminaries

Let w be the vector space of all real or complex sequences. By l_∞, c and c_0; we denote the classes of all bounded, convergent and null sequences; respectively. Also, we write bs, cs and l_p to denote the spaces of all bounded, convergent series and p-absolutely summable sequences, respectively, where $1 \leq p < \infty$. We use the convention that any term with a negative subscript is equal to zero.
Let X and Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the $A-$transformation from X into Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk} x_k$.
By $A \in (X : Y)$ we mean the characterizations of matrices $A : X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called the A-limit of x. For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{1.1}$$

The theory of matrix transformations is a wide field in summability theory. It deals with the characterizations of classes of matrix mappings between sequence spaces

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by giving necessary and sufficient conditions on the entries of the infinite matrices. The classical summability theory deals with a generalization of convergence of sequences and series. One original idea was to assign a limit to divergent sequences or series. Toeplitz [29] was the first to study summability methods as a class of transformations of complex sequences by complex infinite matrices.

Let $A = (a_{nk})$ be any matrix. Then a sequence x is said to be summable to l, written $x_k \to l$, if and only if $A_n x = \sum_k a_{nk} x_k$ exists for each n and $A_n x \to l$ $(n \to \infty)$.

For example, if $A_n = I$, the unit matrix for all n, then $x_k \to l(I)$ means precisely that $x_k \to l$ $(k \to \infty)$, in the ordinary sense of convergence.

An infinite matrix $A = (a_{nk})$ is said to be regular ([11], page:165) if and only if the following conditions (or Toplitz conditions) hold:

(i) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1,$

(ii) $\lim_{n\to\infty} a_{nk} = 0, \quad (k = 0, 1, 2, ...)$

(iii) $\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} |a_{nk}| < \infty.$

Let (q_k) be a sequence of strictly positive numbers and let us write, $Q_n = \sum_{k=0}^{n} q_k$ for $n \in \mathbb{N}$. Then the matrix $R^q = (r^q_{nk})$ of the Riesz mean (R, q_n) is given by

$$r^q_{nk} = \begin{cases} \frac{q_k}{Q_n}, & \text{if } 0 \le k \le n, \\ \\ 0 & \text{if } k > n. \end{cases}$$

The Riesz mean (R, q_n) is regular if and only if $Q_n \to \infty$ as $n \to \infty$ (see, Petersen [22], p.10).

The sequence space $r^q(u, p)$ is introduced by Sheikh and Ganie [26] as:

$$r^q(u,p) = \Big\{ x = (x_k) \in w : \sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k} u_j q_j x_j \Big|^{p_k} < \infty \Big\},$$

where $0 \le p_k \le D < \infty$.

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $\sup_k p_k = D$ and $H = \max\{1, D\}$. Then, the linear spaces $l(p)$ and $l_\infty(p)$ were defined by Maddox [13] (see also, [27],[30]) as follows:

$$l(p) = \{x = (x_k) : \sum_k |x_k|^{p_k} < \infty\}$$

and

$$l_\infty(p) = \{x = (x_k) : \sup_k |x_k|^{p_k} < \infty\}$$

which are complete spaces paranormed by

$$g_1(x) = \Big[\sum_k |x_k|^{p_k} \Big]^{\frac{1}{H}} \text{ and } g_2(x) = \sup_k |x_k|^{\frac{p_k}{H}}$$

if and only if $\inf p_k > 0$ for all k.

Throughout the paper we shall assume that ${p_k}^{-1} + \{p_k^{'}\}^{-1} = 1$ provided $1 <$

$\inf p_k \leq D < \infty$ and we denote the collection of all finite subsets of $\mathbb{N}$ by F where $\mathbb{N} = \{0, 1, 2, ...\}$.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, $k > 0$ and for $L > 1$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function (see [14], [19]). A sequence $\mathcal{N} = (N_k)$ is defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k)$$

and $x = (x_k) \in t_{\mathcal{M}}$.
We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [8], who studied the difference sequence spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$. The notion was further generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\triangle^m)$, $c(\triangle^m)$ and $c_0(\triangle^m)$.
Let n, m be non-negative integers, then for Z a given sequence space, we have

$$Z(\triangle_n^m) = \{x = (x_k) \in w : (\triangle_n^m x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\triangle_n^m x = (\triangle_n^m x_k) = (\triangle_n^{m-1} x_k - \triangle_n^{m-1} x_{k+1})$ and $\triangle^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $n = 1$, we get the spaces $l_\infty(\triangle^m), c(\triangle^m)$ and $c_0(\triangle^m)$ studied by Et and Çolak [5]. Taking $m = n = 1$, we get the spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ introduced and studied by Kizmaz [8]. Mursaleen et al. ([15], [16], [17], [18]) used the idea of Orilcz function and study different sequence spaces. Esi et al. ([1], [3], [4]) work on these type of sequence spaces. For more details about sequence spaces and matrix transformations (see [2], [7], [12], [20], [21], [23], [24], [25], [28]) and references there in.

2. The Riesz Sequence Space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ of Non-absolute Type

Let X be a linear metric space. A function $g : X \to \mathbb{R}$ is called paranorm, if

1. $g(x) \geq 0$, for all $x \in X$,
2. $g(-x) = g(x)$, for all $x \in X$,
3. $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ *as* $n \to \infty$, then $g(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, g) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [31], Theorem 10.4.2, P-183).
Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then we define new difference sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ as follows:

$$r^q(\mathcal{M}, \Delta_n^m, u, p) = \Big\{ x = (x_k) \in w : \sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k} M_j(|u_j q_j \Delta_n^m x_j|) \Big|^{p_k} < \infty \Big\},$$

where $0 < p_k \leq D < \infty$.
With the definition of matrix domain (1.1), the sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ may be redefined as

$$r^q(\mathcal{M}, \Delta_n^m, u, p) = \{l(p)\}_{R^q(\mathcal{M}, \Delta_n^m, u)}$$

where $R^q(\mathcal{M}, \Delta_n^m, u)$ denotes the matrix $R^q(\mathcal{M}, \Delta_n^m, u) = r_{nk}^q(\mathcal{M}, \Delta_n^m, u)$ defined by

$$r_{nk}^q(\mathcal{M}, \Delta_n^m, u) = \begin{cases} \frac{1}{Q_n}(M_k(u_k q_k) - M_{k+1}(u_{k+1} q_{k+1})), & \text{if } 0 \leq k \leq n-1 \\ \frac{M_n(u_n q_n)}{Q_n}, & \text{if } k = n \\ 0, & \text{if } k > n. \end{cases}$$

Define the sequence $y = (y_k)$ which will be used by the $R^q(\mathcal{M}, \Delta_n^m, u)$-transform of a sequence $x = (x_k)$, we have

$$y_k = \frac{1}{Q_k} \sum_{j=0}^{k} M_j(|u_j q_j \Delta_n^m x_j|). \tag{2.1}$$

The main purpose of this paper is to study some new difference sequence spaces generated by Riesz Mean and Musielak-Orlicz function. We shall show that these spaces are complete and paranormed spaces. We have also discuss the $\alpha-, \beta-$duals of these spaces in section third of this paper. Finally, we discuss the matrix transformations on these spaces in the last section of this paper.

Theorem 2.1. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then $r^q(\mathcal{M}, \Delta_n^m, u, p)$ is a complete linear metric space paranormed by*

$$g(x) = \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) x_j + \frac{M_k(u_k q_k)}{Q_k} x_k \Big|^{p_k} \Big]^{\frac{1}{H}}$$

with $0 \leq p_k \leq D < \infty$ and $H = \max\{1, D\}$.

Proof. The linearity of $r^q(\mathcal{M}, \Delta_n^m, u, p)$ follows from the inequality. For $x, y \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ (see [11], p.30)

(2.2)

$$\Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1}))(x_j + y_j) + \frac{M_k(u_k q_k)}{Q_k} (x_k + y_k) \Big|^{p_k} \Big]^{\frac{1}{H}}$$

$$\begin{aligned} &\leq \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) x_j + \frac{M_k(u_k q_k)}{Q_k} x_k \Big|^{p_k} \Big]^{\frac{1}{H}} \\ &+ \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) y_j + \frac{M_k(u_k q_k)}{Q_k} y_k \Big|^{p_k} \Big]^{\frac{1}{H}} \end{aligned}$$

and for any $\alpha \in \mathbb{R}$ (See [12])

$$|\alpha|^{p_k} \leq \max(1, |\alpha|^H). \tag{2.3}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Again the inequality (2.2) and (2.3) yield the subadditivity of g and

$$g(\alpha x) \leq \max(1, |\alpha|) g(x).$$

Let $\{x^n\}$ be any sequence of points of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ such that $g(x^n - x) \to 0$ and (α^n) is a sequence of scalars such that $\alpha^n \to \alpha$. Then since the inequality,

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$g(\alpha_n x^n - \alpha x) = \Big[\sum_k \Big|\frac{1}{Q_k}\sum_{j=0}^{k}(M_j(u_jq_j) - M_{j+1}(u_{j+1}q_{j+1}))(\alpha_n x_j^n + \alpha x_j)\Big|^{p_k}\Big]^{\frac{1}{H}}$$
$$\leq |\alpha_n - \alpha|^{\frac{1}{H}} g(x^n) + |\alpha|^{\frac{1}{H}} g(x^n - x)$$

which tends to zero as $n \to \infty$. This proves that the scalar multiplication is continuous. Hence g is paranorm on the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$.

Now we prove the completeness of $r^q(\mathcal{M}, \Delta_n^m, u, p)$:

Let $\{x^i\}$ be any Cauchy sequence in the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$, where $x^i = \{x_0^i, x_1^i, ...\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \epsilon \ \ \forall \ \ i, j \geq n_0(\epsilon). \tag{2.4}$$

Using definition of g and for each fixed $k \in \mathbb{N}$ that

$$|(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k|$$

$$\leq \Big[\sum_k |(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k|^{p_k}\Big]^{\frac{1}{H}} < \epsilon \ \text{ for } \ i, j \geq n_0(\epsilon)$$

which yields that $\{(R^q(\mathcal{M}, \Delta_n^m, u)x^0)_k, (R^q(\mathcal{M}, \Delta_n^m, u)x^1)_k, ...\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges say

$$(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k \to (R^q(\mathcal{M}, \Delta_n^m, u)x)_k \ \text{ as } \ i \to \infty.$$

Using these infinitely many limits $(R^q(\mathcal{M}, \Delta_n^m, u)x)_0, (R^q(\mathcal{M}, \Delta_n^m, u)x)_1, ...$, we define the sequence $\{(R^q(\mathcal{M}, \Delta_n^m, u)x)_0, (R^q(\mathcal{M}, \Delta_n^m, u)x)_1, ...\}$. From (2.4) for each $t \in \mathbb{N}$ and $i, j \geq n_0(\epsilon)$,

$$\sum_{k=0}^{t} |(R^q(\mathcal{M}, \Delta_n^m, u)x^i)_k - (R^q(\mathcal{M}, \Delta_n^m, u)x^j)_k|^{p_k} \leq g(x^i - x^j)^H < \epsilon^H. \tag{2.5}$$

Take any $i, j \geq n_0(\epsilon)$. First, let $j \to \infty$ in (2.5) and then $t \to \infty$, we obtain

$$g(x^i - x) \leq \epsilon.$$

Finally, taking $\epsilon = 1$ in (2.5) and letting $i \geq n_0(1)$, we have by Minkowski's inequality for each $t \in \mathbb{N}$ that

$$\Big[\sum_{k=0}^{t} |(R^q(\mathcal{M}, \Delta_n^m, u)x)_k|^{p_k}\Big]^{\frac{1}{H}} \leq g(x^i - x) + g(x^i) \leq 1 + g(x^i)$$

which implies that $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Since $g(x - x^i) \leq \epsilon$ for all $i \geq n_0(\epsilon)$, it follows that $x^i \to x$ as $i \to \infty$. Hence, the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ is complete. $\square$

Theorem 2.2. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Then the sequence space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ of non-absolute type is linearly isomorphic to the space $l(p)$, where $0 < p_k \leq D < \infty$.*

Proof. To show that the spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and $l(p)$ are linearly isomorphic, we have to prove that there exists a linear bijection between these spaces. Define a linear transformation $T : r^q(\mathcal{M}, \Delta_n^m, u, p) \to l(p)$ by $x \to y = Tx$ by using equation (2.2). The linearity of T is trivial. Further, it is obvious that $x = \theta$ whenever $T(x) = T(\theta)$ and hence T is injective. Let $y \in l(p)$ and define the sequence $x = (x_k)$ by

$$x_k = \sum_{n=0}^{k-1} \Big(\frac{1}{M_n(u_n q_n)} - \frac{1}{M_{n+1}(u_{n+1} q_{n+1})} \Big) Q_k y_k + \frac{Q_k}{M_k(u_k q_k)} y_k$$

for $k \in \mathbb{N}$. Then

$$g(x) = \Big[\sum_k \Big| \frac{1}{Q_k} \sum_{j=0}^{k-1} (M_j(u_j q_j) - M_{j+1}(u_{j+1} q_{j+1})) x_j + \frac{M_k(u_k q_k)}{Q_k} x_k \Big|^{p_k} \Big]^{\frac{1}{H}}$$

$$\begin{aligned} &= \Big[\sum_k \Big| \sum_{j=0}^{k} \delta_{kj} y_j \Big|^{p_k} \Big]^{\frac{1}{H}} \\ &= \Big[\sum_k \Big| y_k \Big|^{p_k} \Big]^{\frac{1}{H}} \\ &= g_1(y) < \infty, \end{aligned}$$

where

$$\delta_{kj} = \begin{cases} 1, & \text{if } k = j \\ 0, & \text{if } k \neq j. \end{cases}$$

Thus, we have $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. Consequently, T is surjective and paranorm preserving. Hence, T is linear bijection and this shows that the spaces $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and $l(p)$ are linearly isomorphic. □

3. Basis and $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$

In this section, we compute $\alpha-, \beta-$ and $\gamma-$ duals of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and finally we give the basis for the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$.

For the sequence space X and Y, define the set

$$S(X : Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}.$$

The $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space X, respectively denoted by X^α, X^β and X^γ which are defined by

$$X^\alpha = S(X : l_1), X^\beta = S(X : cs) \text{ and } X^\gamma = S(X : bs).$$

Firstly, we state some lemmas which are required in proving our theorems:

Lemma 3.1. [6] *(i) Let $1 < p_k \le D < \infty$. Then $A \in (l(p) : l_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{k \in F} \sum_k \Big| \sum_{n \in k} \alpha_{nk} B^{-1} \Big|^{p'_k} < \infty.$$

(ii) Let $0 < p_k \le 1$. Then $A \in (l(p) : l_1)$ if and only if

$$\sup_{k \in F} \sup_k \Big| \sum_{n \in k} \alpha_{nk} B^{-1} \Big|^{p_k} < \infty.$$

Lemma 3.2. [10] *(i) Let $1 < p_k \le D < \infty$. Then $A \in (l(p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_n \sum_k \Big| \alpha_{nk} B^{-1} \Big|^{p'_k} < \infty. \tag{3.1}$$

(ii) Let $0 < p_k \le 1$ for every $k \in \mathcal{N}$. Then $A \in (l(p) : l_\infty)$ if and only if

$$\sup_{n,k} \Big| \alpha_{nk} \Big|^{p_k} < \infty. \tag{3.2}$$

Lemma 3.3. [8] *Let $0 < p_k \le D < \infty$ for every $k \in \mathcal{N}$. Then $A \in (l(p) : c)$ if and only if (3.1) and (3.2) hold along with*

$$\lim_n \alpha_{nk} = \beta_k \text{ for } k \in \mathcal{N} \tag{3.3}$$

also holds.

Theorem 3.1. *Let $\mathcal{M} = (M_j)$ be a Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_1(\mathcal{M}, \Delta_n^m, u, p)$ and $D_2(\mathcal{M}, \Delta_n^m, u, p)$ as follows:*

$$D_1(\mathcal{M}, \Delta_n^m, u, p) = \bigcup_{B>1} \Big\{ \alpha = (\alpha_k) \in w : \sup_{k \in F} \sum_k \Big| \sum_{n \in k} \Big[\Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) Q_k \alpha_n + \frac{Q_n}{M_n(u_n q_n))} \alpha_n \Big] B^{-1} \Big|^{p'_k} < \infty \Big\}$$

and

$$D_2(\mathcal{M}, \Delta_n^m, u, p) = \bigcup_{B>1} \Big\{ \alpha = (\alpha_k) \in w : \sum_k \Big| \Big[\Big(\frac{\alpha_k}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_i \Big) Q_k \Big] B^{-1} \Big|^{p'_k} < \infty \Big\}$$

Then

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p) \Big]^\alpha = D_1(\mathcal{M}, \Delta_n^m, u, p)$$

and

$$\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^\beta = D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs.$$

Proof. Let us take any $\alpha=(\alpha_k)\in w$. We can easily derive with (2.1) that

$$(3.4)\qquad \alpha_n x_n = \sum_{k=0}^{n-1}\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\alpha_n Q_k y_k + \frac{\alpha_n}{M_n(u_nq_n)}Q_n y_n$$
$$= (Cy)_n,$$

where $C=(c_{nk})$ is defined as

$$c_{nk}=\begin{cases}\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\alpha_nQ_k, & \text{if } 0\le k\le n-1\\ \frac{\alpha_n}{M_n(u_nq_n)}Q_n, & \text{if } k=n\\ 0, & \text{if } k>n,\end{cases}$$

for all $n,k\in\mathcal{N}$. Thus, we observe by combining (3.4) with (i) of lemma (3.1) that $\alpha x=(\alpha_nx_n)\in l_1$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Cy\in l_1$ whenever $y\in l_p$. This gives the result that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^\alpha = D_1(\mathcal{M},\Delta_n^m,u,p)$. Further, consider the equation

(3.5)
$$\sum_{k=0}^{n}\alpha_kx_k=\sum_{k=0}^{n}\Big[\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k\Big]y_k$$
$$= (Dy)_n,$$

where $D=(d_{nk})$ is defined as

$$d_{nk}=\begin{cases}\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k, & \text{if } 0\le k\le n\\ 0, & \text{if } k>n.\end{cases}$$

Thus, we deduce from Lemma (3.3) with (3.5) that $\alpha x=(\alpha_nx_n)\in cs$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Dy\in c$ whenever $y\in l(p)$. Therefore, we derive from (3.1) that

(3.6)
$$\sum_k\Big|\Big[\Big(\frac{\alpha_k}{M_k(u_kq_k)}+\Big(\frac{1}{M_k(u_kq_k)}-\frac{1}{M_{k+1}(u_{k+1}q_{k+1})}\Big)\sum_{i=k+1}^{n}\alpha_i\Big)Q_k\Big]B^{-1}\Big|^{p_k'}<\infty$$

and $\lim_n d_{nk}$ exists and hence shows that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^\beta = D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs$. From lemma (3.2) together with (3.5) that $\alpha x=(\alpha_kx_k)\in bs$ whenever $x=(x_n)\in r^q(\mathcal{M},\Delta_n^m,u,p)$ if and only if $Dy\in l_\infty$ whenever $y=(y_k)\in l(p)$. Therefore, we again obtain the condition (3.6) which means that $\Big[r^q(\mathcal{M},\Delta_n^m,u,p)\Big]^\gamma = D_2(\mathcal{M},\Delta_n^m,u,p)\cap cs$ and the proof of theorem is complete. □

Theorem 3.2. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sets $D_3(\mathcal{M}, \Delta_n^m, u, p)$ and $D_4(\mathcal{M}, \Delta_n^m, u, p)$ as follows:*

$$D_3(\mathcal{M}, \Delta_n^m, u, p) =$$

$$\Big\{\alpha = (\alpha_k) \in w : \sup_{k \in F} \sup_k \Big| \sum_{n \in k} \Big[\Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})}\Big) Q_k \alpha_n + \frac{Q_n}{M_n(u_n q_n)} \alpha_n\Big]\Big|^{p_k} < \infty\Big\}$$

and

$$D_4(\mathcal{M}, \Delta_n^m, u, p) =$$

$$\Big\{\alpha = (\alpha_k) \in w : \sup_k \Big|\Big[\Big(\frac{\alpha_k}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})}\Big) \sum_{i=k+1}^{n} \alpha_i\Big) Q_k\Big]\Big|^{p_k} < \infty\Big\}.$$

Then

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p)\Big]^\alpha = D_3(\mathcal{M}, \Delta_n^m, u, p)$$

and

$$\Big[r^q(\mathcal{M}, \Delta_n^m, u, p)\Big]^\beta = D_4(\mathcal{M}, \Delta_n^m, u, p) \cap cs.$$

Proof. This is obtained by proceeding in proof of Theorem (3.1), by using second parts of lemmas (3.1), (3.2) and (3.3) instead of the first parts so we exclude the details. □

Theorem 3.3. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers. Define the sequence $b^{(k)}(q) = \{b_n^{(k)}(q)\}$ of the elements of the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_n^{(k)}(q) = \begin{cases} \Big(\frac{1}{M_n(u_n q_n)} - \frac{1}{M_{n+1}(u_{n+1} q_{n+1})}\Big) Q_n + u_n^{-1} \frac{Q_k}{M_k(u_k q_k)}, & \text{if } 0 \le n \le k-1 \\ 0, & \text{if } n > k-1. \end{cases}$$

Then the sequence $\{b^{(k)}(q)\}$ is a basis for the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ and any $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ has a unique representation of the form

$$x = \sum_k \lambda_k(q) b^{(k)}(q), \tag{3.7}$$

where $\lambda_k(q) = (R^q(\mathcal{M}, \Delta_n^m, u)x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \le D < \infty$.

Proof. It is clear that $\{b^{(k)}(q)\} \subset r^q(\mathcal{M}, \Delta_n^m, u, p)$, since

$$R^q(\mathcal{M}, \Delta_n^m, u) b^{(k)}(q) = e^{(k)} \in l(p) \ \text{ for } \ k \in \mathbb{N} \tag{3.8}$$

and $0 < p_k \le D < \infty$, where $e^{(k)}$ is the sequence whose only non-zero term is 1 in kth place for each $k \in \mathbb{N}$.

Let $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ be given. For every non-negative integer t, we put

$$x^{[t]} = \sum_{k=0}^{t} \lambda_k(q) b^{(k)}(q). \tag{3.9}$$

Then, we obtain by applying $R^q(\mathcal{M}, \Delta_n^m, u)$ to (3.9) with (3.8) that

$$R^q(\mathcal{M}, \Delta_n^m, u)x^{[t]} = \sum_{k=0}^{t} \lambda_k(q) R^q(\mathcal{M}, \Delta_n^m, u) b^{(k)}(q) = \sum_{k=0}^{t} (R^q(\mathcal{M}, \Delta_n^m, u)x)_k e^{(k)}$$

and

$$\Big(R^q(\mathcal{M}, \Delta_n^m, u)(x - x^{[t]})\Big)_i = \begin{cases} 0, & \text{if } 0 \le i \le t \\ (R^q(\mathcal{M}, \Delta_n^m, u)x)_i, & \text{if } i > t, \end{cases}$$

where $i, t \in \mathbb{N}$. Given $\epsilon > 0$, there exists an integer t_0 such that

$$\Big(\sum_{i=t}^{\infty} \Big| (R^q(\mathcal{M}, \Delta_n^m, u)x)_i \Big|^{p_k} \Big)^{\frac{1}{H}} < \frac{\epsilon}{2} \ \ \forall \ \ t \ge t_0.$$

Hence,

$$\begin{aligned} g(x - x^{[t]}) &= \Big(\sum_{i=t}^{\infty} \Big| (R^q(\mathcal{M}, \Delta_n^m, u)x)_i \Big|^{p_k} \Big)^{\frac{1}{H}} \\ &\le \Big(\sum_{i=t_0}^{\infty} \Big| (R^q(\mathcal{M}, \Delta_n^m, u)x)_i \Big|^{p_k} \Big)^{\frac{1}{H}} \\ &< \frac{\epsilon}{2} \\ &< \epsilon, \end{aligned}$$

for all $t \ge t_0$ which proves that $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ is represented as equation (3.7).

Let us show that the uniqueness of the representation for $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ given by equation (3.6). Suppose, on the contrary that there exists a representation $x = \sum_k \mu_k(q) b^{(k)}(q)$. Since the linear transformation T from $r^q(\mathcal{M}, \Delta_n^m, u, p)$ to $l(p)$ used in the Theorem (2.2) is continuous, we have

$$(R^q(\mathcal{M}, \Delta_n^m, u)x)_n = \sum_k \mu_k(q) (R^q(\mathcal{M}, \Delta_n^m, u) b^{(k)}(q))_n = \sum_k \mu_k(q) e_n^{(k)} = \mu_n(q)$$

for $n \in \mathbb{N}$, which contradicts the fact that $(R^q(\mathcal{M}, \Delta_n^m, u)x)_n = \lambda_n(q) \ \ \forall \ \ n \in \mathcal{N}$. Hence, the representation (3.7) is unique. □

4. Matrix Mappings on the Space $r^q(\mathcal{M}, \Delta_n^m, u, p)$

In this section, we characterize the matrix mappings from the space $r^q(\mathcal{M}, \Delta_n^m, u, p)$ to the space l_∞.

Theorem 4.1. *Let $\mathcal{M} = (M_j)$ be Musielak-Orlicz function, $u = (u_j)$ be a sequence of strictly positive real numbers and $p = (p_k)$ be a bounded sequence of positive real numbers.*

(i) Let $1 < p_k < D < \infty$ for $k \in \mathbb{N}$. Then $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ if and only if there exists an integer $B > 1$ such that

(4.1)
$$C(B) = \sup_n \sum_k \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_{ni} \Big) Q_k \Big] B^{-1} \Big|^{p'_k} < \infty$$

and $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

(ii) Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ if and only if

$$\text{(4.2)} \quad \sup_{n,k} \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{n} \alpha_{ni} \Big) Q_k \Big] \Big|^{p_k} < \infty$$

and $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in cs$ for each $n \in \mathbb{N}$.

Proof. We shall prove only (i) and the proof of (ii) will follow on applying similar argument. Let $A \in (r^q(\mathcal{M}, \Delta_n^m, u, p) : l_\infty)$ and $1 < p_k \leq D < \infty$ for every $k \in \mathbb{N}$. Then Ax exists for $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$ and implies that $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\mathcal{M}, \Delta_n^m, u, p)\}^\beta$ for each $n \in \mathbb{N}$. Hence necessity of (4.1) holds. Conversely, suppose that (4.1) holds and $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$, since $\{\alpha_{nk}\}_{k\in\mathbb{N}} \in \{r^q(\mathcal{M}, \Delta_n^m, u, p)\}^\beta$ for every fixed $n \in \mathbb{N}$, so the $A-$ transform of x exists. Consider the following equality obtained by using the relation (3.4) that

(4.3)
$$\sum_{k=0}^{t} \alpha_{nk} x_k = \sum_{k=0}^{t} \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{t} \alpha_{ni} \Big) Q_k \Big] y_k.$$

Taking into account the assumptions, we derive from (3.3) as $t \to \infty$ that

(4.4)
$$\sum_{k} \alpha_{nk} x_k = \sum_{k} \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{\infty} \alpha_{ni} \Big) Q_k \Big] y_k.$$

Now by combining (4.4) and the inequality which holds for any $B > 0$ and any complex numbers a, b

$$|ab| \leq B\Big(|aB^{-1}|^{p'} + |b|^p \Big)$$

with $p^{-1} + \{p'\}^{-1} = 1$ [10], we can see that

$$\sup_{n\in\mathcal{N}} \Big| \sum_k \alpha_{nk} x_k \Big| \leq \sup_{n\in\mathbb{N}} \sum_k \Big| \Big[\Big(\frac{\alpha_{nk}}{M_k(u_k q_k)} + \Big(\frac{1}{M_k(u_k q_k)} - \frac{1}{M_{k+1}(u_{k+1} q_{k+1})} \Big) \sum_{i=k+1}^{\infty} \alpha_{ni} \Big) Q_k \Big] \Big| |y_k|$$

$$\begin{aligned} &\leq B[C(B) + h_1^B(y)] \\ &< \infty. \end{aligned}$$

This shows that $Ax \in l_\infty$ whenever $x \in r^q(\mathcal{M}, \Delta_n^m, u, p)$. The proof is complete. □

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