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BLASCHKE APPROACH TO EULER-SAVARY FORMULAE FOR ONE PARAMETER DUAL HYPERBOLIC SPHERICAL MOTION

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Abstract. In this paper, we have introduced one parameter dual hyperbolic spherical motions in the dual Lorentzian space. This examination is given using Blaschke frame of axodes corresponding to the curves on the unit dual hyperbolic sphere. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae for one parameter dual hyperbolic spherical motions. At the end of this study, by obtaining orthogonal rotation matrices in the sense of dual Lorentzian type, we have found real and dual invariants of fixed and moving axodes.

1. Introduction

Line trajectories have an important place in the kinematic design and mechanism. In spatial motions, trajectories of directed lines connected in a moving rigid body are ruled surface. Differential geometry of ruled surfaces has been widely used in spatial mechanism, Computer Aided Geometric Design (CAGD), kinematic modeling of analytical tools of robot science and manufacturing of mechanical products. On dual geometry, many applications of ruled surfaces is studied by using transference principle or E. Study mapping. By this transfer, ruled surfaces can be represented by dual spherical curves lying on unit dual sphere of dual space. Then, a motion of a line in the 3-dimensional space can be studied by the motion of a unit dual vector of dual space and the properties of this motion can be obtained [2,3,4,12,19,20,30,32]. On the one parameter spatial motion, instantaneous screw axis ISA which a pair of ruled surface generates moving axode in the moving space and fixed axode in the fixed space. Kinematics and geometry of these axodes with corresponding to dual curves have investigated by some mathematician [2,3,4,14,19]. In the planar kinematics, there exists only one curvature circle and the position of point is given in the moving plane, then the radius and center of this circle can be determined by the famous Euler-Savary formulae. Euler-Savary formulae of a line trajectory were studied. This formula have introduced on the spherical kinematics [2, 3,14,30]. Furthermore, Lorentzian space kinematics is more different and more

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interesting than the Euclidean case. Differential geometry of curves and surfaces in the Lorentzian space are studied [1,13,17,21,23,26,27,28,29]. In this space, the spherical motions are studied according to the Lorentzian casual characters of the lines. Then, the spherical motion is called hyperbolic spherical motion if the motion is determinated by moving and fixed unit hyperbolic spheres and the spherical motion is called Lorentzian spherical motion if it is determinate by moving and fixed unit Lorentzian spheres [16,22]. Similar to the Euclidean case, by considering the E. Study mapping of timelike and spacelike lines, the motions of these lines are studied in dual Lorentzian space and the properties of these motions are obtained [25]. One parameter spherical motion have investigated at reel and dual Lorentzian spaces [5,8,16,24,25]. The purpose of this paper is to introduce one parameter dual hyperbolic spherical motions on the dual Lorentzian space. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae for one parameter dual hyperbolic spherical motions. At the end of this study, we have found real and dual invariants of fixed and moving axodes by using orthogonal rotation matrices in the sense of dual Lorentzian type 3×3.

2. Lorentz Space

Let R_1^3 be a 3-dimensional Minkowski space over the field of real numbers R with the Lorentzian inner product $\langle\, , \rangle$ given by

$$\left\langle \vec{a}, \vec{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in R^3$. A vector $\vec{a} = (a_1, a_2, a_3)$ of IR_1^3 is said to be timelike if $\langle \vec{a}, \vec{a} \rangle < 0$, spacelike if $\langle \vec{a}, \vec{a} \rangle > 0$ or $\vec{a} = 0$, and lightlike (null) if $\langle \vec{a}, \vec{a} \rangle = 0$ and $\vec{a} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha}(s)$ in R_1^3 is spacelike, timelike or lightlike (null), if all of its velocity vectors $\vec{\alpha}'(s)$ are spacelike, timelike or lightlike (null), respectively [15]. The norm of a vector $\vec{a}$ is defined by $\|\vec{a}\| = \sqrt{|\langle \vec{a}, \vec{a} \rangle|}$. Now, let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ be two vectors in IR_1^3. Then the Lorentzian cross product of $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2\, ,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 - a_1 b_2).$$

The sets of the unit timelike and spacelike vectors are called hyperbolic unit sphere and Lorentzian unit sphere and denoted by

$$H_0^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in R_1^3 : \ \langle \vec{a}, \vec{a} \rangle = -1 \right\},$$

and

$$S_1^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in R_1^3 : \ \langle \vec{a}, \vec{a} \rangle = 1 \right\},$$

respectively [28].

3. Dual Space

A dual number has the form $\bar{\lambda} = \lambda + \varepsilon \lambda^*$, where λ and λ^* are real numbers and ε is called dual unit which is subject to following rules:

$$\varepsilon \neq 0\, ,\ \varepsilon^2 = 0,\ 0\varepsilon = \varepsilon 0 = 0,\ 1\varepsilon = \varepsilon 1 = \varepsilon\, .$$

We denote the set of all dual numbers by D:

$$\mathrm{D} = \left\{ \bar{\lambda} = \lambda + \varepsilon\lambda^* : \lambda, \lambda^* \in R, \varepsilon^2 = 0 \right\}.$$

Equality, addition and multiplication are defined in D by
(i) $\lambda + \varepsilon\lambda^* = \beta + \varepsilon\beta^*$ if and only if $\lambda = \beta$ and $\lambda^* = \beta^*$.
(ii)$(\lambda + \varepsilon\lambda^*) + (\beta + \varepsilon\beta^*) = (\lambda + \beta) + \varepsilon(\lambda^* + \beta^*)$.
(iii) $(\lambda + \varepsilon\lambda^*)(\beta + \varepsilon\beta^*) = (\lambda\beta) + \varepsilon(\lambda^*\beta + \beta^*\lambda)$.
respectively. Then it is easy to show that (D , + , .) is a commutative ring with unity [20].

The dual number $\bar{a} = a + \varepsilon a^*$ divide by dual number $\bar{b} = b + \varepsilon b^*$, with $b \neq 0$, is defined by

$$\frac{\bar{a}}{\bar{b}} = \frac{a}{b} + \varepsilon\frac{a^*b - ab^*}{b^2}.$$

Let f be a differentiable function with dual variable $\bar{x} = x + \varepsilon x^*$. Then the Maclaurin series generated by f is

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x),$$

where $f'(x)$ is the derivative of f with respect to x.

Let D^3 be the set of all triples of dual numbers, i.e.

$$\mathrm{D}^3 = \{ \tilde{a} = (\bar{a}_1, \bar{a}_2, \bar{a}_3) \mid \bar{a}_i \in \mathrm{D}, \ 1 \leq i \leq 3 \}.$$

The elements of D^3 are called dual vectors. A dual vector $\tilde{a}$ may be expressed in the form $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$, where $\vec{a}$ and $\vec{a}^*$ are the vectors of R^3. Let $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$, $\tilde{b} = \vec{b} + \varepsilon\vec{b}^* \in \mathrm{D}^3$ and $\bar{\lambda} = \lambda + \varepsilon\lambda^* \in \mathrm{D}$. Then we define

$$\tilde{a} + \tilde{b} = \vec{a} + \vec{b} + \varepsilon(\vec{a}^* + \vec{b}^*),$$
$$\bar{\lambda}\tilde{a} = \lambda\vec{a} + \varepsilon(\lambda\vec{a}^* + \lambda^*\vec{a}).$$

By these operations, D^3 becomes a unitary module and it is called D-module or dual space (See [7,9]).

For any dual vectors $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$ and $\tilde{b} = \vec{b} + \varepsilon\vec{b}^*$ in D^3, scalar product and vector product are defined by

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon\left(\left\langle \vec{a}, \vec{b}^* \right\rangle + \left\langle \vec{a}^*, \vec{b} \right\rangle \right),$$

and

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon\left(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b} \right),$$

respectively, where $\left\langle \vec{a}, \vec{b} \right\rangle$ and $\vec{a} \times \vec{b}$ are inner product and vector product of the vectors $\vec{a}$ and $\vec{b}$ in R^3, respectively.

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| = \sqrt{\langle \tilde{a}, \tilde{a} \rangle} = \|\vec{a}\| + \varepsilon\frac{\langle \vec{a}, \vec{a}^* \rangle}{\|\vec{a}\|}, \vec{a} \neq \vec{0}.$$

Definition 3.1 (7,30)**.** The set of all unit dual vectors is called unit dual sphere, and is denoted by $\tilde{S}^2$ and this sphere is defined by

$$\tilde{S}^2 = \left\{ \tilde{a} \in \mathrm{D}^3 \middle| \ ||\tilde{a}|| = (1, 0) \right\}.$$

Theorem 3.2. *(E. Study's Mapping): There exists a one-to-one correspondence between the points of unit dual sphere $\tilde{S}^2$ and the directed lines of the space R^3 [7].*

4. Dual Lorentzian Space

The Lorentzian inner product of two dual vectors $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^*$, $\tilde{b} = \vec{b} + \varepsilon\vec{b}^* \in \mathrm{D}^3$ is defined by

$$\left\langle \tilde{a},\ \tilde{b} \right\rangle = \left\langle \vec{a}\,,\, \vec{b} \right\rangle + \varepsilon\left(\left\langle \vec{a}\,,\, \vec{b}^* \right\rangle + \left\langle \vec{a}^*,\, \vec{b} \right\rangle \right),$$

where $\left\langle \vec{a}\,,\, \vec{b} \right\rangle$ is the Lorentzian inner product of the vectors $\vec{a}$ and $\vec{b}$ in the Minkowski 3-space R_1^3. Then, a dual vector $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^*$ is said to be dual timelike if $\vec{a}$ is timelike, dual spacelike if $\vec{a}$ is spacelike or $\vec{a} = 0$ and dual lightlike (null) if $\vec{a}$ is lightlike (null) and $\vec{a} \neq 0$ [25].

The set of all dual Lorentzian vectors is called dual Lorentzian space and it is denoted by

$$\mathrm{D}_1^3 = \left\{ \tilde{a} = \vec{a} + \varepsilon\,\vec{a}^* :\ \vec{a}\,,\, \vec{a}^* \in R_1^3 \right\}.$$

The Lorentzian cross product of dual vectors $\tilde{a}$,$\tilde{b} \in \mathrm{D}_1^3$ is defined by

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon\,(\vec{a}^* \times \vec{b} + \vec{a} \times \vec{b}^*)\,,$$

where $\vec{a} \times \vec{b}$ is the Lorentzian cross product in R_1^3.

Let $\tilde{a} = \vec{a} + \varepsilon\,\vec{a}^* \in \mathrm{D}_1^3$. Then $\tilde{a}$ is said to be unit dual timelike (resp. spacelike) vector if the vectors $\vec{a}$ and $\vec{a}^*$ satisfy the following equations:

$$< \vec{a}, \vec{a} >= -1 \ \ (resp.\ \ < \vec{a}, \vec{a} >= 1), \ \ < \vec{a}\,,\, \vec{a}^* > = 0.$$

The set of all unit dual timelike vectors is called dual hyperbolic unit sphere, and is denoted by $\tilde{H}_0^2$. Similarly, the set of all unit dual spacelike vectors is called dual Lorentzian unit sphere, and is denoted by $\tilde{S}_1^2$ and these spheres are defined by

$$\tilde{H}_0^2 = \left\{ \tilde{a} \in \mathrm{D}_1^3 :\ \langle \tilde{a}, \tilde{a} \rangle = -1 \right\}, \quad \tilde{S}_1^2 = \left\{ \tilde{a} \in \mathrm{D}_1^3 :\ \langle \tilde{a}, \tilde{a} \rangle = 1 \right\}$$

respectively (See [21,25,28]).

Definition 4.1 (18,31)**.** ***(i)*** Dual hyperbolic angle: Let $\tilde{a}$ and $\tilde{b}$ be dual timelike vectors in D_1^3. Then the dual angle between $\tilde{a}$ and $\tilde{b}$ is defined by $< \tilde{a}, \tilde{b} >= -\left\| \tilde{a} \right\| \left\| \tilde{b} \right\| \cosh \bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual hyperbolic angle*. The geometric interpretation of dual hyperbolic angle is that θ is the real hyperbolic angle between timelike lines $L_1,\ L_2$ corresponding to the dual timelike unit vectors $\tilde{a},\ \tilde{b}$, respectively, and θ^* is the shortest distance between those lines.

(ii) Dual central angle: Let $\tilde{a}$ and $\tilde{b}$ be dual spacelike vectors in D_1^3 that span a dual timelike vector subspace. The dual angle between $\tilde{a}$ and $\tilde{b}$ is defined by $\left| < \tilde{a}, \tilde{b} > \right| = \left\| \tilde{a} \right\| \left\| \tilde{b} \right\| \cosh \bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual central angle*. The geometric interpretation of dual central angle is that θ is the real central angle between spacelike lines $L_1,\ L_2$ corresponding to the dual spacelike unit vectors $\tilde{a},\ \tilde{b}$ in D_1^3 that span a dual timelike vector subspace, respectively, and θ^* is the shortest distance between those lines.

(iii) Dual spacelike angle: Let $\tilde{a}$ and $\tilde{b}$ be dual spacelike vectors in D_1^3 that span a dual spacelike vector subspace. Then the angle between $\tilde{a}$ and $\tilde{b}$ is defined by $<\tilde{a}, \tilde{b}>= \|\tilde{a}\| \left\|\tilde{b}\right\| \cos\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual spacelike angle*. The geometric interpretation of dual spacelike angle is that θ is the real spacelike angle between spacelike lines $L_1,\ L_2$ corresponding to the dual spacelike unit vectors $\tilde{a},\ \tilde{b}$ in D_1^3 that span a dual spacelike vector subspace, respectively, and θ^* is the shortest distance between those lines.

(iv) Dual timelike angle: Let $\tilde{a}$ be a dual spacelike vector and $\tilde{b}$ be a dual timelike vector in D_1^3. Then the angle between $\tilde{a}$ and $\tilde{b}$ is defined by $\left|<\tilde{a}, \tilde{b}>\right| = \|\tilde{a}\| \left\|\tilde{b}\right\| \sinh\bar{\theta}$. The dual number $\bar{\theta} = \theta + \varepsilon\theta^*$ is called the *dual timelike angle*. The geometric interpretation of dual timelike angle is that θ is the real timelike angle between spacelike line L_1 and timelike line L_2 corresponding to the dual spacelike unit vector $\tilde{a}$ and timelike unit vector $\tilde{b}$, respectively, and θ^* is the shortest distance between those lines.

Theorem 4.2 (E. Study's Mapping for Lorentzian Space)**.** *: The dual timelike (respectively spacelike) unit vectors of the dual hyperbolic (respectively Lorentzian) unit sphere $\tilde{H}_0^2$ (respectively $\tilde{S}_1^2$) are in one-to-one correspondence with the directed timelike (respectively spacelike) lines of the Minkowski 3-space IR_1^3 [25].*

5. Differential Geometry of Dual Hyperbolic Spherical Curves

$\tilde{q} = \vec{q}(t) + \varepsilon\vec{q}^*(t)$ be a unit dual timelike vector is connected to a real parameter t , this vector draws a curve on the unit dual hyperbolic sphere $\tilde{H}_0^2$. Applying Study's map, this curve represents a timelike ruled surface M. If the ruling $\vec{q}$ is timelike, then the ruled surface M is said to be of type M_-^1 [11]. Therefore, differential geometry of dual hyperbolic spherical curves corresponds to differential geometry of timelike ruled surface M_-^1.

Let $d\bar{\theta} = d\theta + \varepsilon d\theta^*$ dual arc-length of dual hyperbolic spherical curve $\tilde{q} = \tilde{q}(t)$. Thus, we have

$$d\bar{\theta}^2 = \langle d\vec{q}, d\vec{q}\rangle + 2\varepsilon \langle d\vec{q}, d\vec{q}^*\rangle \tag{5.1}$$

Hence we obtain

$$d\theta^2 = \langle d\vec{q}, d\vec{q}\rangle\,,\ \ d\theta d\theta^* = \langle d\vec{q}, d\vec{q}^*\rangle\,. \tag{5.2}$$

Therefore, differential invariant of timelike ruled surface M_-^1 given by

$$\delta_q = \frac{d\theta^*}{d\theta} = \frac{\langle d\vec{q}, d\vec{q}^*\rangle}{\langle d\vec{q}, d\vec{q}\rangle} = \frac{\left\langle \vec{q}', \vec{q}^{*\prime}\right\rangle}{\langle \vec{q}', \vec{q}'\rangle}. \tag{5.3}$$

The invariant δ_q is said to be distribution parameter (or drall) of the timelike ruled surface. If $\left\langle \vec{q}', \vec{q}'\right\rangle = 0$, the ruled surface is said to be timelike cylindrical and we except this case [17,21].

We now give an orthonormal moving frame of a dual hyperbolic spherical curve as follows:

$$\tilde{q}=\tilde{q}(t),\quad \tilde{h}=\frac{\tilde{q}'}{\|\tilde{q}'\|}\quad,\quad \tilde{a}=-\tilde{q}\times\tilde{h}. \tag{5.4}$$

This frame is called the Blaschke frame, and the corresponding lines intersect at the striction point of timelike ruled surface M_-^1. The set of the striction points constitute a curve $C=C(t)$ lying on the timelike ruled surface M_-^1 and is called striction curve. $\tilde{h}$ and $\tilde{a}$ are known as the central tangent and the central normal of the timelike ruled surface M_-^1. So, Blaschke formula is given by

$$\begin{cases} \tilde{q}'=\bar{k}_1\tilde{h}, & \bar{k}_1=\sqrt{\langle\tilde{q}',\tilde{q}'\rangle} \\ \tilde{h}'=\bar{k}_1\tilde{q}+\bar{k}_2\tilde{a}, & \bar{k}_2=-\frac{(\tilde{q},\tilde{q}',\tilde{q}'')}{\langle\tilde{q}',\tilde{q}'\rangle} \\ \tilde{a}'=-\bar{k}_2\tilde{h} & \end{cases} \tag{5.5}$$

and

$$\frac{dC}{dt}=\cosh\bar{\phi}\tilde{q}+\sinh\bar{\phi}\tilde{a} \tag{5.6}$$

where $\bar{k}_1,\ \bar{k}_2$ are called the Blaschke's invariants. From (5.5) for dual vector $\tilde{\psi}=\vec{\psi}+\varepsilon\vec{\psi}^*=-\bar{k}_2\tilde{q}-\bar{k}_1\tilde{a}$ we can write

$$\tilde{q}'=\tilde{\psi}\times\tilde{q},\quad \tilde{h}'=\tilde{\psi}\times\tilde{h},\quad \tilde{a}'=\tilde{\psi}\times\tilde{a},$$

where dual vector $\tilde{\psi}=\vec{\psi}+\varepsilon\vec{\psi}^*=-\bar{k}_2\tilde{q}-\bar{k}_1\tilde{a}$ is called the dual instantaneous Pfaffian vector. The pole vector and the Steiner vector of the motion are given by

$$\tilde{\psi}=\left\|\tilde{\psi}\right\|\tilde{P},\quad \tilde{d}=\oint\tilde{\psi}, \tag{5.7}$$

respectively [17,21].

6. One Parameter Dual Hyperbolic Spherical Motions

Let two coordinate systems $\left\{O';\vec{q}_f,\vec{h}_f,\vec{a}_f\right\}$ and $\left\{O;\vec{q}_m,\vec{h}_m,\vec{a}_m\right\}$ be orthonormal coordinate systems which one represents fixed space L_2 and which one represents moving space L_3 in R_1^3 , respectively, where $\vec{q}_f$ and $\vec{q}_m$ are assumed as timelike vectors. In order to introduce the motion L_3/L_2 let take the coordinate system $\left\{Q;\vec{q},\vec{h},\vec{a}\right\}$ as an orthonormal relative system which represent the relative space L_1. Let Σ_1, Σ_2 and Σ_3 be unit dual hyperbolic spheres with same center O. According to the E. Study mapping, the points of unit dual hyperbolic spheres Σ_1, Σ_2 and Σ_3 can be represented by dual orthogonal systems $\left\{O;\tilde{q},\tilde{h},\tilde{a}\right\}$, $\left\{O;\tilde{q}_f,\tilde{h}_f,\tilde{a}_f\right\}$ and $\left\{O;\tilde{q}_m,\tilde{h}_m,\tilde{a}_m\right\}$, respectively. Therefore, the motions L_1/L_2, L_1/L_3 and L_3/L_2 can be considered as dual hyperbolic spherical motions Σ_1/Σ_2, Σ_1/Σ_3 and Σ_3/Σ_2, respectively.

Let A_f and A_m be a unit dual Lorentzian orthogonal matrices of type 3×3 and we can write

$$\Sigma_1=A_f\Sigma_2,\qquad \Sigma_1=A_m\Sigma_3, \tag{6.1}$$

where

$$\Sigma_1 = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \ \Sigma_2 = \begin{bmatrix} \tilde{q}_f \\ \tilde{h}_f \\ \tilde{a}_f \end{bmatrix}, \ \Sigma_3 = \begin{bmatrix} \tilde{q}_m \\ \tilde{h}_m \\ \tilde{a}_m \end{bmatrix}$$

are dual column matrices. The elements of the matrices A_f and A_m are continuous and differentiable functions of dual parameter $\bar{t} = t + \varepsilon t^*$. In order to introduce one parameter hyperbolic motion we assume that $t^* = 0$.

Differential of the relative orthonormal coordinate frame Σ_1 with respect to unit dual fixed and moving hyperbolic spheres Σ_2 and Σ_3 are

$$d\Sigma_{1f} = dA_f\Sigma_2 = dA_f(A_f)^{-1}\Sigma_1, \quad d\Sigma_{1m} = dA_m\Sigma_3 = dA_m(A_m)^{-1}\Sigma_1. \tag{6.2}$$

By choosing $\tilde{\Omega}_f = dA_f(A_f)^{-1}, \quad \tilde{\Omega}_m = dA_m(A_m)^{-1}$ Eq. (6.2) can be rewritten as follows

$$d\Sigma_{1f} = \tilde{\Omega}_f\Sigma_1, \quad d\Sigma_{1m} = \tilde{\Omega}_m\Sigma_1 \tag{6.3}$$

where $\tilde{\Omega}_f$ and $\tilde{\Omega}_m$ matrices are anti-symmetric in the sense of Lorentzian.

During the one parameter dual hyperbolic motion Σ_3/Σ_2 the differential velocity vector of a fixed dual hyperbolic point $\tilde{X}_i = \vec{x}_i + \varepsilon\vec{x}_i^* \ (1 \leq i \leq 3)$ on Σ_3 is

$$\frac{d\tilde{X}}{dt} = \tilde{\Omega} \times \tilde{X} \tag{6.4}$$

where $\tilde{\Omega} = \vec{\omega} + \varepsilon\vec{\omega}^*$ is called the instantaneous dual hyperbolic Pfaffian vector of the motion Σ_3/Σ_2. The Pfaffian dual vector $\tilde{\Omega}$ of the motion Σ_3/Σ_2, at the instant t, is like to the Darboux vector of space curves in the differential geometry. In this case ω and ω^* correspond to instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of corresponding hyperbolic motion L_3/L_2, respectively. The dual number $\left\|\tilde{\Omega}\right\| = \bar{\Omega} = \omega + \varepsilon\omega^*$ is said to be dual angular speed of the one parameter dual hyperbolic motion Σ_3/Σ_2.

We consider the following identification

$$\bar{\Omega} = \begin{bmatrix} 0 & \bar{\Omega}_3 & -\bar{\Omega}_2 \\ \bar{\Omega}_3 & 0 & -\bar{\Omega}_1 \\ -\bar{\Omega}_2 & \bar{\Omega}_1 & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \bar{\Omega}_1 \\ \bar{\Omega}_2 \\ \bar{\Omega}_3 \end{bmatrix} = \tilde{\Omega}. \tag{6.5}$$

Lemma 6.1. *For a one parameter dual hyperbolic spherical motion the following conditions are provided:*

(i) *The skew-symmetric in the sense of Lorentzian matrix of type* 3×3 *determined by* $\tilde{\Omega}_m(t) = A^{-1}A'$ *is called the moving polode.*

(ii) *The skew-symmetric in the sense of Lorentzian matrix of type* 3×3 *determined by* $\tilde{\Omega}_f(t) = A'A^{-1}$ *is called the fixed polode.*

(iii) *The moving and fixed polodes are related by* $\tilde{\Omega}_f(t) = adA(t)\tilde{\Omega}_m(t)$*, where* $adA\tilde{\Omega}_m = A\tilde{\Omega}_mA^{-1}$.

(iv) $\left\|\tilde{\Omega}_f\right\| = \left\|\tilde{\Omega}_m\right\|$.

(v) $\tilde{q}_f(t) = \frac{\tilde{\Omega}_f(t)}{\|\tilde{\Omega}_f(t)\|}$ *and* $\tilde{q}_m(t) = \frac{\tilde{\Omega}_m(t)}{\|\tilde{\Omega}_m(t)\|}$ *are called the fixed axode and moving axodes of the one parameter dual hyperbolic motion* Σ_3/Σ_2*, respectively.*

(vi) $\frac{d\tilde{q}_f}{dt} = adA\frac{d\tilde{q}_m}{dt} \Leftrightarrow \frac{d\tilde{q}_f}{dt} = A\frac{d\tilde{q}_m}{dt}A^{-1}$ *[5].*

During the dual hyperbolic motion Σ_3/Σ_2, the differentiable curve

$$t \in \mathrm{R} \to \tilde{q}_m(t) \in \Sigma_3 \tag{6.6}$$

states a differentiable family of straight lines on the moving axode. Now give an orthonormal moving frame along curve $\tilde{q}_m(t)$;

$$\tilde{q}_m = \tilde{q}_m(t)\,(timelike), \tilde{h}_m = \left(\frac{d\tilde{q}_m}{dt}\right)\left\|\frac{d\tilde{q}_m}{dt}\right\|^{-1}, \quad \tilde{a}_m = -\tilde{q}_m \times \tilde{h}_m. \tag{6.7}$$

This frame is called the Blaschke frame, and the corresponding lines intersect at the striction point of the axode $\tilde{q}_m = \tilde{q}_m(t)$. $\tilde{a}_m$ and $\tilde{h}_m$ are described as the central tangent and central normal of the timelike ruled surface $\tilde{q}_m = \tilde{q}_m(t)$, respectively. Let Σ_1^m be a dual unit hyperbolic sphere generated by the set $\left\{O; \tilde{q}_m, \tilde{h}_m, \tilde{a}_m\right\}$. Therefore, the motion Σ_1^m/Σ_3 is given by

$$\begin{bmatrix} d\tilde{q}_m \\ d\tilde{h}_m \\ d\tilde{a}_m \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1m} & 0 \\ \bar{k}_{1m} & 0 & \bar{k}_{2m} \\ 0 & -\bar{k}_{2m} & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_m \\ \tilde{h}_m \\ \tilde{a}_m \end{bmatrix} \tag{6.8}$$

where dual functions

$$\bar{k}_{1m} = k_{1m} + \varepsilon k_{1m}^* = \left\|\frac{d\tilde{q}_m}{dt}\right\|, \quad \bar{k}_{2m} = k_{2m} + \varepsilon k_{2m}^* = -\frac{\det\left(\tilde{q}_m, \frac{d\tilde{q}_m}{dt}, \frac{d^2\tilde{q}_m}{dt^2}\right)}{\bar{k}_{1m}^2} \tag{6.9}$$

are called Blaschke invarians of the moving axode. Striction curve is given by

$$\frac{dC^m}{dt} = \bar{k}_{2m}^*\tilde{q}_m + \bar{k}_{1m}^*\tilde{a}_m. \tag{6.10}$$

In this case dual functions in Eq. (6.9) abide by

$$\bar{k}_{1m} = k_{1m} + \varepsilon \sinh \bar{\sigma}_m, \quad \bar{k}_{2m} = k_{2m} + \varepsilon \cosh \bar{\sigma}_m \tag{6.11}$$

where $\bar{\sigma}_m$ is the striction angle measuring the derivation of the generating lines of $\tilde{q}_m(t)$ from the striciton curve. The distribution of timelike moving axode is

$$\lambda_m = \frac{k_{1m}^*}{k_{1m}} = \frac{\sinh \bar{\sigma}_m}{k_{1m}}. \tag{6.12}$$

During the one parameter dual hyperbolic motion Σ_3/Σ_2, the ISA on fixed hyperbolic sphere Σ_2 generates the fixed polode which accepts the Blaschke frame

$$\tilde{q}_f = \tilde{q}_f(t)(timelike), \; \tilde{h}_f = \left(\frac{d\tilde{q}_f(t)}{dt}\right)\left\|\frac{d\tilde{q}_f}{dt}\right\|^{-1}, \quad \tilde{a}_f = -\tilde{q}_f \times \tilde{h}_f. \tag{6.13}$$

Similarly, the set $\left\{O; \tilde{q}_f, \tilde{h}_f, \tilde{a}_f\right\}$ describes a unit dual hyperbolic sphere Σ_1^f, and the hyperbolic spherical motion Σ_1^f/Σ_2 is given by

$$\begin{bmatrix} d\tilde{q}_f \\ d\tilde{h}_f \\ d\tilde{a}_f \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1f} & 0 \\ \bar{k}_{1f} & 0 & \bar{k}_{2f} \\ 0 & -\bar{k}_{2f} & 0 \end{bmatrix} \begin{bmatrix} \tilde{q}_f \\ \tilde{h}_f \\ \tilde{a}_f \end{bmatrix} \tag{6.14}$$

where the dual functions

$$\bar{k}_{1f} = k_{1f} + \varepsilon k_{1f}^* = \left\| \frac{d\tilde{q}_f}{dt} \right\|, \quad \bar{k}_{2f} = k_{2f} + \varepsilon k_{2f}^* = -\frac{\det\left(\tilde{q}_f, \frac{d\tilde{q}_f}{dt}, \frac{d^2\tilde{q}_f}{dt^2}\right)}{\bar{k}_{1f}^2} \tag{6.15}$$

are the Blaschke invariants of fixed polode. Striction curve is given by

$$\frac{dC^f}{dt} = \bar{k}_{2f}^* \tilde{q}_f + \bar{k}_{1f}^* \tilde{a}_f. \tag{6.16}$$

Likewise the dual functions in (6.15) are

$$\bar{k}_{1f} = k_{1f} + \varepsilon \sinh \bar{\sigma}_f, \quad \bar{k}_{2f} = k_{2f} + \varepsilon \cosh \bar{\sigma}_f, \tag{6.17}$$

where $\bar{\sigma}_f$ is the striction angle between the lines of $\tilde{q}_f(t)$ and the striction curve. Therefore, the distribituon parameter of the fixed axode is

$$\lambda_f = \frac{k_{1f}^*}{k_{1f}} = \frac{\sinh \bar{\sigma}_f}{k_{1f}}. \tag{6.18}$$

Theorem 6.2. *Relations between Blaschke invariants of the timelike axodes are given by the equalities*

$$\bar{k}_{1m} = \bar{k}_{1f}, \quad \bar{k}_{2m} - \bar{k}_{2f} = \left\| \tilde{\Omega} \right\|. \tag{6.19}$$

Proof. Using (6.8) and (6.14) and Lemma (6.1) can be easily proved. □

Consequently, the following corollary can be given.

Corollary 6.3. During the one parameter hyperbolic spherical motion Σ_3/Σ_2, the moving polode is contact with the fixed polode along ISA in the first order at any instant t. The common distribution parameter of timelike axodes is

$$\lambda := \lambda_m = \lambda_f = \frac{k_1^*}{k_1}. \tag{6.20}$$

Let Σ_1 be unit dual hyperbolic sphere generated by the system $\left\{O; \tilde{q}(timelike), \tilde{h}, \tilde{a}\right\}$. In this system, $\tilde{a}(t) = a(t) + \varepsilon a^*(t)$ is the common perpendicular of $\tilde{q}(t)$ and $\tilde{q}(t+dt)$ and $\tilde{a}(t) = a(t) + \varepsilon a^*(t) = -\tilde{q} \times \tilde{h}$ and; $\tilde{q}, \tilde{h}$ and $\tilde{a}$ correspond to orthogonal lines in the Minkowski 3-space R_1^3. Then, the derivative equations of the one parameter dual hyperbolic spherical motions Σ_1/Σ_3 and Σ_1/Σ_2 are

$$\left.\frac{d\tilde{q}}{dt}\right|_m = C(M)\tilde{q}(t), \quad \tilde{q}(t) = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \quad C(M) = \begin{bmatrix} 0 & \bar{k}_1 & 0 \\ \bar{k}_1 & 0 & \bar{k}_{2m} \\ 0 & -\bar{k}_{2m} & 0 \end{bmatrix}, \tag{6.21}$$

and

$$\left.\frac{d\tilde{q}}{dt}\right|_f = C(F)\tilde{q}(t), \qquad \tilde{q}(t) = \begin{bmatrix} \tilde{q} \\ \tilde{h} \\ \tilde{a} \end{bmatrix}, \quad C(F) = \begin{bmatrix} 0 & \bar{k}_1 & 0 \\ \bar{k}_1 & 0 & \bar{k}_{2f} \\ 0 & -\bar{k}_{2f} & 0 \end{bmatrix}, \tag{6.22}$$

respectively,where

$$\bar{k}_1 = k_1 + \varepsilon k_1^*, \ \ \bar{k}_{2m} = k_{2m} + \varepsilon k_{2m}^*, \ \ \bar{k}_{2f} = k_{2f} + \varepsilon k_{2f}^* \tag{6.23}$$

are the Blaschke invariants of the one parameter dual hyperbolic spherical motion.

7. The approach to a timelike ruled surface with axodes

In this section, we introduce geometrical and kinematic meanings of dual invariants of hyperbolic polodes. In order to this analysis we consider a timelike point $\tilde{X}$ on the unit dual hyperbolic sphere such that its coordinates are

$$-\bar{X}_1^2 + \bar{X}_2^2 + \bar{X}_3^2 = -1, \quad \tilde{X} = X^T\tilde{q} \quad \tilde{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \\ \bar{X}_3 \end{bmatrix}. \tag{7.1}$$

If $\tilde{X}$ is a function of t, the velocity of $\tilde{X}$ at the instant t with according to the moving unit dual hyperbolic sphere Σ_3 and fixed unit dual hyperbolic sphere Σ_2 are

$$\left.\frac{d\tilde{X}}{dt}\right|_m = \frac{d\tilde{X}^T}{dt}\tilde{q} + \tilde{X}^T \left.\frac{d\tilde{q}}{dt}\right|_m \tag{7.2}$$

and

$$\left.\frac{d\tilde{X}}{dt}\right|_f = \frac{d\tilde{X}^T}{dt}\tilde{q} + \tilde{X}^T \left.\frac{d\tilde{q}}{dt}\right|_f \tag{7.3}$$

respectively. From (6.21) and (6.22), we get

$$\left.\frac{d\tilde{X}}{dt}\right|_m = \left(\frac{d\tilde{X}^T}{dt} + \tilde{X}^T C(M)\right)\tilde{q} \tag{7.4}$$

and

$$\left.\frac{d\tilde{X}}{dt}\right|_f = \left(\frac{d\tilde{X}^T}{dt} + \tilde{X}^T C(F)\right)\tilde{q}\,. \tag{7.5}$$

If the line $\tilde{X}$ is fixed relative to the moving unit dual hyperbolic sphere, then the derivative $\left.\frac{d\tilde{X}}{dt}\right|_m = 0$. That is we have

$$\frac{d\tilde{X}^T}{dt} = -\tilde{X}^T C(M). \tag{7.6}$$

Now, assume that $\tilde{X}$ is fixed according to the moving unit dual hyperbolic sphere Σ_3 and let us compute its velocity according to the fixed unit dual hyperbolic sphere Σ_2. Then we obtain equation

$$\frac{d\tilde{X}}{dt} = \tilde{X}^T(C(F) - C(M))\tilde{q}. \tag{7.7}$$

Let us define a matrix $C(R)$ by

$$C(R) = C(F) - C(M). \tag{7.8}$$

Then (7.7) can be rewritten as

$$\frac{d\tilde{X}}{dt} = \tilde{X}^T(C(R))\tilde{q}. \tag{7.9}$$

We have an axial dual vector $\tilde{D}_r = d + \varepsilon d^*$ such that

$$C(R)\tilde{X} = \tilde{D}_r \times \tilde{X}. \tag{7.10}$$

Therefore (7.9) can be stated as

$$\frac{d\tilde{X}}{dt} = \tilde{D}_r \times \tilde{X}, \quad \tilde{D}_r = \tilde{D}_f - \tilde{D}_m = -\bar{\Omega}\tilde{q}, \tag{7.11}$$

where $\left\|\tilde{\Omega}\right\| = \bar{\Omega} = \omega + \varepsilon\omega^*$. Then from Theorem 6.2 and (7.11) we have

$$\frac{d\tilde{X}}{dt} = (-\bar{X}_3\bar{\Omega})\tilde{h} + (\bar{X}_2\bar{\Omega})\tilde{a}. \tag{7.12}$$

From (7.11) and (7.12), it follows that the acceleration of $\tilde{X}$ is given by

$$\frac{d^2\tilde{X}}{dt^2} = (-\bar{\Omega}\bar{k}_1\bar{X}_3)\tilde{q} + (-\bar{\Omega}'\bar{X}_3 - \bar{\Omega}^2\bar{X}_2)\tilde{h} + (-\bar{\Omega}\bar{k}_1\bar{X}_1 + \bar{\Omega}'\bar{X}_2 - \bar{\Omega}^2\bar{X}_3)\tilde{a}. \tag{7.13}$$

8. Line complex during one parameter hyperbolic spherical motion

In this section, we investigate timelike ruled surface generated by the timelike line $\tilde{X}$. Now we describe a frame moving along the curve $\tilde{X}(t)$ on the unit hyberbolic sphere Σ_2. According to transference principle, this curve corresponds to a timelike ruled surface in the fixed Lorentzian space L_2. The Blaschke frame along $\tilde{X}(t)$ is defined as follows:

$$\tilde{E}_1 = \tilde{X} = \bar{X}_1\tilde{q} + \bar{X}_2\tilde{h} + \bar{X}_3\tilde{a}, (time) \tag{8.1}$$

$$\tilde{E}_2 = \frac{\tilde{X}'}{\left\|\tilde{X}'\right\|} = \frac{-\bar{X}_3\tilde{h} + \bar{X}_2\tilde{a}}{\sqrt{\bar{X}_2^2 + \bar{X}_3^2}} \tag{8.2}$$

$$\tilde{E}_3 = -(\tilde{E}_1 \times \tilde{E}_2) = -\left(\frac{(1 + \bar{X}_1^2)\tilde{q} + \bar{X}_1\bar{X}_2\tilde{h} + \bar{X}_1\bar{X}_3\tilde{a}}{\sqrt{\bar{X}_2^2 + \bar{X}_3^2}}\right). \tag{8.3}$$

The unit dual timelike vector $\tilde{E}_1$ is one-to-one correspondence with the directed timelike line of the Minkowski 3-space IR_1^3 and dual spacelike unit vectors $\tilde{E}_2$, $\tilde{E}_3$ are one-to-one correspondence with the directed spacelike lines of the Minkowski 3-space. The Blaschke derivative formulas are

$$\frac{d}{dt}\begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{k}_{1x} & 0 \\ \bar{k}_{1x} & 0 & \bar{k}_{2x} \\ 0 & -\bar{k}_{2x} & 0 \end{bmatrix} \begin{bmatrix} \tilde{E}_1 \\ \tilde{E}_2 \\ \tilde{E}_3 \end{bmatrix} \tag{8.4}$$

where

$$\begin{aligned} \bar{k}_{1x} &= k_{1x} + \varepsilon k_{1x}^* = \left\| \tfrac{d\tilde{X}}{dt} \right\| = \bar{\Omega}\sqrt{\bar{X}_2^2 + \bar{X}_3^2}, \\ \bar{k}_{2x} &= k_{2x} + \varepsilon k_{2x}^* = -\tfrac{\det(\tilde{X}, \tilde{X}', \tilde{X}'')}{(\bar{k}_{1x})^2} = -(\bar{\Omega}\bar{X}_1 + \tfrac{\bar{k}_{1x}\bar{X}_3}{\bar{X}_2^2 + \bar{X}_3^2}) \end{aligned} \tag{8.5}$$

are Blaschke invariants of the timelike curve $\tilde{X}(t)$.

Theorem 8.1. *During the one parameter dual hyperbolic spherical motion Σ_3/Σ_2, consider a set of lines are contact with the timelike moving axode and these lines are generators of timelike ruled surfaces having the same distribution parameter in the fixed Lorentzian space L_2. Therefore this set of lines belongs to a quadratic line complex.*

Proof. The distribution parameter of the timelike ruled surface generated by the line $\tilde{X}$ from (8.5) can be stated by

$$\lambda_x = \frac{\bar{k}_{1x}^*}{\bar{k}_{1x}} = \frac{x_2 x_2^* + x_3 x_3^* + h(x_2^2 + x_3^2)}{(x_2^2 + x_3^2)}. \tag{8.6}$$

This equation can be applied to determine those lines of timelike moving axode that trace timelike ruled surfaces having the same distribution parameter. This set of timelike lines is called a line complex and is stated by the equation

$$x_2 x_2^* + x_3 x_3^* + (h - \lambda_x)(x_2^2 + x_3^2) = 0. \tag{8.7}$$

This equation shows a quadratic line complex. □

Now let $p(x, y, z)$ be the position vector of an arbitrary point on the timelike line $\tilde{X}$. In order to introduce (8.7) If we use Lorentzian cross product then,

$$x^* = p \times x$$

$$(x_1^*, x_2^*, x_3^*) = \begin{bmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 \\ x & y & z \\ x_1 & x_2 & x_3 \end{bmatrix} = (yx_3 - zx_2, xx_3 - zx_1, yx_1 - xx_2). \tag{8.8}$$

After that, substituting (8.8) into (8.7) we have

$$x_1 x_3 y - x_1 x_2 z + (h - \lambda_x)(x_2^2 + x_3^2) = 0. \tag{8.9}$$

This equation represent that the timelike lines $\tilde{X}$ of timelike moving axode that trace timelike ruled surfaces with the same distribution parameter lie on a plane parallel to the ISA of the one parameter Lorentzian spatial motion L_3/L_2.

From (8.9), we have two different cases: In the case of $\lambda_x = h$ the distribution parameter is associated with the lines in planes passing through the ISA. In the case of $\lambda_x = 0$, the timelike line $\tilde{X}$ of the timelike moving axode, generate a developable timelike ruled surface, (8.9) reduces to

$$x_1x_3y - x_1x_2z + h(x_2^2 + x_3^2) = 0. \tag{8.10}$$

Now, kinematic investigation of Blaschke frame is given by using Blaschke invariants $\bar{k}_{1x} = k_{1x} + \varepsilon k_{1x}^*$ and $\bar{k}_{2x} = k_{2x} + \varepsilon k_{2x}^*$. To realize this, we define dual vector

$$\tilde{D}_x = -\bar{k}_{2x}\tilde{E}_1 - \bar{k}_{1x}\tilde{E}_3 \tag{8.11}$$

known as Darboux's vector. $\left\|\tilde{D}\right\| = \sqrt{\bar{k}_{1x}^2 - \bar{k}_{2x}^2} = \omega_x + \varepsilon\omega_x^*$ is the angular speed of timelike line $\tilde{E}_1$ about the Darboux vector.

$$\omega_x = \sqrt{\left|k_{1x}^2 - k_{2x}^2\right|}\,,\;\; \omega_x^* = \frac{k_{1x}k_{1x}^* - k_{2x}k_{2x}^*}{\sqrt{\left|k_{1x}^2 - k_{2x}^2\right|}} \tag{8.12}$$

are the rotational angular speed and translational angular speed of timelike line $\tilde{E}_1$, respectively. The pitch of $\tilde{E}_1$ along the Darboux vector is

$$h_x = \frac{\omega_x^*}{\omega_x} = \frac{k_{1x}k_{1x}^* - k_{2x}k_{2x}^*}{k_{1x}^2 - k_{2x}^2}. \tag{8.13}$$

Disteli axis is axis of hyperbolic motion of the timelike line $\tilde{E}_1$ and it's defined by

$$\tilde{U} = \frac{\tilde{D}_x}{\left\|\tilde{D}_x\right\|} = \frac{-\bar{k}_{2x}\tilde{E}_1 - \bar{k}_{1x}\tilde{E}_3}{\sqrt{\bar{k}_{1x}^2 - \bar{k}_{2x}^2}} \quad . \tag{8.14}$$

From (8.14), the Disteli axis is parallel to tangent plane of timelike ruled surface $\tilde{X} = \tilde{X}(t)$, and is unit dual timelike vector. Then the ISA of one parameter hyperbolic spherical motion Σ_3/Σ_2 and the Disteli axis lie on a single great dual hyperbolic circle determined by the intersection of $\tilde{E}_1\tilde{E}_3$-plane and the unit dual hyperbolic sphere Σ_2. Now let $\Delta = \delta + \varepsilon\delta^*$ be the dual hyperbolic angle between the Disteli axis and the timelike line $\tilde{X}$; then we have

$$\tilde{U} = -\cosh\Delta\,\tilde{E}_1 - \sinh\Delta\,\tilde{E}_3, \tag{8.15}$$

where $\Delta = \delta + \varepsilon\delta^*$ is dual hyperbolic spherical radius of curvature. For differential of (8.15) we have

$$\tilde{U}' = (-sinh\Delta\,\tilde{E}_1 - \cosh\Delta\,\tilde{E}_3)\Delta' + (\bar{k}_{2x}\sinh\Delta - \bar{k}_{1x}\cosh\Delta)\tilde{E}_2 \tag{8.16}$$

and

$$\coth \Delta = \frac{\bar{k}_{2x}}{\bar{k}_{1x}}. \tag{8.17}$$

This equation shows that the relationship between the dual hyperbolic spherical curvature $\bar{\rho}$ and the dual hyperbolic spherical radius of curvature is

$$\bar{\rho} = \rho + \varepsilon\rho^* = \coth \Delta. \tag{8.18}$$

9. During one parameter hyperbolic spherical motion line trajectories and Euler Savary formulae

In this section, by using dual hyperbolic angle we give a different method for deriving a new Euler-Savary formula of Lorentzian spatial kinematics. This means that we investigate an oriented timelike line in the moving Lorentzian space L_3 with a fixed hyperbolic angle with respect to a given timelike line in the fixed Lorentzian space L_2.

Theorem 9.1. *Let Σ_3/Σ_2 be the one parameter dual hyperbolic motion. In this case, the relation between the spherical radii of curvature of the pole curves is given by*

$$(\coth \bar{\theta}_c - \coth \bar{\theta}) \sin \bar{\phi} = \bar{\rho} = \frac{\bar{\Omega}}{\bar{k}_1} = \coth \bar{\gamma}_f - \coth \bar{\gamma}_m, \tag{9.1}$$

where $\bar{\gamma}_f$ and $\bar{\gamma}_m$ are the dual hyperbolic spherical curvatures, $\bar{\Omega}$ is the dual screw velocity and $\bar{k}_{1m} = \bar{k}_{1f}$ are dual invariants.

Proof. For instantaneous fixed timelike line $\tilde{X}$ of the hyperbolic motion Σ_3/Σ_2, we present the dual hyperbolic angle $\bar{\theta} = \theta + \varepsilon\theta^*$ and dual spacelike angle $\bar{\phi} = \phi + \varepsilon\phi^*$ to determine the direction of timelike line $\tilde{X}$. Because $\tilde{X}$ is a unit dual timelike vector, we can give the components of $\tilde{X}$ in the following form:

$$\tilde{X} = \cosh \bar{\theta}\tilde{q} + \sinh \bar{\theta}\tilde{L}, \quad \tilde{L} = \cos \bar{\phi}\tilde{h} + \sin \bar{\phi}\tilde{a}. \tag{9.2}$$

The dual hyperbolic angle $\bar{\theta} = \theta + \varepsilon\theta^*$ describes the position of timelike line $\tilde{X}$ relative to the ISA of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2.

A similar set of coordinates may be used to determine the timelike Disteli axis $\tilde{U}$ of the timelike ruled surface $\tilde{X} = \tilde{X}(t)$. Since central normal $\tilde{E}_2$ is also normal to the timelike Disteli axis, it is determined by the same dual central angle $\bar{\varphi}$ about the ISA of the hyperbolic motion Σ_3/Σ_2. Describing its dual hyperbolic angle with the ISA by $\bar{\theta}_c = \theta_c + \varepsilon\theta_c^*$, we can write

$$\tilde{U} = \cosh \bar{\theta}_c\, \tilde{q} + \sinh \bar{\theta}_c \cos \bar{\varphi}\, \tilde{h} + \sinh \bar{\theta}_c \sin \bar{\varphi}\, \tilde{a}. \tag{9.3}$$

From (9.2) and (9.3) we have

$$\left\langle \tilde{X}, \tilde{U} \right\rangle = -\cosh(\bar{\theta}_c - \bar{\theta}). \tag{9.4}$$

This equation describes a hyperbolic circle on the dual hyperbolic unit sphere Σ_2 where $(\bar{\theta}_c - \bar{\theta})$ a given dual hyperbolic spherical radius is and $\tilde{U}$ is a fixed dual

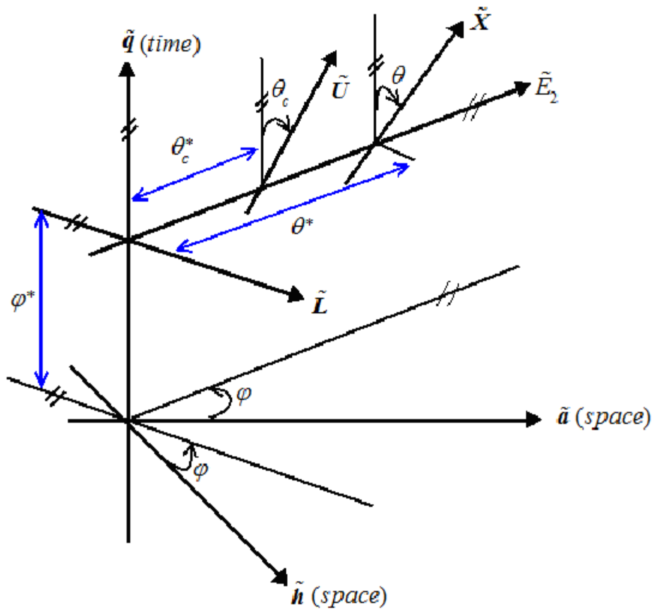

FIGURE 1. The moved timelike line $\tilde{X}$ and its timelike Disteli axis $\tilde{U}$

unit timelike vector which identifies the hyperbolic circle's center. According to E. Study's map (9.4) defines the set of all oriented timelike lines $\tilde{X}$. Like this a set of timelike lines depends on two parameters and is called linear timelike line congruence. Since osculating hyperbolic circle should have contact of at least second order with the curve, timelike Disteli axis $\tilde{U}$ and $(\bar{\theta}_c - \bar{\theta})$ remain constant up to second order at $t = t_0$, that is

$$\left.\frac{d(\bar{\theta}_c - \bar{\theta})}{dt}\right|_{t=t_0} = 0, \qquad \left.\frac{d\tilde{U}}{dt}\right|_{t=t_0} = 0 \tag{9.5}$$

and

$$\left.\frac{d^2(\bar{\theta}_c - \bar{\theta})}{dt^2}\right|_{t=t_0} = 0, \qquad \left.\frac{d^2\tilde{U}}{dt^2}\right|_{t=t_0} = 0. \tag{9.6}$$

From differentiation of (9.4) and equation (9.5) we have

$$\left\langle \frac{d\tilde{X}}{dt}, \tilde{U} \right\rangle = 0. \tag{9.7}$$

We have second order

$$\left\langle \frac{d^2\tilde{X}}{dt^2}, \tilde{U} \right\rangle = 0. \tag{9.8}$$

We substitute from (7.13) and (9.3) into (9.8) and obtain:

$$(\coth \bar{\theta}_c - \coth \bar{\theta}) \sin \bar{\phi} = \frac{\bar{\Omega}}{\bar{k}_1}. \tag{9.9}$$

This equation is dual hyperbolic Euler-Savary equation of one parameter dual hyperbolic spherical motion Σ_3/Σ_2 [24]. By using (8.18) we can rewrite Euler-Savary equation the form as desired

$$(\coth \bar{\theta}_c - \coth \bar{\theta}) \sin \bar{\phi} = \bar{\rho}. \tag{9.10}$$

If this equation separate real and dual part then we have

$$(\coth \theta_c - \coth \theta) \sin \phi = \rho \tag{9.11}$$

and

$$(\coth \theta_c - \coth \theta)\phi^* \cos \phi - \left(\frac{\theta_c^*}{\sinh^2 \theta_c} - \frac{\theta^*}{\sinh^2 \theta} \right) \sin \phi = \rho^*. \tag{9.12}$$

Lorentzian Euler-Savary Eq. (9.11) together with (9.12) is called Disteli formulae of axode of dual hyperbolic spherical motion. (9.11) is Euler-Savary equation for axode of real hyperbolic spherical motion in the Lorentzian space. In order to Eq. (9.12) simplified to reduce by using (9.11) we have

$$\rho\phi^* \cot \phi - \left(\frac{\theta_c^*}{\sinh^2 \theta_c} - \frac{\theta^*}{\sinh^2 \theta} \right) \sin \phi = \rho^*. \tag{9.13}$$

□

10. Example

In this section we display the use of dual Lorentzian vectors for denoting the ISA of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2. The one parameter dual hyperbolic spherical motion Σ_3/Σ_2 can be denoted analytically by the matrix equation

$$\tilde{x}_f(t) = A(t)\tilde{x}_m(t) + \tilde{m}_f(t) \quad , \quad \tilde{x}_m(t) = A^{-1}(t)\tilde{x}_f(t) + \tilde{m}_m(t) \tag{10.1}$$

where $\tilde{x}_f$, $\tilde{x}_m$ are vectors of a same point, with respect to the orthonormal frames of the moving space and fixed space, respectively, and $\tilde{m}_f$, $\tilde{m}_m$ and A are differentiable functions of a dual parameter $\bar{t} = t + \varepsilon t^*$, since we study one parameter hyperbolic spherical motion we consider the case $t^* = 0$. Also we know that

$$\tilde{m}_f = -A\,\tilde{m}_m \quad , \quad \tilde{m}_m = -A^{-1}\,\tilde{m}_f \tag{10.2}$$

where A and A^{-1} matrices are anti-symmetric in the sense of Lorentzian.

The velocity of a fixed point $\tilde{x}_m \in \Sigma_3$ is

$$\tilde{x}'_f = A'\tilde{x}_m + \tilde{m}'_f\,. \tag{10.3}$$

From (10.1) we get

$$\tilde{x}'_f = A'A^{-1}\tilde{x}_f + (\tilde{m}'_f - A'A^{-1}\tilde{m}_f). \tag{10.4}$$

If we consider matrix $\omega = A'A^{-1}$ is anti-symmetric in the sense of Lorentzian, then Eq. (10.4) can be rewritten in the form

$$\tilde{x}'_f = \omega\,\tilde{x}_f + (\tilde{m}'_f - \omega\,\tilde{m}_f). \tag{10.5}$$

As a consequence of this equation, there is a dual vector

$$\tilde{\Omega}(t) = \omega(t) + \varepsilon\,\omega^*(t) \tag{10.6}$$

such that

$$\omega x_f = \omega \times x_f; \quad \omega^* = (m' - \omega \times m). \tag{10.7}$$

Now we give a simple example using by above statement. First we consider the one parameter dual hyperbolic spherical motion Σ_3/Σ_2 denoting by the dual Lorentzian orthogonal matrix

$$A = R_1 \cdot R_2 = \begin{pmatrix} \cosh^2\phi & -\sinh\phi & -\cosh\phi\sinh\phi \\ -\sinh\phi\cosh\phi & \cosh\phi & \sinh^2\phi \\ -\sinh\phi & 0 & \cosh\phi \end{pmatrix} \tag{10.8}$$

such that

$$R_1 = \begin{pmatrix} \cosh\bar{\theta} & -\sinh\bar{\theta} & 0 \\ -\sinh\bar{\theta} & \cosh\bar{\theta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cosh\bar{\phi} & 0 & -\sinh\bar{\phi} \\ 0 & 1 & 0 \\ -\sinh\bar{\phi} & 0 & \cosh\bar{\phi} \end{pmatrix} \tag{10.9}$$

where we assume that $\bar{\theta} = \bar{\phi}$, $\theta^* = \phi^* = 0$. Also we consider an anti-symmetric in the sense of Lorentzian matrix

$$m(\phi) = \begin{pmatrix} 0 & 0 & \mu\sinh\phi \\ 0 & 0 & -\mu\cosh\phi \\ \mu\sinh\phi & \mu\cosh\phi & 0 \end{pmatrix}, \tag{10.10}$$

where we assume that $\mu > 1$. Since $\tilde{q}$, $\tilde{q}_m$, $\tilde{q}_f$ are timelike vectors we can write

$$m(\phi) = \begin{pmatrix} \mu\cosh\phi \\ \mu\sinh\phi \\ 0 \end{pmatrix}. \tag{10.11}$$

If we substitute the (10.8) and (10.10) in (10.7), we have

$$\omega(\phi) = \begin{pmatrix} -\sinh\phi \\ -\cosh\phi \\ 1 \end{pmatrix}, \quad \omega^*(\phi) = \begin{pmatrix} 2\mu\sinh\phi \\ 2\mu\cosh\phi \\ \mu \end{pmatrix}. \tag{10.12}$$

Therefore the dual hyperbolic Pfaffian dual vector $\tilde{\Omega}$ at the instant ϕ of the one parameter dual hyperbolic spherical motion Σ_3/Σ_2 is

$$\tilde{\Omega}(\phi) = \omega(\phi) + \varepsilon\omega^*(\phi) = \begin{pmatrix} -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ 1+\varepsilon\mu \end{pmatrix}. \tag{10.13}$$

Fixed axode is given by

$$\tilde{q}_f(\phi) = \frac{\tilde{\Omega}}{\left\|\tilde{\Omega}\right\|} = \frac{1}{\sqrt{2-2\varepsilon\mu}} \begin{pmatrix} -\sinh\phi + 2\varepsilon\mu\sinh\phi \\ -\cosh\phi + 2\varepsilon\mu\cosh\phi \\ 1+\varepsilon\mu \end{pmatrix}. \tag{10.14}$$

Moving polode on Σ_3 is denoted by

$$\Omega_m = \frac{dM^{-1}}{d\phi} \cdot M;\ \ M = (A + \varepsilon m A) \tag{10.15}$$

where

$$M = \begin{pmatrix} \cosh^2\phi + \varepsilon\mu(-\sinh^2\phi) & -\sinh\phi & -\sinh\phi\cosh\phi + \varepsilon\mu(\sinh\phi\cosh\phi) \\ -\sinh\phi\cosh\phi + \varepsilon\mu(\sinh\phi\cosh\phi) & \cosh\phi & \sinh^2\phi - \varepsilon\mu(\cosh^2\phi) \\ -\sinh\phi & \varepsilon\mu & \cosh\phi \end{pmatrix}.$$

Therefore the moving axode is given by

$$\tilde{q}_m(\phi) = \frac{\tilde{\Omega}_m}{\left\|\tilde{\Omega}_m\right\|} = \frac{1}{\sqrt{2-2\varepsilon\mu}} \begin{pmatrix} \sinh\phi \\ 1-\varepsilon\mu \\ -\cosh\phi \end{pmatrix}. \tag{10.16}$$

Now we introduce the Blaschke invariants of the fixed axode $\tilde{q} = \tilde{q}_f(\phi)$. For the one parameter hyperbolic spherical motion Σ_3/Σ_2, from (10.14), we can give

$$\tilde{\Omega}_f(\phi) = \bar{\Omega}\,\tilde{q}(\phi);\quad \bar{\Omega} = \sqrt{2-2\varepsilon\mu}. \tag{10.17}$$

For differential of (10.17) with respect to ϕ, we have

$$\frac{d\tilde{\Omega}_f}{d\phi} = \tilde{\Omega}_f' = \bar{\Omega}'\tilde{q} + \bar{k}_1\bar{\Omega}\tilde{h} \tag{10.18}$$

and by writing the (6.22) in the differentiation of (10.18)we obtain

$$\tilde{\Omega}_f'' = (\bar{\Omega}'' + \bar{k}_1^2\bar{\Omega})\tilde{q} + (2\bar{k}_1\bar{\Omega}' + \bar{k}_1'\bar{\Omega})\tilde{h} + (\bar{k}_1\bar{\Omega}\bar{k}_{2f})\tilde{a}. \tag{10.19}$$

Further, if we consider Lorentzian vectorial product of (10.18) and (10.19) we find

$$\tilde{\Omega}_f(\phi) \times \tilde{\Omega}_f'(\phi) = -\bar{k}_1\bar{\Omega}^2\tilde{a}. \tag{10.20}$$

And then by using following Lorentzian property

$$\left\|\tilde{\Omega}_f(\phi) \times \tilde{\Omega}_f'(\phi)\right\| = -\left\langle \tilde{\Omega}_f(\phi), \tilde{\Omega}_f(\phi)\right\rangle \left\langle \tilde{\Omega}_f'(\phi), \tilde{\Omega}_f'(\phi)\right\rangle + \left(\left\langle \tilde{\Omega}_f(\phi), \tilde{\Omega}_f'(\phi)\right\rangle\right)^2 \tag{10.21}$$

we find that

$$-\langle\bar{\Omega}\tilde{q},\bar{\Omega}\tilde{q}\rangle\left\langle\bar{\Omega}'\tilde{q}+\bar{k}_1\bar{\Omega}\tilde{h}\,,\,\bar{\Omega}'\tilde{q}+\bar{k}_1\bar{\Omega}\tilde{h}\right\rangle+\left(\left\langle\bar{\Omega}\tilde{q},\bar{\Omega}'\tilde{q}+\bar{k}_1\bar{\Omega}\tilde{h}\right\rangle\right)^2=\bar{k}_1^2\bar{\Omega}^4. \tag{10.22}$$

Finally, we have

$$\det(\tilde{\Omega}_f,\tilde{\Omega}'_f,\tilde{\Omega}''_f)=\bar{k}_1^2\bar{\Omega}^3\bar{k}_{2f}. \tag{10.23}$$

From (10.13) we can give

$$\tilde{\Omega}'_f(\phi)=\begin{pmatrix}-\cosh\phi+2\varepsilon\mu\cosh\phi\\ -\sinh\phi+2\varepsilon\mu\sinh\phi\\ 0\end{pmatrix} \tag{10.24}$$

and

$$\tilde{\Omega}''_f(\phi)=\begin{pmatrix}-\sinh\phi+2\varepsilon\mu\sinh\phi\\ -\cosh\phi+2\varepsilon\mu\cosh\phi\\ 0\end{pmatrix}. \tag{10.25}$$

From (10.13) and (10.14) we obtain

$$\left\langle\tilde{\Omega}_f(\phi),\tilde{\Omega}'_f(\phi)\right\rangle=0 \tag{10.26}$$

and so

$$\left(\left\langle\tilde{\Omega}_f(\phi),\tilde{\Omega}'_f(\phi)\right\rangle\right)^2=0. \tag{10.27}$$

Besides, we have

$$\left\langle\tilde{\Omega}'_f(\phi),\tilde{\Omega}'_f(\phi)\right\rangle=-1+4\varepsilon\mu. \tag{10.28}$$

Substituting the (10.13), (10.27) and (10.28) in (10.22), we find

$$-(2-2\varepsilon\mu)(-1+4\varepsilon\mu)=\bar{k}_1^2\bar{\Omega}^4. \tag{10.29}$$

If we separate the real and dual parts the (10.29), we have

$$k_1=\pm\frac{1}{\sqrt{2}},\;\; k_1^*=-\frac{3\sqrt{2}\mu}{4}. \tag{10.30}$$

By using (6.20) we find that the common distribution parameter of the axodes is given by

$$\lambda=\frac{3\mu}{2}. \tag{10.31}$$

From (10.13), (10.23), (10.24) and (10.25), we find that

$$\det(\tilde{\Omega}_f,\tilde{\Omega}'_f,\tilde{\Omega}''_f)=1-3\varepsilon\mu=\bar{k}_1^2\bar{\Omega}^3\bar{k}_{2f}. \tag{10.32}$$

If we separate that the real and dual parts of above equations, we have

$$k_{2f} = \frac{\sqrt{2}}{2}, \quad k_{2f}^{*} = \mu\frac{3\sqrt{2}}{4}. \tag{10.33}$$

By means of (6.19) and (10.33) we get

$$k_{2m} = \frac{3\sqrt{2}}{2}, \; k_{2m}^{*} = \mu\frac{\sqrt{2}}{4}. \tag{10.34}$$

Therefore we obtain real and dual parts of the integral invariants of the axodes.

11. Conclusion

In this paper, we have introduced one parameter dual hyperbolic spherical motions in the dual Lorentzian space. By considering Disteli axis on the Blaschke frame we have obtained Euler Savary formulae of dual hyperbolic spherical motions. At the end of study, for given orthogonal rotation matrices in the sense of dual Lorentzian type 3×3, we have found real and dual invariants of fixed and moving axodes.

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