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ON THE PARANORMED TAYLOR SEQUENCE SPACES

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Abstract. In this paper, the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type which are the generalization of the Maddox sequence spaces have been introduced and it is proved that the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are linearly isomorphic to spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively. Furthermore, the $\alpha-,\beta-$ and $\gamma-$duals of the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ have been computed and their bases have been constructed and some topological properties of these spaces have been investigated. Besides this, the class of matrices $(t_0^r(p):\mu)$ has been characterized, where μ is one of the sequence spaces ℓ_∞, c and c_0 and derives the other characterizations for the special cases of μ.

1. Introduction

By w, we shall denote the space of all real-valued sequences. Any vector subspace of w is called a sequence space. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and $p-$absolutely convergent series, respectively, where $1 < p < \infty$.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta)=0, g(x)=g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n-\alpha| \to 0$ and $g(x_n-x)\to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequences of strictly positive real numbers with $\sup p_k = H$ and $L=\max\{1,H\}$. Then, the linear spaces $\ell_\infty(p), c(p), c_0(p)$ and $\ell(p)$ were defined by Maddox [12] (see also Simons [14] and

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Nakano [13]) as follows:

$$\ell_\infty(p) = \{x = (x_k) \in w : \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\},$$
$$c(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{R}\},$$
$$c_0(p) = \{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\},$$
$$\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\},$$

which are the complete spaces paranormed by

$$g_1(x) = \sup_{k\in\mathbb{N}} |x_k|^{p_k/L} \Longleftrightarrow \inf p_k > 0 \text{ and } g_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/L},$$

respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $\mathcal{F}$ and $\mathbb{N}_k$, we shall denote the collection of all finite subsets of $\mathbb{N}$ and the set of all $n \in \mathbb{N}$ such that $n \geq k$, respectively. We shall assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ provided $1 < \inf p_k \leq H < \infty$.

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from λ into μ, and we denote it by $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$, the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x, is in μ, where

$$(Ax)_n = \sum_k a_{nk} x_k, \ (n \in \mathbb{N}). \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \mu$. A sequence x is said to be $A-$summable to α if Ax converges to α which is called the $A-$limit of x.

2. The Sequence Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of Non-Absolute Type

In this section, we define the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, and prove that $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are the complete paranormed linear spaces.

For a sequence space λ, the matrix domain λ_A of an infinite matrix A is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}. \tag{2.1}$$

In [5], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ which consists of all sequences such that $S-$transforms are in $\ell_{(p)}$, where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1 & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n. \end{cases}$$

Başar and Altay [3] have studied the space $bs(p)$ which is formerly defined by Başar in [4] as the set of all series whose sequences of partial sums are in $\ell_\infty(p)$.

More recently, Altay and Başar have studied the sequence spaces $r^t(p), r^t_\infty(p)$ in [1] and $r^t_c(p), r^t_0(p)$ in [2] which are derived by the Riesz means from the sequence spaces $\ell(p), \ell_\infty(p), c(p)$ and $c_0(p)$ of Maddox, respectively.

With the notation of (2.1), the spaces $\overline{\ell(p)}, bs(p), r^t(p), r^t_\infty(p), r^t_c(p)$ and $r^t_0(p)$ may be redefined by

$$\overline{\ell(p)} = [\ell(p)]_S, bs(p) = [\ell_\infty(p)]_S, r^t(p) = [\ell(p)]^t_R$$
$$r^t_\infty(p) = [\ell_\infty(p)]^t_R, r^t_c(p) = [c(p)]^t_R, r^t_0(p) = [c_0(p)]^t_R.$$

In [6], Demiriz and Çakan have defined the sequence spaces $e^r_0(u,p)$ and $e^r_c(u,p)$ which consists of all sequences such that $E^{r,u}$- transforms are in $c_0(p)$ and $c(p)$, respectively $E^{r,u} = \{e^r_{nk}(u)\}$ is defined by

$$e^r_{nk}(u) = \begin{cases} \binom{n}{k}(1-r)^{n-k}r^k u_k & , \quad (0 \leq k \leq n), \\ 0 & , \quad (k > n) \end{cases}$$

for all $k, n \in \mathbb{N}$ and $0 < r < 1$.

In [9], the Taylor sequence spaces t^r_0 and t^r_c of non-absolute type, which are the matrix domains of Taylor mean T^r of order r in the sequence spaces c_0 and c, respectively, are introduced, some inclusion relations and Schauder basis for the spaces t^r_0 and t^r_c are given, and the $\alpha-, \beta-$ and $\gamma-$ duals of those spaces are determined. The main purpose of this paper is to introduce the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ of nonabsolute type which are the set of all sequences whose T^r-transforms are in the spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively; where T^r denotes the matrix $T^r = \{t^r_{nk}\}$ defined by

$$t^r_{nk} = \begin{cases} \binom{k}{n}(1-r)^{n+1}r^{k-n} & , \quad (k \geq n), \\ 0 & , \quad (0 \leq k < n) \end{cases}$$

where $0 < r < 1$. Also, we have constructed the basis and computed the $\alpha-, \beta-$ and $\gamma-$duals and investigated some topological properties of the spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$.

Following Choudhary and Mishra [5], Başar and Altay [3], Altay and Başar [1, 2], Demiriz [6], Kirişçi [9], we define the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$, as the sets of all sequences such that T^r-transforms of them are in the spaces $c_0(p)$,$c(p)$ and $\ell(p)$, respectively, that is,

$$t^r_0(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \right|^{p_n} = 0 \right\},$$

$$t^r_c(p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k - l \right|^{p_n} = 0 \text{ for some } l \in \mathbb{R} \right\}$$

and

$$t^r(p) = \left\{ x = (x_k) \in w : \sum_n \left| \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \right|^{p_n} < \infty \right\}.$$

In the case $(p_n) = e = (1, 1, 1, ...)$, the sequence spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ are, respectively, reduced to the sequence spaces t^r_0 and t^r_c which are introduced by Kirişçi [9] and $t^r(p)$ is reduced to the sequence space t^r_p. With the notation of (2.1), we may redefine the spaces $t^r_0(p)$, $t^r_c(p)$ and $t^r(p)$ as follows:

$$t^r_0(p) = [c_0(p)]_{T^r}, \ t^r_c(p) = [c(p)]_{T^r} \text{ and } t^r(p) = [\ell(p)]_{T^r}. \tag{2.2}$$

Define the sequence $y = \{y_k(r)\}$, which will be frequently used, as the T^r-transform of a sequence $x = (x_k)$, i.e.,

$$y_k(r) := \sum_{k=n}^{\infty} \binom{k}{n} (1-r)^{n+1} r^{k-n} x_k \text{ for all } k \in \mathbb{N}. \tag{2.3}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 2.1. *$t_0^r(p)$ and $t_c^r(p)$ are the complete linear metric space paranormed by g, defined by*

$$g(x) = \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k/L}.$$

Also, $t_p^r(p)$ is the complete linear metric space paranormed by h, defined by

$$h(x) = \left(\sum_{k=0}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M}. \tag{2.4}$$

Proof. Since the proof is similar for $t_0^r(p)$ and $t_c^r(p)$, we give the proof only for the space $t_0^r(p)$. The linearity of $t_0^r(p)$ with respect to the co-ordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, z \in t_0^r(p)$ (see Maddox [11, p.30])

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} (x_j + z_j) \right|^{p_k/L} \\ &\leq \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k/L} + \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} z_j \right|^{p_k/L} \end{aligned} \tag{2.5}$$

and for any $\alpha \in \mathbb{R}$ (see [14])

$$|\alpha|^{p_k} \leq \max\{1, |\alpha|^L\}. \tag{2.6}$$

It is clear that $g(\theta) = 0$ and $g(x) = g(-x)$ for all $x \in t_0^r(p)$. Again the inequalities (2.5) and (2.6) yield the subadditivity of g and

$$g(\alpha x) \leq \max\{1, |\alpha|^L\} g(x).$$

Let $\{x^n\}$ be any sequence of the points $x^n \in t_0^r(p)$ such that $g(x^n - x) \to 0$ and (α_n) also be any sequence of scalars such that $\alpha_n \to \alpha$. Then, since the inequality

$$g(x^n) \leq g(x) + g(x^n - x)$$

holds by the subadditivity of g, $\{g(x^n)\}$ is bounded and we thus have

$$\begin{aligned} g(\alpha^n x^n - \alpha x) &= \sup_{k\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} (\alpha^n x_j^n - \alpha x_j) \right|^{p_k/L} \\ &\leq |\alpha_n - \alpha| g(x^n) + |\alpha| g(x^n - x), \end{aligned}$$

which tends to zero as $n \to \infty$. This means that the scalar multiplication is continuous. Hence, g is paranorm on the space $t_0^r(p)$.

It remains to prove the completeness of the space $t_0^r(p)$. Let $\{x^i\}$ be any Cauchy sequence in the space $t_0^r(p)$, where $x^i = \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \ldots\}$. Then, for a given $\epsilon > 0$ there exists a positive integer $n_0(\epsilon)$ such that

$$g(x^i - x^j) < \frac{\epsilon}{2}$$

for all $i, j > n_0(\epsilon)$. Using the definition of g we obtain for each fixed $k \in \mathbb{N}$ that

$$|(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} \leq \sup_{k\in\mathbb{N}} |(T^r x^i)_k - (T^r x^j)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{2.7}$$

for every $i, j > n_0(\epsilon)$ which leads to the fact that $\{(T^r x^0)_k, (T^r x^1)_k, (T^r x^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(T^r x^i)_k \to (T^r x)_k$ as $i \to \infty$. Using these infinitely many limits $(T^r x)_0, (T^r x)_1, \ldots$, we define the sequence $\{(T^r x)_0, (T^r x)_1, \ldots\}$. From (2.7) with $j \to \infty$, we have

$$|(T^r x^i)_k - (T^r x)_k|^{p_k/L} \leq \frac{\epsilon}{2} \ (i, j > n_0(\epsilon)) \tag{2.8}$$

for every fixed $k \in \mathbb{N}$. Since $x^i = \{x_k^{(i)}\} \in t_0^r(p)$ for each $i \in \mathbb{N}$, there exists $k_0(\epsilon) \in \mathbb{N}$ such that

$$|(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2} \tag{2.9}$$

for every $k \geq k_0(\epsilon)$ and for each fixed $i \in \mathbb{N}$. Therefore, taking a fixed $i > n_0(\epsilon)$ we obtain by (2.8) and (2.9) that

$$|(T^r x)_k|^{p_k/L} \leq |(T^r x)_k - (T^r x^i)_k|^{p_k/L} + |(T^r x^i)_k|^{p_k/L} < \frac{\epsilon}{2}$$

for every $k > k_0(\epsilon)$. This shows that $x \in t_0^r(p)$. Since $\{x^i\}$ was an arbitrary Cauchy sequence, the space $t_0^r(p)$ is complete and this concludes the proof.

Now, $t^r(p)$ is the complete linear metric space paranormed by h defined by (2.4). It is easy to see that the space $t^r(p)$ is linear with respect to the coordinate-wise addition and scalar multiplication. Therefore, we first show that it is a paranormed space with the paranorm h defined by (2.4).

It is clear that $h(\theta) = 0$ where $\theta = (0, 0, 0, ...)$ and $h(x) = h(-x)$ for all $x \in t^r(p)$.

Let $x, y \in t^r(p)$; then by Minkowski's inequality we have

$$\begin{aligned} h(x+y) &= \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}(x_j+y_j)\right|^{p_k}\right)^{1/M} \\ &= \left(\sum_{k=0}^{\infty}\left[\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}(x_j+y_j)\right|^{p_k/M}\right]^{M}\right)^{1/M} \\ &\leq \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M} \\ &\qquad + \left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}y_j\right|^{p_k}\right)^{1/M} \\ &= h(x)+h(y) \end{aligned} \tag{2.10}$$

Let $\{x^n\}$ be any sequence of the points $x^n \in t^r(p)$ such that $h(x^n - x) \to 0$ and (λ_n) also be any sequence of scalars such that $\lambda_n \to \lambda$. We observe that

$$\begin{aligned} h(\lambda^n x^n - \lambda x) &\leq h[(\lambda^n - \lambda)(x^n - x)] \\ &+ h[\lambda(x^n - x)] \\ &+ h[(\lambda^n - \lambda)x]. \end{aligned} \tag{2.11}$$

It follows from $\lambda^n \to \lambda (n \to \infty)$ that $|\lambda^n - \lambda| < 1$ for all sufficiently large n; hence

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)(x^n - x)] \leq \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.12}$$

Furthermore, we have

$$\lim_{n\to\infty} h[\lambda(x^n - x)] \leq \max\{1, |\lambda|^M\} \lim_{n\to\infty} h(x^n - x) = 0. \tag{2.13}$$

Also, we have

$$\lim_{n\to\infty} h[(\lambda_n - \lambda)x)] \leq \lim_{n\to\infty} |\lambda_n - \lambda| h(x) = 0. \tag{2.14}$$

Then, we obtain from (2.11), (2.12), (2.13) and (2.14) that $h(\lambda^n x^n - \lambda x) \to 0$, as $n \to \infty$. This shows that h is a paranorm on $t^r(p)$.

Furthermore, if $h(x) = 0$, then $\left(\sum_{k=0}^{\infty}\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M} = 0$. Therefore $\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k} = 0$ for each $k \in \mathbb{N}$. Since $0 < r < 1$, we have $\binom{j}{k}(1-r)^{k+1}r^{j-k} > 0$. Then, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. That is, $x = \theta$. This shows that h is a total paranorm.

Now, we show that $t^r(p)$ is complete. Let $\{x^n\}$ be any Cauchy sequence in the space $t^r(p)$, where $x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, ...\}$. Then, for a given $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that $h(x^n - x^m) < \epsilon$ for all $n, m > n_0(\epsilon)$. Since for each fixed $k \in \mathbb{N}$ that

$$\begin{aligned} |(T^r x^n)_k - (T^r x^m)_k| &\leq \left[\sum_k |(T^r x^n)_k - (T^r x^m)_k|^{p_k}\right]^{\frac{1}{M}} \\ &= h(x^n - x^m) < \epsilon \qquad (2.15) \end{aligned}$$

for every $n, m > n_0(\epsilon)$, $\{(T^r x^0)_k, (T^r x^1)_k, (T^r x^2)_k, ...\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(T^r x^n)_k \to (T^r x)_k$ as $n \to \infty$. Using these infinitely many limits $(T^r x)_0, (T^r x)_1, ...$, we define the sequence $\{(T^r x)_0, (T^r x)_1, ...\}$. For each $K \in \mathbb{N}$ and $n, m > n_0(\epsilon)$

$$\left[\sum_{k=0}^{K} |(T^r x^n)_k - (T^r x^m)_k|^{p_k}\right]^{\frac{1}{M}} \leq h(x^n - x^m) < \epsilon. \qquad (2.16)$$

By letting $m, K \to \infty$, we have for $n > n_0(\epsilon)$ that

$$h(x^n - x) = \left[\sum_k |(T^r x^n)_k - (T^r x)_k|^{p_k}\right]^{\frac{1}{M}} < \epsilon. \qquad (2.17)$$

This shows that $x^n - x \in t^r(p)$. Since $t^r(p)$ is a linear space, we conclude that $x \in t^r(p)$; it follows that $x^n \to x$, as $n \to \infty$ in $t^r(p)$, thus we have shown that $t^r(p)$ is complete. □

Note that the absolute property does not hold on the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, since there exists at least one sequence in the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ and such that $g(x) \neq g(|x|)$, where $|x| = (|x_k|)$. This says that $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ are the sequence spaces of non-absolute type.

Theorem 2.2. *The sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type are linearly isomorphic to the spaces $c_0(p)$, $c(p)$ and $\ell(p)$, respectively, where $0 < p_k \leq H < \infty$.*

Proof. To avoid repetition of similar statements, we give the proof only for $t_0^r(p)$. We should show the existence of a linear bijection between the spaces $t_0^r(p)$ and $c_0(p)$. With the notation of (2.3), define the transformation T from $t_0^r(p)$ and $c_0(p)$ by $x \mapsto y = Tx$. The linearity of T is trivial. Furthermore, it is obvious that $x = \theta$ whenever $Tx = \theta$, and hence T is injective.

Let $y \in c_0(p)$ and define the sequence

$$x_k(r) := \sum_{j=k}^{\infty} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} y_j; \quad k \in \mathbb{N}.$$

Then, we have

$$g(x) = \sup_{k \in \mathbb{N}} \left|\sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j\right|^{p_k/L} = \sup_{k \in \mathbb{N}} |y_k|^{p_k/L} = g_1(y) < \infty.$$

Thus, we have that $x \in t_0^r(p)$ and consequently T is surjective. Hence, T is a linear bijection and this says that the spaces $t_0^r(p)$ and $c_0(p)$ are linearly isomorphic, as was desired.

□

Theorem 2.3. *Convergence in $t^r(p)$ is stronger than coordinate-wise convergence.*

Proof. First we show that $h(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^n \to x_k$; as $n \to \infty$ for every $k \in \mathbb{N}$. We fix k, then we have

$$\begin{aligned} &\lim_{n\to\infty} \left| \sum_{n=k}^{\infty} \binom{n}{k} (1-r)^{k+1} r^{n-k} [x_k^{(n)} - x_k] \right|^{p_k} \\ &\leq \lim_{n\to\infty} \sum_k \left| \sum_{n=k}^{\infty} \binom{n}{k} (1-r)^{k+1} r^{n-k} [x_k^{(n)} - x_k] \right|^{p_k} \\ &= \lim_{n\to\infty} [h(x^n - x)]^M = 0. \end{aligned} \tag{2.18}$$

Hence, we have for $k = 0$ that

$$\lim_{n\to\infty} \left| \sum_{n=0}^{\infty} (1-r) r^n [x_0^{(n)} - x_0] \right| = 0$$

which gives the fact that $|x_0^{(n)} - x_0| \to 0$, as $n \to \infty$. Similarly, for each $k \in \mathbb{N}$, we have $x_k^n \to x_k$; as $n \to \infty$.

A sequence space λ with a linear topology is called a K-space provided each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A K-space λ is called an FK-space provided λ is complete linear metric space. An FK-space whose topology is normable is called a BK-space. Given a BK-space $\lambda \supset \phi$, we denote the n th section of a sequence $x = (x_k) \in \lambda$ by $x^{[n]} := \sum_{k=0}^{n} x_k e^{(k)}$, and we say that $x = (x_k)$ has the property AK if $\lim_{n\to\infty} ||x - x^{[n]}||_\lambda = 0$. If AK property holds for every $x \in \lambda$, then we say that the space λ is called AK-space (cf. [7]). Now, we may give the following. □

Theorem 2.4. *The space $t^r(p)$ has AK.*

Proof. For each $x = (x_k) \in t^r(p)$, we put

$$x^{<m>} = \sum_{k=0}^{m} x_k e^{(k)}, \forall m \in \{1, 2, \ldots\}. \tag{2.19}$$

Let $\epsilon > 0$ and $x \in t^r(p)$ be given. Then, there is $N = N(\epsilon) \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} < \epsilon^M. \tag{2.20}$$

Then we have for all $m \geq N$,

$$\begin{aligned} h(x - x^{<m>}) &= h\left(x - \sum_{k=0}^{m} x_k e^{(k)} \right) \\ &= \left(\sum_{k=m+1}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M} \\ &\leq \left(\sum_{k=N}^{\infty} \left| \sum_{j=k}^{\infty} \binom{j}{k} (1-r)^{k+1} r^{j-k} x_j \right|^{p_k} \right)^{1/M} < \epsilon. \end{aligned} \tag{2.21}$$

This shows that $x = \sum_k x_k e^{(k)}$.

Now, we have to show that this representation is unique. We assume that $x = \sum_k \lambda_k e^{(k)}$. Then for each k,

$$\left(\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j - \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M}$$

$$\leq \left(\sum_k\left|\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j - \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j\right|^{p_k}\right)^{1/M}$$

$$= h(x-x) = 0 \tag{2.22}$$

Hence, $\sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}\lambda_j = \sum_{j=k}^{\infty}\binom{j}{k}(1-r)^{k+1}r^{j-k}x_j$ for each j. Then, $\lambda_j = x_j$ for each j. Therefore, the representation is unique. □

3. The Basis for the Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

Let (λ, h) be a paranormed space. Recall that a sequence (b_k) of the elements of λ is called a basis for λ if and only if, for each $x \in \lambda$, there exists a unique sequence (α_k) of scalars such that

$$h\left(x - \sum_{k=0}^{n}\alpha_k b_k\right) \to 0 \text{ as } n \to \infty.$$

The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x = \sum \alpha_k b_k$. Since it is known that the matrix domain λ_A of a sequence space λ has a basis if and only if λ has a basis whenever $A = (a_{nk})$ is a triangle (cf. [8, Remark 2.4]), we have the following. Because of the isomorphism T is onto, defined in the proof of Theorem 2.2, the inverse image of the basis of those spaces $c_0(p)$, $c(p)$ and $\ell(p)$ are the basis of the new spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$, respectively. Therefore, we have the following:

Theorem 3.1. *Let $\lambda_k(r) = (T^r x)_k$ for all $k \in \mathbb{N}$ and $0 < p_k \leq H < \infty$. Define the sequence $b^{(k)}(r) = \{b^{(k)}(r)\}_{k\in\mathbb{N}}$ of the elements of the space $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ by*

$$b^{(k)}(r) = \begin{cases} \binom{k}{n}(1-r)^{-(k+1)}(-r)^{k-n} & , \quad k \geq n \\ 0 & , \quad 0 \leq k < n \end{cases}$$

for every fixed $k \in \mathbb{N}$. Then

- **(a):** *The sequence $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ is a basis for the space $t_0^r(p)$, and any $x \in t_0^r(p)$ has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r),$$

- **(b):** *The set $e, b^{(1)}(r), b^{(2)}(r), ...$ is a basis for the space $t_c^r(p)$, and any $x \in t_c^r(p)$ has a unique representation of the form*

$$x = le + \sum_k [\lambda_k(r) - l] b^{(k)}(r),$$

where $l = \lim_{k\to\infty}(T^r x)_k$.

(c): *The sequence* $\{b^{(k)}(r)\}_{k\in\mathbb{N}}$ *is a basis for the space* $t^r(p)$, *and any* $x \in t^r(p)$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(r) b^{(k)}(r).$$

4. The $\alpha-, \beta-$ and $\gamma-$Duals of the Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

In this section, we state and prove the theorems determining the $\alpha-, \beta-$ and $\gamma-$duals of the sequence spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type.

We shall firstly give the definition of $\alpha-, \beta-$ and $\gamma-$duals of sequence spaces and after quoting the lemmas which are needed in proving the theorems given in Section 4.

The set $S(\lambda,\mu)$ defined by

$$S(\lambda,\mu) = \{z = (z_k) \in w : xz = (x_k z_k) \in \mu \text{ for all } x = (x_k) \in \lambda\} \tag{4.1}$$

is called the multiplier space of the sequence spaces λ and μ. One can easily observe for a sequence space ν with $\lambda \supset \nu \supset \mu$ that the inclusions

$$S(\lambda,\mu) \subset S(\nu,\mu) \text{ and } S(\lambda,\mu) \subset S(\lambda,\nu)$$

hold. With the notation of (4.1), the alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ are defined by

$$\lambda^\alpha = S(\lambda,\ell_1), \lambda^\beta = S(\lambda, cs) \text{ and } \lambda^\gamma = S(\lambda, bs).$$

The alpha-, beta- and gamma-duals of a sequence space are also referred as Köthe-Toeplitz dual, generalized Köthe-Toeplitz dual and Garling dual of a sequence space, respectively.

For to give the alpha-, beta- and gamma-duals of the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$ of non-absolute type, we need the following Lemma:

Lemma 4.1. [7] *Let* $A = (a_{nk})$ *be an infinite matrix. Then, the following statements hold*

(i): $A \in (c_o(p) : \ell(q))$ *if and only if*

$$\sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} a_{nk} M^{-1/p_k} \right|^{q_n} < \infty, \quad \exists M \in \mathbb{N}_2. \tag{4.2}$$

(ii): $A \in (c(p) : \ell(q))$ *if and only if (4.2) holds and*

$$\sum_n \left| \sum_k a_{nk} \right|^{q_n} < \infty. \tag{4.3}$$

(iii): $A \in (c_0(p) : c(q))$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2, \tag{4.4}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0 \text{ for all } k \in \mathbb{N}, \tag{4.5}$$

$$\exists (\alpha_k) \subset \mathbb{R} \ni \sup_{n\in\mathbb{N}} N^{1/q_n} \sum_k |a_{nk} - \alpha_k| M^{-1/p_k} < \infty, \ \exists M \in \mathbb{N}_2 \text{ and } \forall N \in \mathbb{N}_1. \tag{4.6}$$

(iv): *$A \in (c(p) : c(q))$ if and only if (4.4), (4.5), (4.6) hold and*

$$\exists \alpha \in \mathbb{R} \ni \lim_{n\to\infty} |\sum_k a_{nk} - \alpha|^{q_n} = 0. \tag{4.7}$$

(v): *$A \in (c_o(p) : \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbb{N}} \left(\sum_k |a_{nk}| M^{-1/p_k}\right)^{q_n} < \infty, \ \exists M \in \mathbb{N}_2. \tag{4.8}$$

(vi): *$A \in (\ell(p) : \ell_1)$ if and only if*

(a): *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then*

$$\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} a_{nk}\right|^{p_k} < \infty. \tag{4.9}$$

(b): *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} a_{nk} M^{-1}\right|^{p_k'} < \infty. \tag{4.10}$$

Lemma 4.2. [10] *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold*

(i): *$A \in (\ell(p) : \ell_\infty)$ if and only if*

(a): *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then,*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{4.11}$$

(b): *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, there exists an integer $M > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk} M^{-1}\right|^{p_k'} < \infty. \tag{4.12}$$

(ii): *Let $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.11) and (4.12) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k, \ \forall k \in \mathbb{N}. \tag{4.13}$$

Theorem 4.1. *Let $K \in \mathcal{F}$ and $K^* = \{k \in \mathbb{N} : n \geq k\} \cap K$ for $K \in \mathcal{F}$. Define the sets $T_1^r(p)$, T_2^r, $T_3(p)$ and $T_4(p)$ as follows:*

$$\begin{aligned}
T_1^r(p) &= \bigcup_{M>1} \left\{a = (a_k) \in w : \sup_{K\in\mathcal{F}} \sum_n \left|\sum_{k\in K^*} c_{nk} M^{-1/p_k}\right|^{q_n} < \infty\right\}, \\
T_2^r &= \left\{a = (a_k) \in w : \sum_n \left|\sum_{k=0}^n c_{nk}\right| \text{ exists for each } n \in \mathbb{N}\right\}, \\
T_3(p) &= \bigcup_{M>1} \left\{a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} c_{nk} M^{-1}\right|^{p_k'} < \infty, \right\}, \\
T_4(p) &= \left\{a = (a_k) \in w : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} c_{nk}\right|^{p_k} < \infty\right\},
\end{aligned}$$

where the matrix $C(r)=(c^r_{nk})$ *defined by*

$$c^r_{nk}=\begin{cases}\binom{k}{n}(-r)^{k-n}(1-r)^{-(k+1)}a_n & , \quad (k\geq n),\\ 0 & , \quad (0\leq k<n).\end{cases} \tag{4.14}$$

Then, $[t^r_0(p)]^\alpha=T^r_1(p)$, $[t^r_c(p)]^\alpha=T^r_1(p)\cap T^r_2$ *and*

$$[t^r(p)]^\alpha=\begin{cases}T_3(p) & , \quad 1<p_k\leq H<\infty, \forall k\in\mathbb{N},\\ T_4(p) & , \quad 0<p_k\leq 1, \forall k\in\mathbb{N}.\end{cases} \tag{4.15}$$

Proof. We chose the sequence $a=(a_k)\in w$. We can easily derive that with the (2.3) that

$$a_nx_n=\sum_{k=n}^{\infty}\binom{k}{n}(-r)^{k-n}(1-r)^{-(k+1)}a_ny_k=(C^ry)_n, \quad (n\in\mathbb{N}). \tag{4.16}$$

for all $k,n\in\mathbb{N}$, where $C^r=(c^r_{nk})$ defined by (4.14). It follows from (4.16) that $ax=(a_nx_n)\in\ell_1$ whenever $x\in t^r_0(p)$ if and only if $Cy\in\ell_1$ whenever $y\in c_0(p)$. This means that $a=(a_n)\in[t^r_0(p)]^\alpha$ if and only if $C\in(c_0(p):\ell_1)$. Then, we derive by (4.2) with $q_n=1$ for all $n\in\mathbb{N}$ that $[t^r_0(p)]^\alpha=T^r_1(p)$.

Using the (4.3) with $q_n=1$ for all $n\in\mathbb{N}$ and (4.16), the proof of the $[t^r_c(p)]^\alpha=T^r_1(p)\cap T_2$ can also be obtained in a similar way. Also, using the (4.9),(4.10) and (4.16), the proof of the

$$[t^r(p)]^\alpha=\begin{cases}T_3(p) & , \quad 1<p_k\leq H<\infty, \forall k\in\mathbb{N},\\ T_4(p) & , \quad 0<p_k\leq 1, \forall k\in\mathbb{N},\end{cases}$$

can also be obtained in a similar way. □

Theorem 4.2. *The matrix* $D(r)=(d^r_{nk})$ *is defined by*

$$d^r_{nk}=\begin{cases}\sum_{k=0}^{n}\binom{n}{k}(-r)^{n-k}(1-r)^{-(n+1)}a_k & , \quad (0\leq k\leq n)\\ 0 & , \quad (k>n)\end{cases} \tag{4.17}$$

for all $k,n\in\mathbb{N}$. *Define the sets* $T^r_5(p)$, T^r_6, T^r_7, $T_8(p)$, $T_9(p)$ *and* $T_{10}(p)$ *as follows:*

$$\begin{aligned}
T^r_5(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\sum_k\left|d^r_{nk}M^{-1/p_k}\right|<\infty\right\},\\
T^r_6 &= \left\{a=(a_k)\in w:\lim_{n\to\infty}|d^r_{nk}| \text{ exists for each } k\in\mathbb{N}\right\},\\
T^r_7 &= \left\{a=(a_k)\in w:\lim_{n\to\infty}\sum_{k=0}^{n}|d^r_{nk}| \text{ exists}\right\},\\
T_8(p) &= \bigcup_{M>1}\left\{a=(a_k)\in w:\sup_{n\in\mathbb{N}}\sum_k\left|d_{nk}M^{-1}\right|^{p'_k}<\infty\right\},\\
T_9(p) &= \{a=(a_k)\in w: d_{nk}<\infty\},\\
T_{10}(p) &= \left\{a=(a_k)\in w:\sup_{n,k\in\mathbb{N}}|d_{nk}|^{p_k}<\infty\right\}.
\end{aligned}$$

Then, $[t^r_0(p)]^\beta=T^r_5(p)\cap T^r_6$, $[t^r_c(p)]^\beta=[t^r_0(p)]^\beta\cap T^r_7$ *and*

$$[t^r(p)]^\beta=\begin{cases}T_8(p)\cap T_9(p) & , \quad 1<p_k\leq H<\infty, \forall k\in\mathbb{N},\\ T_9(p)\cap T_{10}(p) & , \quad 0<p_k\leq 1, \forall k\in\mathbb{N}.\end{cases} \tag{4.18}$$

Proof. We give the proof again only for the space $t_0^r(p)$. Consider the equation

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\left[\sum_{k=j}^{\infty}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}y_k\right]a_k \\ &= \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}a_j\right]y_k = (D^r y)_n, \end{aligned} \tag{4.19}$$

where $D^r = (d_{nk}^r)$ defined by (4.17). Thus, we decude from (4.19) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in t_0^r(p)$ if and only if $D^r y \in c$ whenever $y = (y_k) \in c_0(p)$. That is to say that $a = (a_k) \in [t_0^r(p)]^\beta$ if and only if $D^r \in (c_0(p) : c)$. Therefore, we derive from (4.4),(4.5) and (4.6) with $q_n = 1$ for all $n \in \mathbb{N}$ that $[t_0^r(p)]^\beta = T_5^r(u,p) \cap T_6^r(u)$.

Using the (4.4),(4.5), (4.6) and (4.7) with $q_n = 1$ for all $n \in \mathbb{N}$ and (4.19), the proofs of the $[t_c^r(p)]^\beta = [t_0^r(p)]^\beta \cap T_7^r$ can also be obtained in a similar way. Also, using the (4.11),(4.12), (4.13) and (4.19), the proofs of the

$$[t^r(p)]^\beta = \begin{cases} T_8(p) \cap T_9(p) & , \quad 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_9(p) \cap T_{10}(p) & , \quad 0 < p_k \le 1, \forall k \in \mathbb{N}. \end{cases}$$

can also be obtained in a similar way. □

Theorem 4.3. *Define the set $T_6^r(u)$ by*

$$T_{11}^r(u) = \left\{ a = (a_k) \in w : \left\{ \sum_{j=0}^{k}\binom{k}{j}(-r)^{k-j}(1-r)^{-(k+1)}a_j \right\} \in bs \right\}.$$

Then, $[t_0^r(p)]^\gamma = T_5^r(p) \cap T_6^r$, $[t_c^r(p)]^\gamma = [t_0^r(p)]^\gamma \cap T_{11}^r$ *and*

$$[t^r(p)]^\gamma = \begin{cases} T_8(p) & , \quad 1 < p_k \le H < \infty, \forall k \in \mathbb{N}, \\ T_{10}(p) & , \quad 0 < p_k \le 1, \forall k \in \mathbb{N}. \end{cases}$$

Proof. This is obtained in the similar way used in the proof of Theorem 4.2. □

5. Certain Matrix Mappings on the Sequence Spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$

In this section, we characterize some matrix mappings on the spaces $t_0^r(p)$, $t_c^r(p)$ and $t^r(p)$.

We known that, if $t_0^r(p) \cong c_0(p)$, $t_c^r(p) \cong c(p)$ and $t^r(p) \cong \ell(p)$, we can say: The equivalence "$x \in t_0^r(p), t_c^r(p)$ or $t^r(p)$ if and only if $y \in c_0(p), c(p)$ or $\ell(p)$" holds.

In what follows, for brevity, we write,

$$\tilde{a}_{nk} := \sum_{k=0}^{n}\binom{n}{k}(-r)^{n-k}(1-r)^{-(n+1)}a_{nk}$$

for all $k, n \in \mathbb{N}$.

Theorem 5.1. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $E = (e_{nk})$ are connected with the relation*

$$e_{nk} := \tilde{a}_{nk} \tag{5.1}$$

for all $k, n \in \mathbb{N}$ and μ be any given sequence space. Then,

(i): *$A \in (t_0^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$.*

(ii): *$A \in (t_c^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(0)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (c(p) : \mu)$.*

(iii): *$A \in (t^r(p) : \mu)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ for all $n \in \mathbb{N}$ and $E \in (\ell(p) : \mu)$.*

Proof. We prove only part of (i). Let μ be any given sequence space. Suppose that (5.1) holds between $A = (a_{nk})$ and $E = (e_{nk})$, and take into account that the spaces $t_0^r(p)$ and $c_0(p)$ are linearly isomorphic.

Let $A \in (t_0^r(p) : \mu)$ and take any $y = (y_k) \in c_0(p)$. Then $ET(r)$ exists and $\{a_{nk}\}_{k\in\mathbb{N}} \in T_5^r(p) \cap T_6^r$ which yields that $\{e_{nk}\}_{k\in\mathbb{N}} \in c_0(p)$ for each $n \in \mathbb{N}$. Hence, Ey exists and thus

$$\sum_k e_{nk} y_k = \sum_k a_{nk} x_k$$

for all $n \in \mathbb{N}$.

We have that $Ey = Ax$ which leads us to the consequence $E \in (c_0(p) : \mu)$.

Conversely, let $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ for each $n \in \mathbb{N}$ and $E \in (c_0(p) : \mu)$ hold, and take any $x = (x_k) \in t_0^r(p)$. Then, Ax exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{\infty} a_{nk} x_k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{j}{k} (-r)^{j-k} (1-r)^{-(j+1)} a_{nj} \right] y_k$$

for all $n \in \mathbb{N}$, that $Ey = Ax$ and this shows that $A \in (t_0^r(p) : \mu)$. This completes the proof of part of (i). □

Theorem 5.2. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} := \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{(j-n)} a_{jk} \text{ for all } k, n \in \mathbb{N}. \tag{5.2}$$

Let μ be any given sequence space. Then,

(i): *$A \in (\mu : t_0^r(p))$ if and only if $B \in (\mu : c_0(p))$.*

(ii): *$A \in (\mu : t_c^r(p))$ if and only if $B \in (\mu : c(p))$.*

(iii): *$A \in (\mu : t^r(p))$ if and only if $B \in (\mu : \ell(p))$.*

Proof. We prove only part of (i). Let $z = (z_k) \in \mu$ and consider the following equality.

$$\sum_{k=0}^{m} b_{nk} z_k = \sum_{j=n}^{\infty} \binom{j}{n} (1-r)^{n+1} r^{j-n} \left(\sum_{k=0}^{m} a_{jk} z_k \right) \quad \text{for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = \{T(r)(Az)\}_n$ for all $n \in \mathbb{N}$. Therefore, one can observe from here that $Az \in t_0^r(p)$ whenever $z \in \mu$ if and only if $Bz \in c_0(p)$ whenever $z \in \mu$. This completes the proof of part of (i). □

Of course, Theorems 5.1 and 5.2 have several consequences depending on the choice of the sequence space μ. Whence by Theorem 5.1 and Theorem 5.2, the necessary and sufficient conditions for $(t_0^r(p) : \mu)$, $(\mu : t_0^r(p))$, $(t_c^r(p) : \mu)$, $(\mu : t_c^r(p))$ and $(t^r(p) : \mu)$, $(\mu : t^r(p))$ may be derived by replacing the entries of C and A by those of the entries of $E = C\{T(r)\}^{-1}$ and $B = T(r)A$, respectively; where the necessary and sufficient conditions on the matrices E and B are read from the concerning results in the existing literature.

The necessary and sufficient conditions characterizing the matrix mappings between the sequence spaces of Maddox are determined by Grosse-Erdmann [7]. Let N and K denote the finite subset of $\mathbb{N}$, L and M also denote the natural numbers. Prior to giving the theorems, let us suppose that (q_n) is a non-decreasing bounded sequence of positive numbers and consider the following conditions:

$$\lim_n |a_{nk}|^{q_n} = 0, \text{ for all } k \tag{5.3}$$

$$\forall L, \exists M \ni \sup_n L^{1/q_n} \sum_k |a_{nk}| M^{-1/p_k} < \infty, \tag{5.4}$$

$$\sup_n |\sum_k a_{nk}|^{q_n} < \infty, \tag{5.5}$$

$$\lim_n |\sum_k a_{nk}|^{q_n} = 0, \tag{5.6}$$

$$\forall L, \sup_n \sup_{k\in K_1} |a_{nk} L^{1/q_n}|^{p_k} < \infty, \tag{5.7}$$

$$\forall L, \exists M \ni \sup_n \sum_{k\in K_2} |a_{nk} L^{1/q_n} M^{-1}|^{p'_k} < \infty, \tag{5.8}$$

$$\forall M, \lim_n (\sum_k |a_{nk} M^{1/p_k})^{q_n} = 0, \tag{5.9}$$

$$\forall M, \sup_n \sum_k |a_{nk}| M^{1/p_k} < \infty, \tag{5.10}$$

$$\forall M, \exists (\alpha_k) \ni \lim_n (\sum_k |a_{nk} - \alpha_k| M^{1/p_k})^{q_n} = 0, \tag{5.11}$$

$$\forall M, \sup_K \sum_n |\sum_{k\in K} a_{nk} M^{1/p_k}|^{q_n} < \infty. \tag{5.12}$$

Lemma 5.1. *Let $A = (a_{nk})$ be an infinite matrix. Then*

- **(i):** *$A = (a_{nk}) \in (c_0(p) : \ell_\infty(q))$ if and only if (4.8) holds.*
- **(ii):** *$A = (a_{nk}) \in (c(p) : \ell_\infty(q))$ if and only if (4.8) and (5.5) hold.*
- **(iii):** *$A = (a_{nk}) \in (\ell(p) : \ell_\infty)$ if and only if (4.11) and (4.12) hold.*
- **(iv):** *$A = (a_{nk}) \in (c_0(p) : c(q))$ if and only if (4.4), (4.5) and (4.6) hold.*
- **(v):** *$A = (a_{nk}) \in (c(p) : c(q))$ if and only if (4.4), (4.5), (4.6) and (4.7) hold.*
- **(vi):** *$A = (a_{nk}) \in (\ell(p) : c)$ if and only if (4.11), (4.12) and (4.13) hold.*
- **(vii):** *$A = (a_{nk}) \in (c_0(p) : c_0(q))$ if and only if (5.3) and (5.4) hold.*
- **(viii):** *$A = (a_{nk}) \in (c(p) : c_0(q))$ if and only if (5.3), (5.4) and (5.6) hold.*
- **(ix):** *$A = (a_{nk}) \in (\ell(p) : c_0(q))$ if and only if (5.3), (5.7) and (5.8) hold.*
- **(x):** *$A = (a_{nk}) \in (\ell_\infty(p) : c_0(q))$ if and only if (5.9) holds.*
- **(xi):** *$A = (a_{nk}) \in (\ell_\infty(p) : c(q))$ if and only if (5.10) and (5.11) hold.*
- **(xii):** *$A = (a_{nk}) \in (\ell_\infty(p) : \ell(q))$ if and only if (5.12) holds.*
- **(xiii):** *$A = (a_{nk}) \in (c_0(p) : \ell(q))$ if and only if (4.2) holds.*
- **(xiv):** *$A = (a_{nk}) \in (c(p) : \ell(q))$ if and only if (4.2) and (4.4) hold.*

Corollary 5.1. *Let $A = (a_{nk})$ be an infinite matrix. The following statements hold:*

(i): $A \in (t_0^r(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) holds with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

(ii): $A \in (t_0^r(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3) and (5.4) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (t_0^r(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_0^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5) and (4.6) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.2. *Let* $A = (a_{nk})$ *be an infinite matrix. The following statements hold:*

(i): $A \in (t_c^r(p) : \ell_\infty(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.8) and (5.5) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

(ii): $A \in (t_c^r(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.4) and (5.6) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (t_c^r(p) : c(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t_c^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.4), (4.5), (4.6) and (4.7) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

Corollary 5.3. *Let* $A = (a_{nk})$ *be an infinite matrix. The following statements hold:*

(i): $A \in (t^r(p) : \ell_\infty)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.11) and (4.12) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

(ii): $A \in (t^r(p) : c_0(q))$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (5.3), (5.7) and (5.8) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (t^r(p) : c)$ *if and only if* $\{a_{nk}\}_{k\in\mathbb{N}} \in \{t^r(p)\}^\beta$ *for all* $n \in \mathbb{N}$ *and (4.11), (4.12) and (4.13) hold with* $\tilde{a}_{nk}$ *instead of* a_{nk}.

Corollary 5.4. *Let* $A = (a_{nk})$ *be an infinite matrix and* b_{nk} *be defined by (5.2). Then, following statements hold:*

(i): $A \in (\ell_\infty(q) : t_0^r(p))$ *if and only if (5.9) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(ii): $A \in (c_0(q) : t_0^r(p))$ *if and only if (5.3) and (5.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (c(q) : t_0^r(p))$ *if and only if (5.3), (5.4) and (5.6) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.5. *Let* $A = (a_{nk})$ *be an infinite matrix and* b_{nk} *be defined by (5.2). Then, following statements hold:*

(i): $A \in (\ell_\infty(q) : t_c^r(p))$ *if and only if (5.10) and (5.11) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(ii): $A \in (c_0(q) : t_c^r(p))$ *if and only if (4.4), (4.5) and (4.6) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (c(q) : t_c^r(p))$ *if and only if (4.4), (4.5), (4.6) and (4.7) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

Corollary 5.6. *Let* $A = (a_{nk})$ *be an infinite matrix and* b_{nk} *be defined by (5.2). Then, following statements hold:*

(i): $A \in (\ell_\infty(q) : t^r(p))$ *if and only if (5.12) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(ii): $A \in (c_0(q) : t^r(p))$ *if and only if (4.2) holds with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

(iii): $A \in (c(q) : t^r(p))$ *if and only if (4.2) and (4.4) hold with* b_{nk} *instead of* a_{nk} *with* $q = 1$.

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