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PERIODIC SOLUTIONS FOR THIRD ORDER DELAY DIFFERENTIAL EQUATION IMPULSES WITH FREDHOLM OPERATOR OF INDEX ZERO

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Abstract. In this paper the periodic solutions for third order delay differential equation of the form

$$x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \geq 0, t \neq t_k,$$

is investigated. We derive a third order delay differential equation with Fredholm operator of index zero and periodic solution. We obtain the existence of periodic solution and Mawhin's continuation theorem. The delay conditions for the Schwarz inequality of the periodic solutions are also obtained. An example is also furnished which demonstrates validity of main result. Some new positive periodic criteria are given. Therefore it has at least one 2π-periodic solution.

1. Introduction

The theory of impulsive delay differential equations is promising as an important role of investigation, since it is better than the corresponding theory of delay differential equation without impulse effects. Furthermore, such equations may demonstrate several real-world phenomena in physics,chemistry, biology, engineering, etc. In the last few years the theory of periodic solutions and delay differential equations with impulses has been studied by many authors, respectively [3, 5, 7, 8]. There are several books and a lot of papers dealing with the periodic solution of delay differential equations [1, 2, 4, 6, 9]. Periodic solutions of impulsive delay differential equations is a new research area and there are many publications in this field. The paper deals with impulsive equations with constant delay and Fredholm operator of index zero. We obtain the theorems of existence of periodic solution based on the following Mawhin's continuation theorem.

Let $PC(\mathbb{R}, \mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ be continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)\}$,

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$PC^1(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k^-) = x'(t_k)\}$.
$PC^2(\mathbb{R},\mathbb{R}) = \{x : \mathbb{R} \to \mathbb{R}, x(t)$ is continuous everywhere except for some t_k at which $x''(t_k^+)$ and $x''(t_k^-)$ exist and $x''(t_k^-) = x''(t_k)\}$.
Let $X = \{x(t) \in PC^1(\mathbb{R},\mathbb{R}), x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty\}$, where $|x|_\infty = \sup_{t\in[0,T]} |x(t)|$,
$Y = PC(\mathbb{R},\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n$, with norm $\|y\| = \max\{|u|_\infty, |c|\}$, where $u \in PC(\mathbb{R},\mathbb{R}), c = (c_1, \dots c_{2n}) \in R^n \times \mathbb{R}^n$, $|c| = \max_{1\le k\le 2n}\{|c_k|\}$.
$Z = PC(\mathbb{R},\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n$, with norm $\|z\| = \max\{|v|_\infty, |d|\}$, where $v \in PC(\mathbb{R},\mathbb{R}), d = (d_1, \dots d_{2n}) \in R^n \times \mathbb{R}^n$, $|d| = \max_{1\le k\le 2n}\{|d_k|\}$.
Then X, Y and Z are Banach spaces. $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ are a Fredholm operator of index zero, where $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y, R : Z \to Z$ are projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad \ker R = \operatorname{Im} L,$$

$$X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q, \quad Z = \operatorname{Im} L \oplus \operatorname{Im} R.$$

It continues that

$$L|_{D(L)\cap\ker P} : D(L) \cap \ker P \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.
Similarly it follows that

$$L|_{D(L)\cap\ker Q} : D(L) \cap \ker Q \to \operatorname{Im} L$$

is invertible and we assume the inverse of that map by K_q. Let Ω be an open bounded subset of Y, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact in $\overline{\Omega}$, if $RN(\overline{\Omega})$ is bounded and $K_q(I - R)N : \overline{\Omega} \to Y$ is compact.

2. Preliminaries

This paper obtains the existence of periodic solutions for the third-order delay differential equations with impulses

$$\begin{aligned} &x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t - \tau(t)) = p(t), t \ge 0, t \neq t_k,\\ &\Delta x(t_k) = I_k,\\ &\Delta x'(t_k) = J_k,\\ &\Delta x''(t_k) = K_k. \end{aligned} \tag{2.1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$, $x(t_k^-) = \lim_{t\to t_k^-} x(t)$, $x(t_k^-) = x(t_k)$;
$\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$, $x'(t_k^+) = \lim_{t\to t_k^+} x'(t)$, $x'(t_k^-) = \lim_{t\to t_k^-} x'(t)$, $x'(t_k^-) = x'(t_k)$;
$\Delta x''(t_k) = x''(t_k^+) - x''(t_k^-)$, $x''(t_k^+) = \lim_{t\to t_k^+} x''(t)$, $x''(t_k^-) = \lim_{t\to t_k^-} x''(t)$, $x''(t_k^-) = x''(t_k)$.

We assume that the following conditions:

(H1) $f \in C(\mathbb{R}^2,\mathbb{R})$ and $g(t + T, x) = g(t, x)$, $h \in C(\mathbb{R},\mathbb{R})$, $p, \tau \in C(\mathbb{R},\mathbb{R})$ with $\tau(t + T) = \tau(t)$, $p(t + T) = p(t)$;

(H2) $\{t_k\}$ satisfies $t_k < t_{k+1}$ and $\lim_{k\to\pm\infty} t_k = \pm\infty$, $k \in Z$, $I_k(x,y), J_k(x,y), K_k(x,y) \in C(\mathbb{R}^2,\mathbb{R})$, and there is a positive n such that $\{t_k\} \cap [0,T] = \{t_1, t_2, \ldots, t_n\}, t_{k+n} = t_k + T$, $I_{k+n}(x,y) = I_k(x,y), J_{k+n}(x,y) = J_k(x,y), K_{k+n}(x,y) = K_k(x,y)$.

(H3) There are constants $\sigma, \beta \geq 0$ such that

$$|f(t,x)| \leq \sigma|x|, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{2.2}$$

$$xf(t,x) \geq \beta|x|^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}; \tag{2.3}$$

(H4) There are constants $\sigma, \beta \geq 0$ such that

$$|g(t,x)| \leq \sigma|x|, \quad \forall (t,x) \in [0,T] \times \mathbb{R}, \tag{2.4}$$

$$x^2 g(t,x) \geq \beta|x|^2, \quad \forall (t,x) \in [0,T] \times \mathbb{R}; \tag{2.5}$$

(H5) there are constants $\beta_i \geq 0$ $(i = 1,2,3)$ such that

$$|h(x)| \geq \beta_1 + \beta_2|x|, \tag{2.6}$$

$$|h(x) - h(y)| \leq \beta_3|x - y|; \tag{2.7}$$

(H6) there are constants $\gamma_i > 0$ $(i = 1,2,3)$, such that $|\int_x^{x+\lambda J_k(x,y)} h(s)ds| \leq |J_k(x,y)|(\gamma_1 + \gamma_2|x| + \gamma_3|J_k(x,y)|), \quad \forall \lambda \in (0,1)$;

(H7) there are constants $a_k, a_k', a_k'' \geq 0$ such that $|K_k(x,y)| \leq a_k|x|^2 + a_k'|x| + a_k''$;

(H8) $zK_k(x,y) \leq 0$ and there are constants $b_k \geq 0$ such that $|K_k(x,y)| \leq b_k$.

Lemma 2.1. *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. We assume that the following conditions are satisfied:*

(i) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

(ii) $RNx \neq 0$, *for all* $x \in \partial\Omega \cap \ker L$;

(iii) $\deg\{KRNx, \Omega \bigcap \ker L, 0\} \neq 0$, *where* $K : \operatorname{Im} R \to \ker L$ *is an isomorphism.*

Then the abstract equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap D(L)$.

We assume the operators $L : D(L) \subset X \to Y$ and $L : D(L) \subset Y \to Z$ by

$$Lx = (x''', \Delta x(t_1), \ldots, \Delta x(t_n), \Delta x'(t_1), \ldots, \Delta x'(t_n), \Delta x''(t_1), \ldots, \Delta x''(t_n)), \tag{2.8}$$

and $N : X \to Y$, $N : Y \to Z$ by

$$\begin{aligned} Nx = (&-f(t, x''(t)) - g(t, x'(t)) - h(x(t - \tau(t))) + p(t), \\ &I_1(x(t_1)), \ldots, I_n(x(t_n)), J_1(x'(t_1)), \ldots, J_n(x'(t_n)), K_1(x''(t_1)), \ldots, K_n(x''(t_n))). \end{aligned} \tag{2.9}$$

Lemma 2.2. *L is a Fredholm operator of index zero with*

$$\ker L = \{x(t) = c, t \in \mathbb{R}\}, \tag{2.10}$$

and

$$\begin{aligned} &\operatorname{Im} L(y, z, a_1, \ldots, a_n, b_1, \ldots, b_n) \\ &= \int_0^T (y(s) + z(s))ds + \sum_{k=1}^n b_k(T - t_k) + \sum_{k=1}^n a_k + x'(0)T = 0. \end{aligned} \tag{2.11}$$

Let the linear operators $P : X \to X$, $Q : Y \to Y$ and $R : Z \to Z$ be defined by

$$Px = x(0), \tag{2.12}$$

$$
\begin{aligned}
&Q(y, a_1, \dots, a_n, b_1, \dots, b_n) \\
&= \frac{2}{T^2}[\int_0^T (T-s)y(s)ds + \sum_{k=1}^n b_k(T-t_k) + \sum_{k=1}^n a_k + x'(0)T], 0, \dots, 0),
\end{aligned} \tag{2.13}
$$

and

$$
\begin{aligned}
&R(z, a_1, \dots, a_n, b_1, \dots, b_n) \\
&= \frac{2}{T^2}[\int_0^T (T-s)z(s)ds + \sum_{k=1}^n b_k(T-t_k) + \sum_{k=1}^n a_k + x'(0)T], 0, \dots, 0).
\end{aligned} \tag{2.14}
$$

Lemma 2.3. *If $\alpha > 0$, $x(t) \in PC^2(\mathbb{R}, \mathbb{R})$ with $x(t+T) = x(t)$, then*

$$
\int_0^T \int_{t-\alpha}^t |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt \tag{2.15}
$$

and

$$
\int_0^T \int_t^{t+\alpha} |x'(s)|^2 \, ds \, dt = \alpha \int_0^T |x'(t)|^2 dt. \tag{2.16}
$$

Let

$$
\begin{aligned}
A_1(t,\alpha) &= \sum_{t-\alpha \le t_k \le t} a_k, \quad A_2(t,\alpha) = \sum_{t \le t_k \le t+\alpha} a_k, \\
B_1(t,\alpha) &= \sum_{t-\alpha \le t_k \le t} a'_k, \quad B_2(t,\alpha) = \sum_{t \le t_k \le t+\alpha} a'_k, \\
C_1(t,\alpha) &= \sum_{t-\alpha \le t_k \le t} a''_k, \quad C_2(t,\alpha) = \sum_{t \le t_k \le t+\alpha} a''_k, \\
I_1 &= \Big(\int_0^T A_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T A_2^2(t,\alpha)dt\Big)^{1/2}, \\
I_2 &= \Big(\int_0^T B_1^2(t,\alpha)dt\Big)^{1/2} + \Big(\int_0^T B_2^2(t,\alpha)dt\Big)^{1/2}, \\
I_3 &= \int_0^T A_1^2(t,\alpha)dt + \int_0^T A_2^2(t,\alpha)dt, \\
I_4 &= \int_0^T A_1(t,\alpha)B_1(t)dt + \int_0^T A_2(t,\alpha)B_2(t)dt, \\
I_5 &= \int_0^T B_1^2(t,\alpha)dt + \int_0^T B_2^2(t,\alpha)dt
\end{aligned}
$$

The following Lemma is important for us to the delay $\tau(t)$.

Lemma 2.4. *Suppose $\tau(t) \in C(\mathbb{R}, \mathbb{R})$ with $\tau(t+T) = \tau(t)$ and $\tau(t) \in [-\alpha, \alpha]$ for all $t \in [0, T]$, $x(t) \in PC^1(\mathbb{R}, \mathbb{R})$ with $x(t+T) = x(t)$ and there is a positive n such that $\{t_k\} \cap [0, T] = \{t_1, t_2, \dots, t_n\}$, $\Delta x(t_k) = \lambda I_k(x(t_k), x'(t_k))$ for all $\lambda \in (0, 1)$ and $t_{k+n} = t_k + T, I_{k+n}(x, y) = I_k(x, y)$. Furthermore there exist nonnegative constants*

a_k, a_k *such that* $|I_k(x,y)| \le a_k|x| + a'_k$. *Then*

$$\begin{aligned}
&\int_0^T |x(t)-x(t-\tau(t))|^2 dt \\
&\le 2\alpha^2 \int_0^T |x'(t)|^2 dt + 2\alpha I_1 |x(t)|_\infty \Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + 2\alpha I_2 \Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + I_3|x(t)|_\infty^2 + I_4|x(t)|_\infty + I_5.
\end{aligned} \tag{2.17}$$

3. Main results

We establish the theorems of existence of periodic solution based on the following two conditions.

Theorem 3.1. *We assume that* (H1)–(H8) *hold. Then* (3.3) *has at least one* T-*periodic solution and*

$$\sum_{k=1}^n a_k < 1, \tag{3.1}$$

$$\begin{aligned}
&\Big[\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)\Big]M^2 + \beta_3\Big[2|\tau(t)|_\infty^2 \\
&+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M + I_3(|\tau(t)|_\infty)M^2\Big]^{1/2} < \beta,
\end{aligned} \tag{3.2}$$

where

$$M = \frac{1}{1-\sum_{k=1}^n a_k}\Big(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}\Big).$$

Proof. Consider the abstract equation $Lx = \lambda Nx$, with $\lambda \in (0,1)$, where L and N are given by (2.8) and (2.9). Let

$$\Omega_1 = \{x \in D(L) : \ker L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

For $x \in \Omega_1$, we get

$$x'''(t) + f(t, x''(t)) + g(t, x'(t)) + h(x(t-\tau(t)) = p(t), t \ge 0, t \ne t_k, \tag{3.3}$$

Integrating the interval on $[0,T]$, using Schwarz inequality, we get

$$\begin{aligned}
&|\int_0^T h(x(t-\tau(t))dt| \\
&= |\int_0^T p(t)dt - \int_0^T f(t,x''(t))dt - \int_0^T g(t,x'(t))dt + \sum_{k=1}^n K_k(x(t_k), x''(t_k))| \\
&\le T|p(t)|_\infty + \sigma \int_0^T |x''(t)|dt + \sum_{k=1}^n b_k \\
&\le \sigma T^{1/2}\Big(\int_0^T |x''(t)|^2 dt\Big)^{1/2} + T|p(t)|_\infty + \sum_{k=1}^n b_k.
\end{aligned}$$

From the above formula, there is a interval on $t_0 \in [0,T]$ such that

$$|h(x(t_0-\tau(t_0))| \le \frac{\sigma}{T^{1/2}}\Big(\int_0^T |x''(t)|^2 dt\Big)^{1/2} + |p(t)|_\infty + \frac{1}{T}\sum_{k=1}^n b_k.$$

From (2.6),we get

$$\beta_1+\beta_2|x(t_0-\tau(t_0))|\leq \frac{\sigma}{T^{1/2}}(\int_0^T|x''(t)|^2dt)^{1/2}+|p(t)|_\infty+\frac{1}{T}\sum_{k=1}^n b_k.$$

Then

$$|x(t_0-\tau(t_0))|\leq \frac{\sigma}{\beta_2T^{1/2}}\Big(\int_0^T|x''(t)|^2dt\Big)^{1/2}+d,$$

where $d=\big(||p(t)|_\infty+\frac{1}{T}\sum_{k=1}^n b_k-\beta_1|\big)/\beta_2$. So there is an integer m and an interval $t_1\in[0,T]$ such that $t_0-\tau(t_0)=mT+t_1$. Therefore

$$|x(t_1)|=|x(t_0-\tau(t_0))|\leq \frac{\sigma}{\beta_2T^{1/2}}\Big(\int_0^T|x''(t)|^2dt\Big)^{1/2}+d,$$

$$x(t)=x(t_1)+\int_{t_1}^t x''(s)ds+\sum_{t_1\leq t_k<t}K_k(x(t_k),x''(t_k)).$$

Thus

$$\begin{aligned}|x(t)|_\infty&\leq |x(t_1)|+\int_{t_1}^t|x''(s)|ds+\sum_{t_1\leq t_k<t}|K_k(x(t_k))|\\&\leq \frac{\sigma}{\beta_2T^{1/2}}(\int_0^T|x''(t)|^2dt)^{1/2}+d+\int_0^T|x''(t)|dt+\sum_{k=1}^n a_k|x|_\infty+\sum_{k=1}^n a_k'+\sum_{k=1}^n a_k''\\&\leq |x|_\infty\sum_{k=1}^n a_k+(\frac{\sigma}{\beta_2T^{1/2}}+T^{1/2})\Big(\int_0^T|x''(t)|^2dt\Big)^{1/2}+d+\sum_{k=1}^n a_k'+\sum_{k=1}^n a_k''.\end{aligned}$$

It continues that

$$\begin{aligned}|x(t)|_\infty&\leq \frac{d+\sum_{k=1}^n a_k''}{1-\sum_{k=1}^n a_k}+\frac{1}{1-\sum_{k=1}^n a_k}(\frac{\sigma}{\beta_2T^{1/2}}+T^{1/2})(\int_0^T|x''(t)|^2dt)^{1/2}\\&=c_1+M(\int_0^T|x''(t)|^2dt)^{1/2},\end{aligned}\tag{3.4}$$

where c_1 is a positive constant. On the other hand, multiplying both side of (3.3) by $x'(t)$, we have

$$\int_0^T x'''(t)x''(t)dt+\lambda\int_0^T f(t,x''(t))x'(t)dt\quad+\lambda\int_0^T g(t,x'(t))x'(t)dt+\lambda\int_0^T h(t,x(t-\tau(t))x'(t)dt$$
$$=\lambda\int_0^T p(t)x'(t)dt.$$

Since

$$\int_0^T x'''(t)x''(t)dt=-\frac{1}{2}\sum_{i=1}^n[(x''(t_k^+))^2-(x''(t_k))^2],$$

Our assumption (H7) that

$$
\begin{aligned}
&(x'(t_k^+))^2 - (x'(t_k))^2 \\
&= (x'(t_k^+) + x'(t_k))(x'(t_k^+) - (x'(t_k)) \\
&= \Delta x'(t_k)(2x'(t_k) + \Delta x'(t_k)) \\
&= \lambda K_k(x(t_k), x'(t_k))(2x'(t_k) + \lambda K_k(x(t_k), x'(t_k)) \\
&= 2\lambda K_k(x(t_k), x'(t_k))x'(t_k) + [\lambda K_k(x(t_k), x'(t_k))]^2 \le b_k^2.
\end{aligned}
$$

In (2.5), by use Schwarz inequality

(3.5)

$$
\begin{aligned}
&\beta \int_0^T |x''(t)|^2 dt \\
&\le -\int_0^T h(x(t-\tau(t))x'(t)dt + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{k=1}^n b_k^2 \\
&= \int_0^T [h(x(t) - h(x(t-\tau(t))]x'(t)dt - \int_0^T h(x(t))x'(t)dt \\
&\quad + \int_0^T p(t)x'(t)dt + \frac{1}{2}\sum_{i=1}^n b_k^2 \\
&\le \int_0^T |h(x(t)) - h(x(t-\tau(t))||x'(t)|dt + |p(t)|_\infty \int_0^T |x'(t)|dt \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^n b_k^2 \\
&\le \Big[\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2} + |p(t)|_\infty T^{1/2}\Big]\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} \\
&\quad + |\int_0^T h(x(t))x'(t)dt| + \frac{1}{2}\sum_{i=1}^n b_k^2.
\end{aligned}
$$

From (H5) and (H6), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt|\\
&=|\int_{x(0)}^{x(t_1)} h(s)ds+\int_{x(t_1^+)}^{x(t_2)} h(s)ds+\cdots+\int_{x(t_n^+)}^{x(T)} h(s)ds|\\
&=|\int_{x(0)}^{x(T)} h(s)ds-\sum_{k=1}^{n}\int_{x(t_k)}^{x(t_k^+)} h(s)ds|\\
&\le\sum_{k=1}^{n}|\int_{x(t_k)}^{x(t_k)+\lambda K_k(x(t_k),x'(t_k))} h(s)ds|\\
&\le\sum_{k=1}^{n}[|K_k(x(t_k),x'(t_k))|(\gamma_1+\gamma_2|x(t_k)|+\gamma_3|K_k(x(t_k),x'(t_k))|)]\\
&\le[\gamma_2(\sum_{k=1}^{n}a_k)+\gamma_3(\sum_{k=1}^{n}a_k^2)]|x(t)|_\infty^2+c_2|x(t)|_\infty+c_3,
\end{aligned}
$$

where c_2, c_3 are constants. From (3.4), we get

$$
\begin{aligned}
&|\int_0^T h(x(t))x'(t)dt|\\
&\le[\gamma_2(\sum_{k=1}^{n}a_k)+\gamma_3(\sum_{k=1}^{n}a_k^2)]M^2\int_0^T|x'(t)|^2dt+c_4(\int_0^T|x'(t)|^2dt)^{1/2}+c_5,
\end{aligned}
\tag{3.6}
$$

where c_4, c_5 are constants. From Lemma 2.4, we get

$$
\begin{aligned}
&\int_0^T|h(x(t)-h(x(t-\tau(t)))|^2dt\\
&\le\beta_3^2\int_0^T|x(t)-x(t-\tau(t))|^2dt\\
&\le\beta_3^2[2|\tau(t)|_\infty^2\int_0^T|x'(t)|^2dt+2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)|x(t)|_\infty\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}\\
&\quad+2|\tau(t)|_\infty I_2(|\tau(t)|_\infty)\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}+I_3(|\tau(t)|_\infty)|x(t)|_\infty^2\\
&\quad+I_4(|\tau(t)|_\infty)|x(t)|_\infty+I_5(|\tau(t)|_\infty)].
\end{aligned}
$$

Substituting (3.4) into the above inequality, we get

$$
\begin{aligned}
&\int_0^T|h(x(t)-h(x(t-\tau(t)))|^2dt\\
&\le\beta_3^2[2|\tau(t)|_\infty^2+2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M\\
&\quad+I_3(|\tau(t)|_\infty)M^2]\int_0^T|x'(t)|^2dt+c_6\Big(\int_0^T|x'(t)|^2dt\Big)^{1/2}+c_7,
\end{aligned}
$$

where c_6, c_7 are constants. From above inequality

$$
(a+b)^{1/2}\le a^{1/2}+b^{1/2}\quad for\quad a\ge 0, b\ge 0,
\tag{3.7}
$$

we get

$$\Big(\int_0^T |h(x(t)) - h(x(t-\tau(t)))|^2 dt\Big)^{1/2}$$
$$\leq \beta_3[2|\tau(t)|_\infty^2 + 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M$$
$$+ I_3(|\tau(t)|_\infty)M^2]^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/2} + c_6^{1/2}\Big(\int_0^T |x'(t)|^2 dt\Big)^{1/4} + c_7^{1/2}.$$

Substituting the above formula and (3.6) in (3.5), we get

$$\big\{\beta - [\gamma_2(\sum_{k=1}^n a_k) + \gamma_3(\sum_{k=1}^n a_k^2)]M^2 - \beta_3[2|\tau(t)|_\infty^2$$
$$+ 2|\tau(t)|_\infty I_1(|\tau(t)|_\infty)M + I_3(|\tau(t)|_\infty)M^2]^{1/2}\big\}\int_0^T |x'(t)|^2 dt$$
$$\leq c_8(\int_0^T |x'(t)|^2 dt)^{\frac{3}{4}} + c_9(\int_0^T |x'(t)|^2 dt)^{1/2} + c_{10},$$

where c_8, c_9, c_{10} are constants. There is a constant $M_1 > 0$ such that

$$\int_0^T |x'(t)|^2 dt \leq M_1. \tag{3.8}$$

From (3.4), we get

$$|x(t)|_\infty \leq d + M(\int_0^T |x'(t)|^2 dt)^{1/2} \leq d + M(M_1)^{1/2}.$$

Then there is a constant $M_2 > 0$ such that $|x(t)|_\infty \leq M_2$. Therefore, integrating (3.3) on the interval $[0, T]$, using Schwarz inequality, we get

$$\int_0^T |x'''(t)|dt = \int_0^T |-f(t, x''(t)) - g(t, x'(t)) - h(x(t-\tau(t))) + p(t)|dt$$
$$\leq \int_0^T |f(t, x''(t))|dt + \int_0^T |g(t, x''(t))|dt + \int_0^T |h(x(t-\tau(t)))|dt + \int_0^T |p(t)|dt$$
$$\leq \sigma \int_0^T |x''(t)|dt + h_\delta T + T|p(t)|_\infty$$
$$\leq \sigma T^{1/2}(\int_0^T |x''(t)|^2 dt)^{1/2} + h_\delta T + T|p(t)|_\infty$$
$$\leq \sigma T^{1/2}(M_1)^{1/2} + h_\delta T + T|p(t)|_\infty,$$

where $h_\delta = \max_{|x|\leq\delta}|g(x)|$. Then there is a constant $M_3 > 0$ such that

$$\int_0^T |x''(t)|dt \leq M_3. \tag{3.9}$$

From (3.8),then there are $t_2 \in [0, T]$ and $c > 0$ such that $|x'(t_2)| \leq c$ for $t \in [0, T]$

$$|x'(t)|_\infty \leq |x'(t_2)| + \int_0^T |x''(t)|dt + \sum_{k=1}^n b_k. \tag{3.10}$$

Then there is a constant $M_4 > 0$ such that

$$|x'(t)|_\infty \leq M_4. \tag{3.11}$$

It follows that there is a constant $I_2 > \max\{M_2, M_4\}$ such that $\|x\| \le I_2$, Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in \ker L, RNx = 0\}$. If $x \in \Omega_2$, then $x(t) = c \in R$ and satisfies

$$RN(x,0) = (-\frac{2}{T^2}\int_0^T [f(t,0) + g(t,0) + h(c) - p(t)]dt, 0, \dots, 0) = 0. \tag{3.12}$$

we get

$$\int_0^T [f(t,0) + g(t,0) + h(c) - p(t)]dt = 0. \tag{3.13}$$

In (3.13),there must be a interval $t_0 \in [0,T]$ such that

$$h(c) = -f(t_0,0) - g(t_0,0) + p(t_0). \tag{3.14}$$

From (3.14) and assumption (H3), (H4), we get

$$\beta_1 + \beta_2|c| \le |h(c)| \le |f(t_0,0)| + |g(t_0,0)| + |p(t_0)| \le \sigma \times 0 + |p(t)|_\infty. \tag{3.15}$$

Then

$$|c| \le \frac{||p(t)|_\infty - \beta_1|}{\beta_2} \tag{3.16}$$

which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2} \cup \overline{\Omega_3}$, where $\Omega_3 = \{x \in X : |x| < ||p(t)|_\infty - \beta_1|/\beta_2 + 1\}$. By Lemmas 2.2, we can see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument,

(i) $Lx \ne \lambda Nx$ for all $x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
(ii) $RNx \ne 0$ for all $x \in \partial\Omega \cap \ker L$.

Finally we prove that (iii) of Lemma 2.1 is satisfied. We take $H(x,\mu) : \Omega \times [0,1] \to X$,

$$H(x,\mu) = \mu x + \frac{2(1-\mu)}{T^2}\int_0^T [-f(t,x''(t)) - g(t,x'(t)) + h(x(t-\tau(t)) + p(t)]dt.$$

From assumptions (H3) and (H4), we can easily verify $H(x,\mu) \ne 0$, for all $(x,\mu) \in \partial\Omega \cap \ker L \times [0,1]$, which results in

$$\begin{aligned}\deg\{KRNx, \Omega \cap \ker L, 0\} &= \deg\{H(x,0), \Omega \cap \ker L, 0\} \\ &= \deg\{H(x,1), \Omega \cap \ker L, 0\} \ne 0,\end{aligned}$$

where $K(x,0,\dots,0) = x$. Therefore, by Lemma 2.1, Equation (3.3) has at least one T-periodic solution. □

Example 1. Consider the third order delay differential equation with impulses

$$\begin{aligned} x'''(t) + \frac{1}{3}x''(t) + \frac{1}{6}x'(t) + \frac{1}{21}x(t - \frac{1}{10}\cos t) &= \sin t, \quad t \ne k, \\ I_k(x,y) &= \frac{\sin\frac{k\pi}{3}}{120}x + \frac{y}{1+y^2}, \\ J_k(x,y) &= -\frac{2x^2y}{1+x^4y^2}, \\ K_k(x,y) &= -\frac{4x^4y}{1+x^8y^2}, \end{aligned} \tag{3.17}$$

where $t_k = k$, $f(t,x) = \frac{1}{3}x^2$,$g(t,x) = \frac{1}{6}x$, $h(y) = \frac{1}{21}y$, $p(t) = \sin t$, $\tau(t) = \frac{1}{10}\cos t$, it is easy to see that $|\tau(t)|_\infty = \frac{1}{10}$, $T = 2\pi, \{k\} \cap [0, 2\pi] = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $\sigma = \beta = \frac{1}{3}$, $\beta_1 = 0$, $\beta_2 = \beta_3 = \frac{1}{21}$. Since $|I_k(x,y)| \le \frac{1}{120}|x| + \frac{1}{2}$,
$|J_k(x,y)| \le 1$,$|\int_x^{x+I_k(x,y)} h(s)ds| \le |I_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|I_k(x,y)|)$,
$|K_k(x,y)| \le 1$,$|\int_x^{x+J_k(x,y)} h(s)ds| \le |J_k(x,y)|(\frac{1}{21}|x| + \frac{1}{42}|J_k(x,y)|)$,
then we take $a_k = \frac{1}{120}$, $a'_k = \frac{1}{2}$, $b'_k = 1$ $(k = 1, 2, 3, 4, 5, 6, 7, 8)$, $\gamma_1 = 0$, $\gamma_2 = 1/21$, $\gamma_3 = 1/42$.

$$\sum_{k=1}^{8} a_k = \frac{1}{20} < 1,$$

$$M = \frac{1}{1 - \sum_{k=1}^{n} a_k}(\frac{\sigma}{\beta_2 T^{1/2}} + T^{1/2}) = \frac{1}{1 - \frac{1}{20}}(\frac{\frac{1}{3}}{\frac{1}{21}(2\pi)^{1/2}} + (2\pi)^{1/2}) < 8.$$

By Theorem 3.1, Equation (3.17) has at least one 2π-periodic solution.

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