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NEW THEOREMS ON GENERAL INTEGRAL INEQUALITIES, VARIANTS OF THE LEVINSON OR HARDY INTEGRAL INEQUALITY

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Abstract. This article makes contributions to the field of integral inequalities. Under certain assumptions, such as monotonicity and convexity, four theorems show how the Levinson or Hardy integral inequality can be generalized, improved or modified. Multiple functions are involved, and new lower and upper bounds are obtained. Applications are given, with an emphasis on inequalities using the Laplace transform of certain functions.

1. Introduction

Integrals are a fundamental concept in mathematics. They were originally developed to calculate areas under curves and volumes of solids. Their modern use extends far beyond geometry. In particular, they solve complex problems in physics, engineering, finance, economics and environmental science. For more details, see the reference books [5, 14, 16, 12]. A well-known limitation is that most integrals cannot be evaluated in closed form. As a result, some techniques have been developed to analyze them, such as the integral inequality technique. It aims to estimate complex integrals by bounding them with simpler ones. This is also useful for studying convergence and error bounds. Famous inequalities include the Cauchy-Schwarz, Hölder, Minkowski, Young, Hilbert and Hardy integral inequalities. All of these form the theoretical basis for many advanced results in analysis. For more details, see the reference books [7, 3, 17, 10, 2, 19].

In this article, we place particular emphasis on the Hardy integral inequality. Historically, it was first established in [6] and has since attracted much attention. A representative formulation of this inequality is given below. Let $p > 1$ and $f : (0, +\infty) \mapsto (0, +\infty)$ be a (positive) function. For any $t \in (0, +\infty)$, we consider

$$F(t) = \int_0^t f(x)dx,$$

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which corresponds to the natural primitive of f. Then the Hardy integral inequality compares an integral depending on F^p with that depending on f^p, as follows:

$$\int_0^{+\infty} \left[\frac{1}{t}F(t)\right]^p dt \leq \kappa \int_0^{+\infty} [f(t)]^p dt, \tag{1.1}$$

where

$$\kappa = \left(\frac{p}{p-1}\right)^p, \tag{1.2}$$

provided that the integrals in question exist; it is therefore sufficient to assume that $\int_0^{+\infty}[f(t)]^p dt < +\infty$. It is worth noting that the constant κ is the best in this context. A generalization of this inequality to a more flexible interval of integration is known as the Levinson integral inequality. It was originally proposed in [8]. A formal statement of this inequality is given below. Let $(a, b) \in (0, +\infty)$, including the case $b \to +\infty$, with $a < b$ and $f : (a, b) \mapsto (0, +\infty)$ be a function. For any $t \in (a, b)$, we consider

$$F(t) = \int_a^t f(x)dx.$$

Then the Levinson integral inequality states that

$$\int_a^b \left[\frac{1}{t}F(t)\right]^p dt \leq \kappa \int_a^b [f(t)]^p dt, \tag{1.3}$$

where κ is given in Equation (1.2), provided that the integrals in question exist. The constant κ is still the best in this context. The Hardy and Levinson integral inequalities have been extended by various mathematical approaches using intermediate functions or parameters. This has led to some variants that have further enriched the theory. We refer to the contributions in [15, 13, 9, 18, 4, 1].

A key result from [15] serves as the main inspiration for this article. Specifically, it is stated as [15, Theorem 2.1]. For clarity and consistency, we present a slightly modified version of the theorem below.

Theorem 1.1. [15, Theorem 2.1] *Let $b \in (0, +\infty)$, including the case $b \to +\infty$, $f : (0, b) \mapsto (0, +\infty)$, and $\Phi : (0, +\infty) \mapsto (0, +\infty)$ be two functions. For any $t \in (0, b)$, we consider*

$$F(t) = \int_0^t f(x)dx.$$

Then, assuming that f and Φ are non-decreasing, we have

$$\int_0^b \Phi\left[\frac{1}{t}F(t)\right] dt \leq \int_0^b \Phi[f(t)]dt,$$

provided that the integrals in question exist.

Considering $\Phi(x) = x^p$, $x \in (0, b)$, with $p \geq 1$, we obtain

$$\int_0^b \left[\frac{1}{t}F(t)\right]^p dt \leq \int_0^b [f(t)]^p dt, \tag{1.4}$$

still provided that the integrals in question exist.

This theorem, and especially the result presented in Equation (1.4), holds significant importance for the reasons outlined below.

- For any $p \geq 1$, the constant κ in Equation (1.4) satisfies $\kappa > 1$. Therefore, when b is finite, the result in Equation (1.4) shows that the Levinson integral inequality can be significantly improved under an additional assumption on f, i.e., that f is non-decreasing. Thus, the constant κ is indeed the best in the general context, but it can be improved by making some additional assumptions on f.
- The corresponding proof is quite elementary, using only monotonicity arguments, unlike the proof of the Levinson integral inequality.

Some new notes on Theorem 1.1 are given below.

Note 1.: The interval $(0, b)$, with $b > 0$, is taken as the domain of definition for f. We can see how it can be adapted to a more general interval, such as (a, b), with $a < b$ and possible negative values for a and b.

Note 2.: To make sense of f^p with $p \geq 1$, it is assumed that f is positive. We can think about how to adapt this result for functions f that may be negative, especially if we consider a composition function of the form $\Phi(f)$.

Note 3.: The monotonicity assumption on f and Φ is the non-decreasing assumption. A natural question is what happens if f or Φ has a different monotonicity assumption, such that f or Φ is non-decreasing, or if f or Φ is not monotonic at all.

Note 4.: The L_p norm is implicitly used to define the main integral terms. We may consider extending this result to the weighted L_p norm, say $L_p(g)$, with a weight function g that has the least restrictive possible assumptions.

Note 5.: We can also think of finding a lower bound for the main integral term depending on F, under similar assumptions.

Note 6.: The function $\Phi(x) = x^p$, $x \in (0, +\infty)$, is indeed increasing for $p > 0$, relaxing the condition $p \geq 1$ associated with the Hardy integral inequality. The consideration of $p > 0$ can open up some new applications in theory and practice. The case $p < 0$ also deserves special attention, modulo modifications of the assumptions.

Note 7.: The function Φ is assumed to be positive, but this is not really used in the proof; the Jensen integral inequality, which is the main tool of the proof, does not have this requirement. We can relax this assumption.

In this article, we explore in depth the mathematical details raised in these notes. Four theorems are established, each of which proposes new variants of the Levinson or Hardy integral inequality. These variants are very general; they involve multiple functions and rely on broad assumptions of monotonicity and convexity. Thanks to these assumptions, we derive inequalities that generalize, improve or modify the Levinson or Hardy integral inequality, as well as Theorem 1.1. We also complete, in a sense, the article [15], in which several results on integral inequalities have in common the combination of the assumptions of monotonicity and convexity, but with a different perspective than ours; we focus only on the constant factor 1 instead of κ, or on original general bounds which, to the best of our knowledge, have not been presented in the literature. Some proofs use well-known integral inequalities, such as the Hermite-Hadamard and Jensen integral inequalities (see [10, 11]). Concrete examples are given to illustrate some applications of the framework. Emphasis is placed on inequalities involving the Laplace transform of the main functions.

The rest of the article is basically divided into three sections: Section 2 presents the main theorems and illustrative examples. The corresponding proofs are given in detail in Section 3. A conclusion is formulated in Section 4.

2. General theorems

As motivated earlier, several variants of the Levinson or Hardy integral inequality are presented in the theorem below. They have the advantages of dealing with a general integration interval, multiple functions including a weight function in the context of the L_p norm, simple monotonicity assumptions, and the determination of new upper and lower bounds.

Theorem 2.1. *Let $(a, b) \in \mathbb{R}^2$ with $a < b$, including the case $b \to +\infty$, $f : (a, b) \mapsto \mathbb{R}$, $g : (a, b) \mapsto (0, +\infty)$ and $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be three functions. For any $t \in (a, b)$, we consider*

$$F(t) = \int_a^t f(x)dx.$$

Then the four results below hold.

(1) *If we assume that f and Φ are non-decreasing, then we have*

$$\Phi[f(a)] \int_a^b g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \int_a^b \Phi[f(t)]g(t)dt,$$

provided that the integrals in question exist.

(2) *If we assume that f is non-decreasing and Φ is non-increasing, then we have*

$$\int_a^b \Phi[f(t)]g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \Phi[f(a)] \int_a^b g(t)dt,$$

provided that the integrals in question exist.

(3) *If we assume that f and Φ are non-increasing, then we have*

$$\Phi[f(a)] \int_a^b g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \int_a^b \Phi[f(t)]g(t)dt,$$

provided that the integrals in question exist.

(4) *If we assume that f is non-increasing and Φ is non-decreasing, then we have*

$$\int_a^b \Phi[f(t)]g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \Phi[f(a)] \int_a^b g(t)dt,$$

provided that the integrals in question exist.

This statement assumes that $\Phi[f(a)]$ is finite, with $\Phi[f(a)] = \Phi[\lim_{t\to a} f(t)]$. The proof of this theorem, and all the theorems that follow it, is postponed to Section 3.

Some examples of applications of Theorem 2.1 are given below, assuming that all the mathematical quantities involved exist (integrals, functions taken at a point,...).

Examples of the item 1.: if we take $a = 0$, $b \to +\infty$, $\Phi(x) = x^p$, $x \in (0, +\infty)$, with $p > 0$, which is non-decreasing, and $g(t) = e^{-\lambda t}$, $t \in (0, +\infty)$, with $\lambda > 0$, and consider $f : (0, +\infty) \mapsto (0, +\infty)$ non-decreasing, the item 1 of Theorem 2.1 gives

$$\frac{1}{\lambda}[f(0)]^p \leq \int_0^{+\infty} \left[\frac{1}{t}F(t)\right]^p e^{-\lambda t} dt \leq \int_0^{+\infty} [f(t)]^p e^{-\lambda t} dt,$$

where we have used

$$\int_0^{+\infty} g(t)dt = \int_0^{+\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

for the left-hand side term.

An interesting feature of this result is that it can be reformulated in terms of the Laplace transform of the functions involved. More precisely, we have

$$\frac{1}{\lambda}[f(0)]^p \leq \mathcal{L}\left(\chi^p\right)(\lambda) \leq \mathcal{L}\left(f^p\right)(\lambda),$$

where χ denotes the ratio function defined by

$$\chi(t) = \frac{1}{t}F(t), \tag{2.1}$$

and the Laplace transform of a given function, say $\ell : (0, +\infty) \mapsto \mathbb{R}$, is classically defined by

$$\mathcal{L}(\ell)(\lambda) = \int_0^{+\infty} \ell(t) e^{-\lambda t} dt,$$

with $\lambda > 0$.

As another simple example of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = x/(1 - x)$, $x \in (0, 1)$, which is non-decreasing, and $g(t) = 1$, $t \in (0, 1)$, and consider $f : (0, 1) \mapsto (0, 1)$ non-decreasing, we get

$$\frac{f(0)}{1 - f(0)} \leq \int_0^1 \frac{F(t)}{t - F(t)} dt \leq \int_0^1 \frac{f(t)}{1 - f(t)} dt.$$

Examples of the item 2.: If we take $a = 0$, $b \to +\infty$, $\Phi(x) = x^{-p}$, $x \in (0, +\infty)$, with $p > 0$, which is non-increasing, and $g(t) = e^{-\lambda t}$, $t \in (0, +\infty)$, with $\lambda > 0$, and consider $f : (0, +\infty) \mapsto (0, +\infty)$ non-decreasing, the item 2 of Theorem 2.1 gives

$$\int_0^{+\infty} [f(t)]^{-p} e^{-\lambda t} dt \leq \int_0^{+\infty} \left[\frac{1}{t}F(t)\right]^{-p} e^{-\lambda t} dt \leq \frac{1}{\lambda}[f(0)]^{-p}.$$

Using the Laplace Transform, this can be written as follows:

$$\mathcal{L}\left(f^{-p}\right)(\lambda) \leq \mathcal{L}\left(\chi^{-p}\right)(\lambda) \leq \frac{1}{\lambda}[f(0)]^{-p},$$

where χ is given in Equation (2.1).

As another simple example of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = 1/(1 + x)$, $x \in (0, +\infty)$, which is non-increasing, and $g(t) = 1$, $t \in (0, 1)$, and consider $f : (0, 1) \mapsto (0, +\infty)$ non-decreasing, we get

$$\int_0^1 \frac{1}{1 + f(t)} dt \leq \int_0^1 \frac{t}{t + F(t)} dt \leq \frac{1}{1 + f(0)}.$$

Many other examples can be given in the same way. The configuration to be adopted therefore depends on the mathematical context being considered.

The result below presents variants of the Levinson or Hardy integral inequality with a general integration interval, the use of multiple functions and under monotonicity and convexity assumptions.

Theorem 2.2. *Let $(a,b)\in\mathbb{R}^2$ with $a<b$, including the case $b\to+\infty$, $f:(a,b)\mapsto\mathbb{R}$, $g:(a,b)\mapsto(0,+\infty)$ and $\Phi:\mathbb{R}\mapsto\mathbb{R}$ be three functions. For any $t\in(a,b)$, we consider*

$$F(t)=\int_a^t f(x)dx.$$

Then the four results below hold.

(1) *If we assume that f is convex and Φ is non-decreasing, then we have*

$$2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt,$$

provided that the integrals in question exist.

(2) *If we assume that f is convex and Φ is non-increasing, then we have*

$$\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le 2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt,$$

provided that the integrals in question exist.

(3) *If we assume that f is concave and Φ is non-decreasing, then we have*

$$\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le 2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt,$$

provided that the integrals in question exist.

(4) *If we assume that f is concave and Φ is non-increasing, then we have*

$$2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt,$$

provided that the integrals in question exist.

This statement assumes that, when $f(a)$ is used, $\Phi[f(a)]$ is finite. In addition, when $b\to+\infty$, we have

$$2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt=2\int_a^{+\infty}\Phi\left[f(t)\right]g(2t-a)dt,$$

which simplifies the situation.

Some examples of applications of Theorem 2.2 are given below, assuming that all the mathematical quantities involved exist.

Examples of the item 1.: If we take $a = 0$, $b \to +\infty$, $\Phi(x) = x^p$, $x \in (0, +\infty)$, with $p > 0$, which is non-decreasing, and $g(t) = e^{-\lambda t}$, $t \in (0, +\infty)$, with $\lambda > 0$, and consider $f : (0, +\infty) \mapsto (0, +\infty)$ convex, the item 1 of Theorem 2.2 gives

$$2e^{\lambda a} \int_0^{+\infty} [f(t)]^p \, e^{-2\lambda t} dt \leq \int_0^{+\infty} \left[\frac{1}{t} F(t) \right]^p e^{-\lambda t} dt \leq \frac{1}{2^p} \int_0^{+\infty} [f(0) + f(t)]^p e^{-\lambda t} dt.$$

We can rewrite this inequality in terms of the Laplace transform of the functions involved, as follows:

$$2e^{\lambda a} \mathcal{L}\left(f^p\right)(2\lambda) \leq \mathcal{L}\left(\chi^p\right)(\lambda) \leq \frac{1}{2^p} \mathcal{L}\left\{[f(0) + f]^p\right\}(\lambda),$$

where χ is given in Equation (2.1).

As another simple example of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = x/(1 - x)$, $x \in (0, 1)$, which is non-decreasing, and $g(t) = 1$, $t \in (0, 1)$, and consider $f : (0, 1) \mapsto (0, 1)$ convex, we get

$$2 \int_0^{1/2} \frac{f(t)}{1 - f(t)} dt \leq \int_0^1 \frac{F(t)}{t - F(t)} dt \leq \int_0^1 \frac{f(0) + f(t)}{2 - [f(0) + f(t)]} dt.$$

Examples of the item 2.: If we take $a = 0$, $b \to +\infty$, $\Phi(x) = x^{-p}$, $x \in (0, +\infty)$, with $p > 0$, which is non-increasing, and $g(t) = e^{-\lambda t}$, $t \in (0, +\infty)$, with $\lambda > 0$, and consider $f : (0, +\infty) \mapsto (0, +\infty)$ convex, the item 2 of Theorem 2.2 gives

$$2^p \int_0^{+\infty} [f(0) + f(t)]^{-p} e^{-\lambda t} dt \leq \int_0^{+\infty} \left[\frac{1}{t} F(t) \right]^{-p} e^{-\lambda t} dt$$
$$\leq 2e^{\lambda a} \int_0^{+\infty} [f(t)]^{-p} \, e^{-2\lambda t} dt.$$

Using the Laplace transform of the functions involved, this can be expressed as follows:

$$2^p \mathcal{L}\left\{[f(0) + f]^{-p}\right\}(\lambda) \leq \mathcal{L}\left(\chi^{-p}\right)(\lambda) \leq 2e^{\lambda a} \mathcal{L}\left(f^{-p}\right)(2\lambda),$$

where χ is given in Equation (2.1).

As another simple example of the item 2, if we take $a = 0$, $b = 1$, $\Phi(x) = 1/(1 + x)$, $x \in (0, +\infty)$, which is non-increasing, and $g(t) = 1$, $t \in (0, 1)$, and consider $f : (0, 1) \mapsto (0, +\infty)$ convex, we get

$$\int_0^1 \frac{2}{2 + f(0) + f(t)} dt \leq \int_0^1 \frac{t}{t + F(t)} dt \leq 2 \int_0^{1/2} \frac{1}{1 + f(t)} dt.$$

Many other examples can be given in the same way, including for the items 3 and 4. The configuration depends on the mathematical context being considered.

Other variants of the Levinson or Hardy integral inequality are presented in the theorem below. They are innovative in combining the monotonicity and convexity assumptions for some of the functions involved.

Theorem 2.3. *Let $(a, b) \in \mathbb{R}^2$ with $a < b$, including the case $b \to +\infty$, $f : (a, b) \mapsto \mathbb{R}$, $g : (a, b) \mapsto (0, +\infty)$ and $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be three functions. For any $t \in (a, b)$, we consider*

$$F(t) = \int_a^t f(x) dx$$

and

$$G(t)=\int_t^b \frac{1}{x-a}g(x)dx.$$

Then the four results below hold.

(1) *If we assume that* Φ *is convex and, for any* $t\in(a,b)$,

$$\int_a^b\int_a^t \frac{1}{t-a}\left|\Phi\left[f(x)\right]\right|g(t)dxdt \tag{2.2}$$

exists, then we have

$$\left[\int_a^b g(t)dt\right]\Phi\left\{\left[\int_a^b g(t)dt\right]^{-1}\int_a^b \frac{1}{t-a}F(t)g(t)dt\right\}\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\leq \int_a^b \Phi\left[f(t)\right]G(t)dt,$$

provided that the integrals in question exist.

(2) *If we assume that* Φ *is concave and that the double integral in Equation* (2.2) *exists, then we have*

$$\int_a^b \Phi\left[f(t)\right]G(t)dt\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\leq \left[\int_a^b g(t)dt\right]\Phi\left\{\left[\int_a^b g(t)dt\right]^{-1}\int_a^b \frac{1}{t-a}F(t)g(t)dt\right\},$$

provided that the integrals in question exist.

(3) *If we assume that* f *is convex and* Φ *is non-decreasing and convex, then we have*

$$\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt\leq \frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt+\int_a^b \Phi\left[f(t)\right]g(t)dt\right\},$$

provided that the integrals in question exist.

(4) *If we assume that* f *is convex and* Φ *is non-increasing and concave, then we have*

$$\frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt+\int_a^b \Phi\left[f(t)\right]g(t)dt\right\}\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

provided that the integrals in question exist.

This statement assumes that $\Phi\left[f(a)\right]$ is finite when used.

Some examples of applications of Theorem 2.3 are given below, assuming that all the mathematical quantities involved exist.

Examples of the item 1.: If we take $a=0$, $b\to+\infty$, $\Phi(x)=x^p$, $x\in(0,+\infty)$, with $p>1$ or $p<0$, which is convex, and $g(t)=e^{-\lambda t}$, $t\in(0,+\infty)$, with $\lambda>0$, and consider $f:(0,+\infty)\mapsto(0,+\infty)$, the item 1 of Theorem

2.3 gives

$$\lambda^{p-1}\left[\int_0^{+\infty}\frac{1}{t}F(t)e^{-\lambda t}dt\right]^p \le \int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^p e^{-\lambda t}dt$$
$$\le -\int_0^{+\infty}[f(t)]^p \operatorname{Ei}(-\lambda t)dt,$$

where $\operatorname{Ei}(x)$ denotes the (referenced) exponential integral defined by $\operatorname{Ei}(x) = -\int_{-x}^{+\infty}(e^{-t}/t)dt$, $x \in \mathbb{R}$, and we have used

$$G(t) = \int_t^{+\infty}\frac{1}{x}g(x)dx = \int_t^{+\infty}\frac{1}{x}e^{-\lambda x}dx = -\operatorname{Ei}(-\lambda t)$$

for the right-hand side term. We would like to point out the fact that p can be negative, which is not so common for this kind of integral inequality.

We can rewrite the first inequality in terms of the Laplace transform of the functions involved, as follows:

$$\lambda^{p-1}\left[\mathcal{L}(\chi)(\lambda)\right]^p \le \mathcal{L}(\chi^p)(\lambda),$$

where χ is given in Equation (2.1).

As another simple example of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = x/(1-x)$, $x \in (0,1)$, which is convex, and $g(t) = 1$, $t \in (0,1)$, and consider $f : (0,1) \mapsto (0,1)$, we get

$$\frac{\int_0^1[F(t)/t]\,dt}{1-\int_0^1[F(t)/t]\,dt} \le \int_0^1\frac{F(t)}{t-F(t)}dt \le -\int_0^1\frac{f(t)}{1-f(t)}\log(t)dt,$$

where we have used

$$G(t) = \int_t^1\frac{1}{x}g(x)dx = \int_t^1\frac{1}{x}dx = -\log(t).$$

Examples of the item 3.: If we take $a = 0$, $b = 1$, $\Phi(x) = x/(1-x)$, $x \in (0,1)$, which is non-decreasing and convex, and $g(t) = 1$, $t \in (0,1)$, and consider $f : (0,1) \mapsto (0,1)$ convex, the item 3 of Theorem 2.3 gives

$$\int_0^1\frac{F(t)}{t-F(t)}dt \le \frac{1}{2}\left\{\frac{f(0)}{1-f(0)} + \int_0^1\frac{f(t)}{1-f(t)}dt\right\}.$$

As another of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = e^x$, $x \in \mathbb{R}$, which is non-decreasing and convex, and $g(t) = t^\alpha$, $t \in (0,1)$, with $\alpha > 0$, and consider $f : (0,1) \mapsto \mathbb{R}$ convex (note that f can be possibly negative), the item 3 of Theorem 2.3 gives

$$\int_0^1 e^{F(t)/t}t^\alpha dt \le \frac{1}{2}\left[e^{f(0)}\frac{1}{\alpha+1} + \int_0^1 e^{f(t)}t^\alpha dt\right],$$

where we have used

$$\int_0^1 g(t)dt = \int_0^1 t^\alpha dt = \frac{1}{\alpha+1}.$$

These are just a few examples. The items 2 and 4 can be illustrated in the same way.

Complementary and original integral inequalities are given in the theorem below, under general assumptions on the functions involved.

Theorem 2.4. *Let $(a,b)\in\mathbb{R}^2$ with $a<b$, including the case $b\to+\infty$, $f:(a,b)\mapsto\mathbb{R}$, $g:(a,b)\mapsto(0,+\infty)$ and $\Phi:\mathbb{R}\mapsto(0,+\infty)$ be three functions. For any $t\in(a,b)$, we consider*

$$F(t)=\int_a^t f(x)dx.$$

Then the two results below hold.

(1) *We have*

$$\left[\int_a^b g(t)dt\right]^2\left[\int_a^b\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{-1}g(t)dt\right]^{-1}\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

provided that the integrals in question exist.

(2) *We have*

$$\left\{\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]dt\right\}^2\left\{\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]\frac{1}{g(t)}dt\right\}^{-1}$$
$$\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

provided that the integrals in question exist.

Some examples of applications of Theorem 2.4 are given below, assuming that all the mathematical quantities involved exist.

Examples of the item 1.: If we take $a=0$, $b\to+\infty$, $\Phi(x)=x^p$, $x\in(0,+\infty)$, with $p\in\mathbb{R}$ and $g(t)=e^{-\lambda t}$, $t\in(0,+\infty)$, with $\lambda>0$, and consider $f:(0,+\infty)\mapsto(0,+\infty)$, the item 1 of Theorem 2.4 gives

$$\frac{1}{\lambda^2}\left\{\int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^{-p}e^{-\lambda t}dt\right\}^{-1}\le\int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^p e^{-\lambda t}dt.$$

We can rewrite this inequality in terms of the Laplace transform of the functions involved, as follows:

$$\frac{1}{\lambda^2}\left[\mathcal{L}\left(\chi^{-p}\right)(\lambda)\right]^{-1}\le\mathcal{L}\left(\chi^p\right)(\lambda),$$

where χ is given in Equation (2.1).

As another simple example of this item, if we take $a=0$, $b=1$, $\Phi(x)=x/(1-x)$, $x\in(0,1)$, and $g(t)=t^\alpha$, $t\in(0,1)$, with $\alpha\in\mathbb{R}\backslash\{-1\}$, and consider $f:(0,1)\mapsto(0,1)$, we get

$$\frac{1}{(\alpha+1)^2}\left[\int_0^1\frac{t-F(t)}{F(t)}t^\alpha dt\right]^{-1}\le\int_0^1\frac{F(t)}{t-F(t)}t^\alpha dt.$$

Examples of the item 2.: If we take $a=0$, $b\to+\infty$, $\Phi(x)=x^p$, $x\in(0,+\infty)$, with $p\in\mathbb{R}$ and $g(t)=e^{-\lambda t}$, $t\in(0,+\infty)$, with $\lambda>0$, and consider $f:(0,+\infty)\mapsto(0,+\infty)$, the item 2 of Theorem 2.4 gives

$$\left\{\int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^p dt\right\}^2\left\{\int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^p e^{\lambda t}dt\right\}^{-1}\le\int_0^{+\infty}\left[\frac{1}{t}F(t)\right]^p e^{-\lambda t}dt.$$

As another simple example of this item, if we take $a = 0$, $b = 1$, $\Phi(x) = x/(1-x)$, $x \in (0,1)$, and $g(t) = t^{\alpha}$, $t \in (0,1)$, with $\alpha \in \mathbb{R}$, and consider $f : (0,1) \mapsto (0,1)$, we get

$$\left[\int_0^1 \frac{F(t)}{t-F(t)}dt\right]^2 \left[\int_0^1 \frac{F(t)}{t-F(t)}t^{-\alpha}dt\right]^{-1} \leq \int_0^1 \frac{F(t)}{t-F(t)}t^{\alpha}dt.$$

The proofs of the four theorems, with all the details, are given in the next section.

3. Proofs of the theorems

Theorems 2.1, 2.2, 2.3 and 2.4 are demonstrated in turn.

Proof of Theorem 2.1. Let us look at the items 1, 2, 3 and 4 in turn.

(1) Let us assume that f and Φ are non-decreasing. Since f is non-decreasing, for any $x \in [a,t]$, we have $f(a) \leq f(x) \leq f(t)$, which implies that

$$f(a) = f(a)\frac{t-a}{t-a} \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq f(t)\frac{t-a}{t-a} = f(t).$$

Since Φ is also non-decreasing, for any $t \in (a,b)$, we have

$$\Phi[f(a)] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi[f(t)].$$

Now, using the assumption that g is positive, we get

$$\Phi[f(a)]g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right]g(t) \leq \Phi[f(t)]g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\Phi[f(a)]\int_a^b g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt \leq \int_a^b \Phi[f(t)]g(t)dt,$$

which is the desired result.

(2) Let us assume that f is non-decreasing and Φ is non-increasing. Since f is non-decreasing, for any $x \in [a,t]$, we have $f(a) \leq f(x) \leq f(t)$, which implies that

$$f(a) = f(a)\frac{t-a}{t-a} \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq f(t)\frac{t-a}{t-a} = f(t).$$

Since Φ is non-increasing, for any $t \in (a,b)$, we have

$$\Phi[f(t)] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi[f(a)].$$

Now, using the positivity of g, we get

$$\Phi[f(t)]g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right]g(t) \leq \Phi[f(a)]g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we establish that

$$\int_a^b \Phi[f(t)]g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt \leq \Phi[f(a)]\int_a^b g(t)dt,$$

which is the desired result.

(3) Let us assume that f and Φ are non-increasing. Since f is non-increasing, for any $x \in [a,t]$, we have $f(t) \leq f(x) \leq f(a)$, which implies that

$$f(t) = f(t)\frac{t-a}{t-a} \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq f(a)\frac{t-a}{t-a} = f(a).$$

Since Φ is also non-increasing, for any $t \in (a,b)$, we have

$$\Phi[f(a)] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi[f(t)].$$

Now, using the assumption that g is positive, we find that

$$\Phi[f(a)]g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right]g(t) \leq \Phi[f(t)]g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\Phi[f(a)]\int_a^b g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt \leq \int_a^b \Phi[f(t)]g(t)dt,$$

which is the desired result.

(4) Let us assume that f is non-increasing and Φ is non-decreasing. Since f is non-increasing, for any $x \in [a,t]$, we have $f(t) \leq f(x) \leq f(a)$, which implies that

$$f(t) = f(t)\frac{t-a}{t-a} \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq f(a)\frac{t-a}{t-a} = f(a).$$

Since Φ is also non-decreasing, for any $t \in (a,b)$, we have

$$\Phi[f(t)] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi[f(a)].$$

Now, using the fact that g is positive, we get

$$\Phi[f(t)]g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right]g(t) \leq \Phi[f(a)]g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\int_a^b \Phi[f(t)]g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt \leq \Phi[f(a)]\int_a^b g(t)dt,$$

which is the desired result.

The items 1, 2, 3 and 4 are demonstrated. □

Proof of Theorem 2.2. Let us look at the items 1, 2, 3 and 4 in turn.

(1) Let us assume that f is convex and Φ is non-decreasing. We think of using the Hermite-Hadamard integral inequality (see [10]). Since f is convex, the Hermite-Hadamard integral inequality applied to f and the interval $[a,t]$ gives

$$f\left(\frac{a+t}{2}\right) \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq \frac{1}{2}\left[f(a)+f(t)\right].$$

Since Φ is non-decreasing, we get

$$\Phi\left[f\left(\frac{a+t}{2}\right)\right] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\}.$$

Now, by the positivity of g, we have

$$\Phi\left[f\left(\frac{a+t}{2}\right)\right]g(t)\le\Phi\left[\frac{1}{t-a}F(t)\right]g(t)\le\Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\}g(t).$$

Integrating both sides with respect to $t\in(a,b)$, we obtain

$$\int_a^b\Phi\left[f\left(\frac{a+t}{2}\right)\right]g(t)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt.$$

Applying the change of variables $y=(a+t)/2$ in the first term, we find that

$$\int_a^b\Phi\left[f\left(\frac{a+t}{2}\right)\right]g(t)dt=2\int_a^{(a+b)/2}\Phi\left[f(y)\right]g(2y-a)dy. \tag{3.1}$$

After some standardization of the notation, we get

$$2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt,$$

which is the desired result.

(2) Let us assume that f is convex and Φ is non-increasing. Since f is convex, the Hermite-Hadamard integral inequality applied to f and the interval $[a,t]$ gives

$$f\left(\frac{a+t}{2}\right)\le\frac{1}{t-a}\int_a^t f(x)dx=\frac{1}{t-a}F(t)\le\frac{1}{2}\left[f(a)+f(t)\right].$$

Since Φ is non-increasing, we get

$$\Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\}\le\Phi\left[\frac{1}{t-a}F(t)\right]\le\Phi\left[f\left(\frac{a+t}{2}\right)\right].$$

Now, using the fact that g is positive, we establish that

$$\Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\}g(t)\le\Phi\left[\frac{1}{t-a}F(t)\right]g(t)\le\Phi\left[f\left(\frac{a+t}{2}\right)\right]g(t).$$

Integrating both sides with respect to $t\in(a,b)$, with an appropriate change of variables (see Equation (3.1)), we obtain

$$\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt\le\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt$$
$$\le\int_a^b\Phi\left[f\left(\frac{a+t}{2}\right)\right]g(t)dt=2\int_a^{(a+b)/2}\Phi\left[f(t)\right]g(2t-a)dt,$$

which is the desired result.

(3) Let us assume that f is concave and Φ is non-decreasing. Since f is concave, the Hermite-Hadamard integral inequality applied to f and the interval $[a,t]$ gives

$$\frac{1}{2}\left[f(a)+f(t)\right]\le\frac{1}{t-a}\int_a^t f(x)dx=\frac{1}{t-a}F(t)\le f\left(\frac{a+t}{2}\right).$$

Since Φ is non-decreasing, we get

$$\Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\} \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi\left[f\left(\frac{a+t}{2}\right)\right].$$

Now, using the assumption that g is positive, we find that

$$\Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\} g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right] g(t) \leq \Phi\left[f\left(\frac{a+t}{2}\right)\right] g(t).$$

Integrating both sides with respect to $t \in (a, b)$, with an appropriate change of variables (see Equation (3.1)), we obtain

$$\int_a^b \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t)dt \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt$$
$$\leq \int_a^b \Phi\left[f\left(\frac{a+t}{2}\right)\right] g(t)dt = 2\int_a^{(a+b)/2} \Phi\left[f(t)\right] g(2t-a)dt,$$

which is the desired result.

(4) Let us assume that f is concave and Φ is non-increasing. Since f is concave, the Hermite-Hadamard integral inequality applied to f and the interval $[a, t]$ gives

$$\frac{1}{2}\left[f(a)+f(t)\right] \leq \frac{1}{t-a}\int_a^t f(x)dx = \frac{1}{t-a}F(t) \leq f\left(\frac{a+t}{2}\right).$$

Since Φ is non-increasing, we get

$$\Phi\left[f\left(\frac{a+t}{2}\right)\right] \leq \Phi\left[\frac{1}{t-a}F(t)\right] \leq \Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\}.$$

Now, by the positivity of g, we have

$$\Phi\left[f\left(\frac{a+t}{2}\right)\right] g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right] g(t) \leq \Phi\left\{\frac{1}{2}\left[f(a)+f(t)\right]\right\} g(t).$$

Integrating both sides with respect to $t \in (a, b)$, with an appropriate change of variables (see Equation (3.1)), we obtain

$$2\int_a^{(a+b)/2} \Phi\left[f(t)\right] g(2t-a)dt = \int_a^b \Phi\left[f\left(\frac{a+t}{2}\right)\right] g(t)dt$$
$$\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \int_a^b \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t)dt,$$

which is the desired result.

The items 1, 2, 3 and 4 are demonstrated. □

Proof of Theorem 2.3. Let us look at the items 1, 2, 3 and 4 in turn.

(1) Let us assume that Φ is convex. We think of using the Jensen integral inequality (see [11]). For any $t \in (a, b)$, we set

$$j(t) = \left[\int_a^b g(t)dt\right]^{-1} g(t).$$

Then, since g is positive, j is also positive and we have $\int_a^b j(t)dt = 1$. It follows from the Jensen integral inequality applied with the convex function Φ and the probability measure $\mu(I) = \int_I j(t)dt$, with $I \subseteq (a,b)$, that

$$\Phi\left[\int_a^b \frac{1}{t-a}F(t)j(t)dt\right] \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] j(t)dt.$$

Since g is positive (so that its integral is also positive), this inequality can also be expressed as

$$\left[\int_a^b g(t)dt\right] \Phi\left\{\left[\int_a^b g(t)dt\right]^{-1} \int_a^b \frac{1}{t-a}F(t)g(t)dt\right\} \leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt,$$

which is the desired first inequality.

Let us now focus on the second inequality. For any $x \in (a,t)$, we set

$$k(x) = \frac{1}{t-a}.$$

Then k is positive and $\int_a^t k(x)dx = 1$. It follows from the Jensen integral inequality applied with the convex function Φ and the probability measure $\nu(I) = \int_I k(x)dx$, with $I \subseteq (a,t)$, that

$$\Phi\left[\frac{1}{t-a}F(t)\right] = \Phi\left[\frac{1}{t-a}\int_a^t f(x)dx\right] = \Phi\left[\int_a^t f(x)k(x)dx\right]$$
$$\leq \int_a^t \Phi\left[f(x)\right] k(x)dx = \frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx.$$

Now, by the positivity of g, we have

$$\Phi\left[\frac{1}{t-a}F(t)\right] g(t) \leq \left\{\frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx\right\} g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \int_a^b \left\{\frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx\right\} g(t)dt$$
$$= \int_a^b \int_a^t \frac{1}{t-a}\Phi\left[f(x)\right] g(t)dxdt.$$

Since the double integral in Equation (2.2) exists, the change of the order of integration can be done thanks to the Fubini double integral theorem. We find that

$$\int_a^b \int_a^t \frac{1}{t-a}\Phi\left[f(x)\right] g(t)dxdt = \int_a^b \int_x^b \frac{1}{t-a}\Phi\left[f(x)\right] g(t)dtdx$$
$$= \int_a^b \Phi\left[f(x)\right]\left\{\int_x^b \frac{1}{t-a}g(t)dt\right\} dx = \int_a^b \Phi\left[f(x)\right] G(x)dx.$$

After some standardization of the notation, we get

$$\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \int_a^b \Phi\left[f(t)\right] G(t)dt,$$

which is the desired second inequality.

Combining the obtained inequalities, we establish that

$$\left[\int_a^b g(t)dt\right] \Phi \left\{ \left[\int_a^b g(t)dt\right]^{-1} \int_a^b \frac{1}{t-a} F(t)g(t)dt \right\} \leq \int_a^b \Phi \left[\frac{1}{t-a}F(t)\right] g(t)dt$$

$$\leq \int_a^b \Phi\left[f(t)\right] G(t)dt,$$

which is the desired result.

(2) Let us assume that Φ is concave, and adopt the framework of the previous item. It follows from the Jensen integral inequality applied with the concave function Φ and the probability measure $\mu(I) = \int_I j(t)dt$, with $I \subseteq (a,b)$, that

$$\int_a^b \Phi \left[\frac{1}{t-a}F(t)\right] j(t)dt \leq \Phi \left[\int_a^b \frac{1}{t-a} F(t)j(t)dt\right].$$

Since g is positive, this inequality can also be expressed as

$$\int_a^b \Phi \left[\frac{1}{t-a}F(t)\right] g(t)dt \leq \left[\int_a^b g(t)dt\right] \Phi \left\{ \left[\int_a^b g(t)dt\right]^{-1} \int_a^b \frac{1}{t-a} F(t)g(t)dt \right\},$$

which is the desired first inequality.

Let us now focus on the second inequality. The Jensen integral inequality applied with the concave function Φ and the probability measure $\nu(I) = \int_I k(x)dx$, with $I \subseteq (a,t)$, gives

$$\frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx = \int_a^t \Phi\left[f(x)\right] k(x)dx$$

$$\leq \Phi\left[\int_a^t f(x)k(x)dx\right] = \Phi\left[\frac{1}{t-a}\int_a^t f(x)dx\right] = \Phi\left[\frac{1}{t-a}F(t)\right].$$

Since g is positive, we get

$$\left\{\frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx\right\} g(t) \leq \Phi\left[\frac{1}{t-a}F(t)\right] g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\int_a^b \int_a^t \frac{1}{t-a}\Phi\left[f(x)\right] g(t)dxdt = \int_a^b \left\{\frac{1}{t-a}\int_a^t \Phi\left[f(x)\right] dx\right\} g(t)dt$$

$$\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt.$$

Since the double integral in Equation (2.2) exists, the change of the order of integration can be done thanks to the Fubini double integral theorem. We find that

$$\int_a^b \int_a^t \frac{1}{t-a}\Phi\left[f(x)\right] g(t)dxdt = \int_a^b \Phi\left[f(x)\right] G(x)dx,$$

which is the desired second inequality.

After some standardization of the notation, we get

$$\int_a^b \Phi\left[f(t)\right] G(t)dt \le \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt$$

Combining the obtained inequalities, we establish that

$$\int_a^b \Phi\left[f(t)\right] G(t)dt \le \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt$$
$$\le \left[\int_a^b g(t)dt\right] \Phi\left\{\left[\int_a^b g(t)dt\right]^{-1} \int_a^b \frac{1}{t-a}F(t)g(t)dt\right\},$$

which is the desired result.

(3) Let us assume that f is convex and Φ is non-decreasing and convex. Using only the fact that f is convex and Φ is non-decreasing, it follows from the second inequality in the item 1 of Theorem 2.2 that

$$\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \le \int_a^b \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t)dt.$$

Since Φ is convex, by the basic definition of convexity with the weight coefficient $1/2$, we get

$$\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} \le \frac{1}{2}\Phi\left[f(a)\right] + \frac{1}{2}\Phi\left[f(t)\right] = \frac{1}{2}\left\{\Phi\left[f(a)\right] + \Phi\left[f(t)\right]\right\}.$$

Now, using the positivity of g, we have

$$\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t) \le \frac{1}{2}\left\{\Phi\left[f(a)\right] + \Phi\left[f(t)\right]\right\} g(t).$$

Integrating both sides with respect to $t \in (a,b)$, we obtain

$$\int_a^b \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t)dt \le \int_a^b \frac{1}{2}\left\{\Phi\left[f(a)\right] + \Phi\left[f(t)\right]\right\} g(t)dt$$
$$= \frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt + \int_a^b \Phi\left[f(t)\right] g(t)dt\right\}.$$

We thus establish that

$$\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt \le \frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt + \int_a^b \Phi\left[f(t)\right] g(t)dt\right\},$$

which is the desired result.

(4) Let us assume that f is convex and Φ is non-increasing and concave. Using only the fact that f is convex and Φ is non-increasing, it follows from the first inequality in the item 2 of Theorem 2.2 that

$$\int_a^b \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\} g(t)dt \le \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right] g(t)dt.$$

Since Φ is concave, by the basic definition of concavity, we get

$$\frac{1}{2}\left\{\Phi\left[f(a)\right] + \Phi\left[f(t)\right]\right\} = \frac{1}{2}\Phi\left[f(a)\right] + \frac{1}{2}\Phi\left[f(t)\right] \le \Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}.$$

Now, using the positivity of g, we find that

$$\frac{1}{2}\left\{\Phi\left[f(a)\right]+\Phi\left[f(t)\right]\right\}g(t)\leq\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t).$$

Integrating both sides with respect to $t\in(a,b)$, we obtain

$$\begin{aligned}&\frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt+\int_a^b\Phi\left[f(t)\right]g(t)dt\right\}\\&=\int_a^b\frac{1}{2}\left\{\Phi\left[f(a)\right]+\Phi\left[f(t)\right]\right\}g(t)dt\leq\int_a^b\Phi\left\{\frac{1}{2}[f(a)+f(t)]\right\}g(t)dt.\end{aligned}$$

We thus establish that

$$\frac{1}{2}\left\{\Phi\left[f(a)\right]\int_a^b g(t)dt+\int_a^b\Phi\left[f(t)\right]g(t)dt\right\}\leq\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

which is the desired result.

The items 1, 2, 3 and 4 are demonstrated. □

Proof of Theorem 2.4. Let us look at the items 1 and 2 in turn.

(1) Since g and Φ are positive, using a suitable decomposition of the integral of g and the Cauchy-Schwarz integral inequality (or the Hölder integral inequality with parameter 2), we obtain

$$\begin{aligned}&\int_a^b g(t)dt=\int_a^b\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{-1/2}[g(t)]^{1/2}\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{1/2}[g(t)]^{1/2}dt\\&\leq\left[\int_a^b\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{-1}g(t)dt\right]^{1/2}\left\{\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt\right\}^{1/2}.\end{aligned}$$

This inequality can also be expressed as

$$\left[\int_a^b g(t)dt\right]^2\left[\int_a^b\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{-1}g(t)dt\right]^{-1}\leq\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

which is the desired inequality.

(2) Since g and Φ are positive, using a suitable integral decomposition and the Cauchy-Schwarz integral inequality, we obtain

$$\begin{aligned}&\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]dt\\&=\int_a^b\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{1/2}[g(t)]^{-1/2}\left\{\Phi\left[\frac{1}{t-a}F(t)\right]\right\}^{1/2}[g(t)]^{1/2}dt\\&\leq\left[\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]\frac{1}{g(t)}dt\right]^{1/2}\left\{\int_a^b\Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt\right\}^{1/2}.\end{aligned}$$

This inequality can also be expressed as

$$\left\{\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]dt\right\}^2\left\{\int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]\frac{1}{g(t)}dt\right\}^{-1}$$
$$\leq \int_a^b \Phi\left[\frac{1}{t-a}F(t)\right]g(t)dt,$$

which is the desired inequality.

The items 1 and 2 are demonstrated. □

The next section ends the article.

4. Conclusion

In this article, we have proved new theorems on integral inequalities. Each of these results extends, generalizes, improves or modifies the Levinson or Hardy integral inequalities. These contributions are of interest for several reasons. First, they are very general, involving several intermediate functions. Second, they rely on tractable assumptions, mainly monotonicity and convexity. Third, they extend the current understanding of integral inequalities, which remain essential tools in mathematics and many applied fields. This was demonstrated in the article through applications to the Laplace transform. In particular, several inequalities were established for the Laplace transforms of the functions involved.

Future directions include extending the results to the multivariate case. We also aim to explore assumptions beyond monotonicity and convexity. Another is to apply these results to real problems in applied contexts.

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