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AN EXAMINATION ON THE *MANNHEIM FRENET RULED SURFACE* BASED ON NORMAL VECTOR FIELDS IN E^3

ŞEYDA KILIÇOĞLU

Abstract. In this paper we consider six special Frenet ruled surfaces along to the Mannheim pairs $\{\alpha^*, \alpha\}$. First we define and find the parametric equations of Frenet ruled surfaces which are called *Mannheim Frenet ruled surface*, along Mannheim curve α, in terms of the Frenet apparatus of Mannheim curve α. Later, we find only one matrix gives us all nine positions of normal vector fields of these six Frenet ruled surfaces and *Mannheim Frenet ruled surface* in terms of Frenet apparatus of Mannheim curve α too. Further using that matrix we have some results such as; normal ruled surface and *Mannheim normal ruled surface* of Mannheim curve α have perpendicular normal vector fields along the curve $\varphi_2(s) = \alpha + \frac{\tan\theta}{k_1 \tan\theta - k_2} V_2$, under the condition $\tan\theta \neq \frac{k_2}{k_1}$.

1. Introduction and Preliminaries

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\frac{k_1}{\left(k_1^2+k_2^2\right)}$ is a non-zero constant, k_1 is the curvature and k_2 is the torsion, respectively. Recently, a new definition of the associated curves was given by Liu and Wang in [7]. According to this new definition, if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve. As a result they called these new curves as Mannheim partner curves.

The quantities $\{V_1, V_2, V_3, k_1, k_2\}$ are collectively Frenet-Serret apparatus of the curve $\alpha : I \to E^3$. The Frenet formulae are also well known as

$$\begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

Let $\alpha : I \to E^3$ be the C^2 differentiable unit speed and $\alpha^* : I \to E^3$ be second curve and let $V_1(s), V_2(s), V_3(s)$ and $V_1^*(s^*), V_2^*(s^*), V_3^*(s^*)$ be the Frenet frames of

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the curves α and α^*, respectively. If the principal normal vector V_2 of the curve α is linearly dependent on the binormal vector V_3^* of the curve α^*, then the pair $\{\alpha, \alpha^*\}$ is said to be Mannheim pair, then α is called a Mannheim curve and α^* is called Mannheim partner curve of α where $\langle V_1, V_1^* \rangle = \cos\theta$ and besides the equality $\frac{k_1}{k_1^2+k_2^2} =$ constant; is known the offset property, for some non-zero constant. Mannheim partner curve of α can be represented $\alpha(s^*) = \alpha^*(s^*) + \lambda(s^*)V_3^*(s^*)$ for some function λ, since V_2 and V_3 are linearly dependent, Equation can be rewritten as [8]

$$\alpha^*(s) = \alpha(s) - \lambda V_2(s)$$

where

$$\lambda = \frac{-k_1}{k_1^2 + k_2^2}.$$

Frenet-Serret apparatus of Mannheim partner curve α^*, based in Frenet-Serret vectors of Mannheim curve α are

$$\begin{aligned} V_1^* &= cos\theta\ V_1 - sin\theta\ V_3 \\ V_2^* &= sin\theta\ V_1 + cos\theta\ V_3 \\ V_3^* &= V_2. \end{aligned}$$

The curvature and the torsion have the following equalities,

$$\begin{aligned} k_1^* &= -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta} \\ & \quad and \\ k_2^* &= \frac{k_1}{\lambda k_2}. \end{aligned}$$

we use dot to denote the derivative with respect to the arc-length parameter of the curve α. For more detail see in [8]

Also we can write

$$\frac{ds}{ds^*} = \frac{1}{\sqrt{1 + \lambda k_2}}$$

or

$$\frac{ds}{ds^*} = \frac{1}{\cos\theta}$$

and since $d(\alpha(s), \alpha^*(s)) = \|\alpha(s) - \alpha^*(s)\| = \|\lambda V_2(s)\| = |\lambda|$ we have $|\lambda|$ is the distance between the curves α and α^*.

By using the similiar method we produce a new ruled surface based on the other ruled surface. A ruled surface is one which can be generated by the motion of a straight line in Euclidean $3-space$, ([1]). To illustrate the current situation, we bring here the famous example of L. K. Graves, so called the $B-scroll$, in [3]. A Frenet ruled surface is a ruled surfaces generated by Frenet vectors of the base curve. $Involute\ B-scroll$ is defined in [5]˙The differential geometric elements of the $involute\ \tilde{D}\ scroll$ are examined in [10]. The positions of Frenet ruled surfaces along Bertrand pairs are examined based on their normal vector fields in [6]. Also in [9] Mannheim offsets of ruled surfaces are defined and characterized

Definition 1.1. In the Euclidean $3-space$, let $\alpha(s)$ be the arc-length of a parametrized curve. The equations

$$\begin{cases} \varphi_1(s,u_1) = \alpha(s) + u_1 V_1(s) \\ \varphi_2(s,u_2) = \alpha(s) + u_2 V_2(s) \\ \varphi_3(s,u_3) = \alpha(s) + u_3 V_3(s) \end{cases}$$

are the parametrization of Frenet ruled surfaces which are called $V_1 - scroll$ (tangent ruled surface), $V_2 - scroll$ (normal ruled surface), $V_3 - scroll$ (binormal ruled surface), respectively in [2].

Theorem 1.1. *In the Euclidean* $3 - space$*, let* η_1, η_2, η_3 *be the normal vector fields of ruled surfaces* $\varphi_1, \varphi_2, \varphi_3$ *recpectively, along the curve* α*. They can be expressed by the following matrix;*

$$\begin{aligned} [\eta] &= [A]\,[V] \\ [\eta] &= \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a & 0 & b \\ c & d & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} a &= \frac{-u_2k_2}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}}, \quad c = \frac{-u_3k_2}{\sqrt{(u_3k_2)^2+1}} \\ b &= \frac{(1-u_2k_1)}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}}, \quad d = \frac{-1}{\sqrt{(u_3k_2)^2+1}}. \end{aligned}$$

Proof. The normal vector fields η_1, η_2, η_3 of ruled surfaces $\varphi_1, \varphi_2, \varphi_3$ can be expressed as in the following four equalities

$$\begin{aligned} \eta_1 &= -V_3 \\ \eta_2 &= \frac{-u_2k_2V_1 + (1-u_2k_1)V_3}{\sqrt{(u_2k_2)^2 + (1-u_2k_1)^2}} \\ \eta_3 &= \frac{-u_3k_2V_1 - V_2}{\sqrt{(u_3k_2)^2+1}} \end{aligned}$$

for more detail see in [4]. Same way some results on Frenet Ruled Surfaces along the *evolute-involute* curves, based on normal vector fields are given in [4]. □

2. Mannheim Frenet ruled surfaces

In this section, we found eight special Frenet ruled surfaces along to the Bertrand pairs $\{\alpha^*, \alpha\}$. First we define and find the parametric equations of Frenet ruled surfaces which are called $Bertrandian\ Frenet\ ruled\ surface$, along Bertrand curve α, in terms of the Frenet apparatus of of Bertrand curve α. Later we found only one matrix gives us all sixteen positions of normal vector fields of eight Frenet ruled surfaces and $Bertrandian\ Frenet\ ruled\ surface$ in terms of Frenet apparatus of Bertrand curve α too. Further using that matrix we have some results such as; normal ruled surface and $Bertrandian\ tangent\ ruled\ surface$ have perpendicular normal vector fields along the curve.

Definition 2.1. Let $\{\alpha^*, \alpha\}$ be Mannheim curve pair with $k_1 \neq 0$ and $k_2 \neq 0$. The equations of the ruled surfaces

$$\begin{cases} \varphi_1^* (s, v_1) = \alpha^* (s) + v_1 V_1^* (s), \\ \varphi_2^* (s, v_2) = \alpha^* (s) + v_2 V_2^* (s), \\ \varphi_3^* (s, v_3) = \alpha^* (s) + v_3 V_3^* (s), \end{cases}$$

are the parametrization of Frenet ruled surface of Mannheim pairs $\alpha^* (s)$.

Further we can give these surface equations as in the following way;

$$\begin{cases} \varphi_1^* (s, v_1) = \alpha^* (s) + v_1 V_1^* (s) & = \alpha (s) - \lambda V_2 (s) + v_1 (cos\theta \ V_1 - sin\theta \ V_3) \\ \varphi_2^* (s, v_2) = \alpha^* (s) + v_2 V_2^* (s) & = \alpha (s) - \lambda V_2 (s) + v_2 (sin\theta \ V_1 + cos\theta \ V_3), \\ \varphi_3^* (s, v_3) = \alpha^* (s) + v_3 V_3^* (s) & = \alpha (s) - \lambda V_2 (s) + v_3 V_2 = \alpha (s) + (v_3 - \lambda) V_2, \end{cases}$$

are the parametrization of Frenet ruled surface which are called Mannheim Tangent ruled surface, Mannheim Normal ruled surface, and Mannheim Binormal ruled surface respectively. They are called collectively Mannheim Frenet ruled surface in this study.

Theorem 2.1. *The normal vector fields $\eta_1^*, \eta_2^*, \eta_3^*$, of ruled surfaces $\varphi_1^*, \varphi_2^*, \varphi_3^*$, recpectively, along the curve Mannheim partner α^*, can be expressed by the following matrix;*

$$[\eta^*] = \begin{bmatrix} \eta_1^* \\ \eta_2^* \\ \eta_3^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ a^* & 0 & b^* \\ c^* & d^* & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \end{bmatrix}.$$

where

$$a^* = \frac{-v_2 k_2^*}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad c^* = \frac{-v_3 k_2^*}{\sqrt{(v_3 k_2^*)^2 + 1}}$$
$$b^* = \frac{(1 - v_2 k_1^*)}{\sqrt{(v_2 k_2^*)^2 + (1 - v_2 k_1^*)^2}} \quad d^* = \frac{-1}{\sqrt{(v_3 k_2^*)^2 + 1}}$$

Proof. It is trivial □

Theorem 2.2. *In the Euclidean $3-$ space, the product matrix of the position of the unit normal vector fields η_1, η_2, η_3, and $\eta_1^*, \eta_2^*, \eta_3^*$ of Frenet ruled surfaces, along the Mannheim pairs α and α^* is*

$$[\eta] [\eta^*]^{\mathbf{T}} = \begin{matrix} \langle \eta_1, \eta_1^* \rangle & \langle \eta_1, \eta_2^* \rangle & \langle \eta_1, \eta_3^* \rangle \\ \langle \eta_2, \eta_1^* \rangle & \langle \eta_2, \eta_2^* \rangle & \langle \eta_2, \eta_3^* \rangle \\ \langle \eta_3, \eta_1^* \rangle & \langle \eta_3, \eta_2^* \rangle & \langle \eta_3, \eta_3^* \rangle \end{matrix} \; ... (I)$$

Proof. It is easy from the matrix product;

$$[\eta] [\eta^*]^{\mathbf{T}} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix} \begin{bmatrix} \eta_1^* & \eta_2^* & \eta_3^* \end{bmatrix}.$$

□

Theorem 2.3. *In the Euclidean $3-$ space, the product matrix of the unit normal vector fields η_1, η_2, η_3 and $\eta_1^*, \eta_2^*, \eta_3^*$ of Frenet ruled surfaces, along the Mannheim pairs α and α^*, can be given by the following matrix*

$$[\eta] [\eta^*]^{\mathbf{T}} = \begin{bmatrix} 0 & a^* \sin\theta & c^* \sin\theta - d^* \cos\theta \\ 0 & a^* (a \cos\theta - b \sin\theta) & c^* (a \cos\theta - b \sin\theta) + d^* (a \sin\theta + b \cos\theta) \\ -d & a^* c \cos\theta + d b^* & c^* c \cos\theta + d^* c \sin\theta \end{bmatrix} ... (II)$$

Proof. Let $[\eta] = [A][V]$ and $[\eta^*] = [A^*][V^*]$ hence

$$\begin{aligned} [\eta][\eta^*]^{\mathbf{T}} &= [A][V]([A^*][V^*])^{\mathbf{T}} \\ &= [A]\left([V][V^*]^{\mathbf{T}}\right)[A^*]^{\mathbf{T}}. \end{aligned}$$

Where the matrix product of Frenet vector fields of the Mannheim partner α^*, *and* Mannheim curve α has the following matrix form;

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \begin{bmatrix} V_1^* & V_2^* & V_3^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}$$

Hence

$$\begin{aligned} [\eta][\eta^*]^T &= [A]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{bmatrix}[A^*]^T \\ &= \begin{bmatrix} 0 & a^*\sin\theta & c^*\sin\theta - d^*\cos\theta \\ 0 & a^*(a\cos\theta - b\sin\theta) & c^*(a\cos\theta - b\sin\theta) + d^*(a\sin\theta + b\cos\theta) \\ -d & a^*c\cos\theta + db^* & c^*c\cos\theta + d^*c\sin\theta \end{bmatrix} \end{aligned}$$

this product give us the result. □

In the Euclidean $3-space$, the position of six surface, basicly, can be examined by the position of their unit normal vector fields. We can examine the nine positions of six surfaces, basicly, according to the position of their unit normal vector fields in a matrix. Since the equality of the last two matrice (I) and (II), we have nine interesting results according to the normal vector fields with the following results.

There are two pairs of normal vector fields perpendicular to each other of Frenet ruled surface along the Mannheim pairs $\{\alpha^*, \alpha\}$ as in the following corollary;

Corollary 2.1. *Tangent ruled surface and $Mannheim$ $Tangent$ ruled $surface$ curve α have perpendicular normal vector fields. Normal ruled surface and $Mannheim$ $Tangent$ ruled $surface$ of Mannheim curve α have perpendicular normal vector fields.*

Proof. It is trivial since $\langle \eta_1, \eta_1^* \rangle = 0$ and since $\langle \eta_2, \eta_1^* \rangle = 0$. □

Corollary 2.2. *Tangent ruled surface and $Mannheim$ normal ruled $surface$ of Mannheim curve α have not perpendicular normal vector fields.*

Proof. Since $\langle \eta_1, \eta_2^* \rangle = a^*\sin\theta$ and $v_2 k_2^* \sin\theta \neq 0$ it is trivial. □

Corollary 2.3. *Tangent ruled surface and $Mannheim$ binormal ruled $surface$ of Mannheim curve α have not perpendicular normal vector fields, along the curve $\varphi_3^*(s) = \alpha(s) + \lambda\left(\frac{k_2}{k_1\tan\theta} - 1\right)V_2$.*

Proof. Since $\langle \eta_1, \eta_3^* \rangle = c^*\sin\theta - d^*\cos\theta$ and under the condition $c^*\sin\theta - d^*\cos\theta = 0$

$$\begin{aligned} \frac{-v_3 k_2^* \sin\theta}{\sqrt{(v_3 k_2^*)^2 + 1}} + \frac{\cos\theta}{\sqrt{(v_3 k_2^*)^2 + 1}} &= 0 \\ -v_3 k_2^* \sin\theta + \cos\theta &= 0 \end{aligned}$$

and

$$v_3 = \frac{\lambda k_2}{k_1 \tan\theta}$$

it is trivial. □

Corollary 2.4. *Normal ruled surface and Mannheim normal ruled surface of Mannheim curve α α have perpendicular normal vector fields along the curve* $\varphi_2(s) = \alpha(s) + \frac{\tan\theta}{k_1\tan\theta - k_2}V_2(s)$, $\tan\theta \neq \frac{k_2}{k_1}$.

Proof. Since $\langle \eta_2, \eta_2^* \rangle = a^*(a\cos\theta - b\sin\theta)$ and under the orthogonality condition $-v_2 k_2^*(a\cos\theta - b\sin\theta) = 0$, and $v_2 k_2^* \neq 0$. Hence

$$\begin{aligned} a\cos\theta &= b\sin\theta \\ \tan\theta &= \frac{-u_2 k_2}{(1 - u_2 k_1)} \end{aligned}$$

or

$$u_2 = \frac{\tan\theta}{k_1\tan\theta - k_2},$$

this completes the proof. □

Corollary 2.5. *Normal ruled surface and Mannheim binormal ruled surface of Mannheim curve α α have perpendicular normal vector fields along the curve* $\varphi_3^*(s) = \alpha(s) + \left(\frac{k_2(-u_2k_2\tan\theta - u_2k_1 + 1)}{(k_1^2+k_2^2)(u_2k_1\tan\theta - \tan\theta - u_2k_2)} + \frac{k_1}{(k_1^2+k_2^2)} \right) V_2$ *where* $\tan\theta \neq \frac{u_2k_2}{(u_2k_1-1)}$.

Proof. Since $\langle \eta_2, \eta_3^* \rangle = c^*(a\cos\theta - b\sin\theta) + d^*(a\cos\theta + b\sin\theta)$ and under the orthogonality condition

$$\frac{-v_3k_2^*}{\sqrt{(v_3k_2^*)^2 + 1}}(a\cos\theta - b\sin\theta) + \frac{-1}{\sqrt{(v_3k_2^*)^2+1}}(a\sin\theta + b\cos\theta) = 0$$

$$\begin{aligned} -v_3k_2^*(a\cos\theta - b\sin\theta) &= (a\sin\theta + b\cos\theta) \\ v_3 &= \frac{k_2(-u_2k_2\tan\theta - u_2k_1 + 1)}{(k_1^2 + k_2^2)(u_2k_1\tan\theta - \tan\theta - u_2k_2)} \\ \tan\theta &\neq \frac{u_2k_2}{(u_2k_1 - 1)} \end{aligned}$$

we have the proof. □

Corollary 2.6. *Binormal ruled surface and Mannheim tangent ruled surface of Mannheim curve α have not perpendicular normal vector fields.*

Proof. Since $\langle \eta_3, \eta_1^* \rangle = -d$ and $\frac{-1}{\sqrt{(u_3k_2)^2+1}} \neq 0$ it is trivial. □

Corollary 2.7. *Binormal ruled surface and Mannheim normal ruled surface of Mannheim curve α have perpendicular normal vector fields along* $\varphi_2^*(s) = \alpha(s) + \frac{\cos\theta\sin\theta}{-u_3(k_1^2+k_2^2)\cos^2\theta + \dot\theta}V_1 + \frac{k_1}{(k_1^2+k_2^2)}V_2 + \frac{\cos^2\theta}{-u_3(k_1^2+k_2^2)\cos^2\theta+\dot\theta}V_3$, *except* $u_3 = \frac{(k_1^2+k_2^2)\cos^2\theta}{\dot\theta}$.

Proof. Since$\langle \eta_3, \eta_2^* \rangle = a^* c \cos\theta + db^*$ and under the orthogonality condition $\langle \eta_3, \eta_2^* \rangle = 0$ we have

$$\begin{aligned} -v_2 k_2^* c \cos\theta + d\left(1 - v_2 k_1^*\right) &= 0 \\ -v_2 k_2^* c \cos\theta - d v_2 k_1^* &= -d \\ v_2 &= \frac{\cos\theta}{-u_3\left(k_1^2 + k_2^2\right)\cos^2\theta + \dot{\theta}} \end{aligned}$$

where $k_1^* = -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta}$ and $k_2^* = \frac{k_1}{\lambda k_2}$. □

Corollary 2.8. *Binormal ruled surface and Mannheim binormal ruled surface Mannheim curve α, have perpendicular normal vector fields along the curve $\varphi_3^*(s) = \alpha(s) + \frac{k_2 \tan\theta + k_1}{k_1^2 + k_2^2} V_2$*

Proof. Since $\langle \eta_3, \eta_3^* \rangle = c^* c \cos\theta + d^* c \sin\theta$ and $\langle \eta_3, \eta_3^* \rangle = 0$,we have

$$\begin{aligned} \frac{-v_3 k_2^*}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}} c\cos\theta &= \frac{1}{\sqrt{\left(v_3 k_2^*\right)^2 + 1}} c \sin\theta \\ -v_3 k_2^* c \cos\theta &= c \sin\theta \\ v_3 &= \frac{k_2 \tan\theta}{k_1^2 + k_2^2} \end{aligned}$$

hence we have the proof. □

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Baskent University, Faculty of Education, Department of Elementary Mathematics Education, Ankara-TURKEY

E-mail address: `seyda@baskent.edu.tr`