Turk. J. Math. Comput. Sci.
17(1)(2025) 1–9
© MatDer
DOI : 10.47000/tjmcs.1585388

Multivalued Probabilistic ω-Contraction in b-Menger Spaces with Application

Achtoun Youssef[1,*], Mbarki Abderrahim[2], Tahiri Ismail[3], Sefian Lamarti Mohammed[3]

[1] *Department of Mathematics and Computer Science, Normal Higher School, Abdelmalek Essaadi University, P.O Box 209, Tetouan, Morocco.*
[2] *Department of Mechanics and Applied Mathematics, National School of Applied Sciences of Oujda, Mohammed First University, P.O. Box 669, Oujda, Morocco.*
[3] *Department of Mathematics and Computer Science, Normal Higher School, Abdelmalek Essaadi University, Str. Hassan II, P.O Box 209, Tetouan, Morocco.*

Received: 14-11-2024 • Accepted: 17-12-2024

Abstract. In this article, we extend the concept of multivalued ω-contraction mappings within the framework of b-Menger spaces. We introduce a novel fixed point theorem specific to these mappings, significantly broadening the existing body of knowledge in this area. Our primary result directly leads to the derivation of an equivalent fixed point theorem for fuzzy b-metric spaces, showcasing the versatility and applicability of our findings in more generalized and complex metric structures. Furthermore, we provide a concrete application of the principal theorem in the context of ordinary b-metric spaces, demonstrating the practical implications and effectiveness of our theoretical advancements. This research contributes to the deeper understanding and potential applications of fixed point theory in various scientific and engineering domains, where probabilistic and fuzzy structures play a critical role.

2020 AMS Classification: 47H10, 54E70, 54E35

Keywords: Fixed point, b-Menger spaces, multivalued ω-contraction, fuzzy b-metric spaces, b-metric spaces.

1. Introduction

Menger spaces, named after Karl Menger, are a generalization of metric spaces incorporating probabilistic metrics. These spaces provide a richer structure for dealing with uncertainty and imprecision, making them particularly useful in the study of stochastic processes and fuzzy systems [10]. The fixed point results in Menger spaces have been instrumental in advancing the understanding of various mathematical models that involve randomness and vagueness [15]. The concept of b-Menger spaces, introduced by Mbarki et al. [11], has garnered significant interest, particularly in the context of fixed point theory for single-valued and multi-valued mappings within these spaces [1–3, 12, 13]. Fixed point theory, a fundamental aspect of nonlinear analysis, is crucial for solving various equations where the solution can be interpreted as a point that remains invariant under a given function. This theory has profound implications across diverse fields such as economics, engineering and computer science, offering tools for addressing problems in

*Corresponding Author
Email addresses: achtoun44@outlook.fr (Y. Achtoun), dr.mbarki@gmail.com (A. Mbarki), istahiri@uae.ac.ma (I. Tahiri), lamarti.mohammed.sefian@uae.ac.ma (M. Lamarti Sefian)

optimization, game theory and dynamic systems [4].

Hadzić extended this area by introducing multi-valued probabilistic ω-contraction mappings, proving fixed point results in both Menger spaces and fuzzy metric spaces through the function of non-compactness [6, 7]. These results are significant because multi-valued mappings, where a single input can correspond to multiple outputs, frequently appear in practical applications such as differential inclusions, control theory and economics. Fang further generalized Hadzić's results by replacing the condition of a continuous t-norm with a t-norm of H-type, thereby broadening the applicability of these fixed point theorems [5]. This generalization is crucial for dealing with more complex systems where the underlying probabilistic structure is not necessarily governed by continuous operations.

In this paper, we define the class of ω-contractions in the context of multi-valued mappings in b-Menger spaces, building on the foundational work presented by Fang in Menger spaces. We establish the existence of a fixed point theorem and extend this to fuzzy b-metric spaces, which further enhances the utility and flexibility of these mathematical structures. To demonstrate the practical implications of our theoretical findings, we provide an application in the realm of usual b-metric spaces, illustrating the relevance and impact of our results in real-world scenarios. The development of fixed point theory in b-Menger spaces, particularly with multi-valued mappings, opens new avenues for research and application in various scientific fields. By leveraging the probabilistic nature and the flexibility of Menger spaces, we aim to contribute to the broader understanding and application of fixed point results in complex and dynamic systems.

2. Preliminaries

In this section, we review some of the fundamental terms and lemmas related to the b-Menger space.

Definition 2.1 ([15]). Let Λ^+ be the space of all distance distribution functions $\zeta : [0, +\infty] \to [0, 1]$ such that:

(1) ζ is left continuous on $[0, +\infty]$,
(2) ζ is non-decreasing,
(3) $\zeta(0) = 0$ and $\zeta(+\infty) = 1$.

The subset $\triangle^+ \subset \Lambda^+$ is the set $\triangle^+ = \left\{\zeta \in \Lambda^+ : \lim_{x\to+\infty} \zeta(x) = 1\right\}$.

A unique component of $\triangle^+$ is θ_a function, which is defined as follows:

$$\theta_a(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \in [0, a]\,, \\ 1 & \text{if} \quad \alpha \in (a, +\infty). \end{cases}$$

Definition 2.2 ([15]). A triangular norm, often known as a t-norm, is a binary operation $\mathfrak{H}$ on $[0, 1]$ that satisfies the following requirements for every $\chi, \beta, \eta \in [0, 1]$

(1) $\mathfrak{H}(\chi, \beta) = \mathfrak{H}(\beta, \chi)$,
(2) $\mathfrak{H}(\chi, \mathfrak{H}(\beta, \eta)) = \mathfrak{H}(\mathfrak{H}(\chi, \beta), \eta)$,
(3) $\mathfrak{H}(\chi, \beta) < \mathfrak{H}(\chi, \eta) \quad$ for $\beta < \eta$,
(4) $\mathfrak{H}(\chi, 1) = \mathfrak{H}(1, \chi) = \chi$.

Some fundamental t-norms are: $\mathfrak{H}_M(\beta, \eta) = min(\beta, \eta)$, $\mathfrak{H}_P(\beta, \eta) = \beta \cdot \eta$ and $\mathfrak{H}_L(\beta, \eta) = max(\beta + \eta - 1, 0)$.

Definition 2.3 ([8]). We call a t-norm $\mathfrak{H}$ an H-type t-norm if the family $(\mathfrak{H}^n(x))_{n\in\mathbb{N}}$ is equicontinuous at the point $\chi = 1$, that is,

$$for\ all\ \theta \in (0, 1),\ there\ exists\ \rho \in (0, 1) :\ t > 1 - \rho \Rightarrow \mathfrak{H}^n(t) > 1 - \theta\ \ for\ all\ n \geq 1,$$

where for all $\chi \in [0, 1]$ and $n \in \mathbb{N}$ we write

$$\mathfrak{H}^n(\chi) = \begin{cases} 1 & \text{if } n = 0, \\ \mathfrak{H}(\mathfrak{H}^{n-1}(\chi), \chi) & \text{otherwise.} \end{cases}$$

Definition 2.4 ([11]). The quadruple $(\Omega, \Gamma, \mathfrak{H}, s)$ is called a b-Menger space in which Ω is a nonempty set, Γ is a function from $\Omega \times \Omega$ into Λ^+, $\mathfrak{H}$ is a t-norm, $s \geq 1$ is a real number and the following requirements are met for all $p, q, r \in \Omega$ and $x, y > 0$,

(1) $\Gamma_{p,p} = \theta_0$,
(2) $\Gamma_{p,q} \neq \theta_0\ \ if\ p \neq q$,
(3) $\Gamma_{p,q} = \Gamma_{q,p}$,

(4) $\Gamma_{p,q}(s(x+y)) \geq \mathfrak{H}(\Gamma_{p,r}(x), \Gamma_{r,q}(y))$.

It should be noted that a b-Menger space with $s = 1$ is a Menger space.

Mbarki et al. [11] demonstrated that under the topology produced by the family of (θ,ρ)-neighborhoods, $(\Omega, \Gamma, \mathfrak{H}, s)$ is a Hausdorff topological space if $(\Omega, \Gamma, \mathfrak{H}, s)$ is a b-Menger space under a continuous t-norm $\mathfrak{H}$.

$$N = \left\{N_p(\theta,\rho) : \ p \in \Omega, \ \theta > 0 \ \ and \ \ \rho > 0\right\},$$

where

$$N_p(\theta,\rho) = \left\{q \in \Omega : \ \Gamma_{p,q}(\theta) > 1 - \rho\right\}.$$

Definition 2.5 ([11])**.** In a b-Menger space $(\Omega, \Gamma, \mathfrak{H}, s)$, where $\mathfrak{H}$ is continuous, a sequence $\{\nu_n\}$ is claimed to be:

(1) Convergent to $\nu \in \Omega$ if for all $\theta > 0$ and $\rho > 0$ there exist $N(\theta,\rho) \in \mathbb{N}$ satisfying $\Gamma_{\nu_n,\nu}(\rho) > 1 - \theta$ for any $n \geq N$.
(2) Strong Cauchy sequence if for each $\theta > 0$ and $\rho > 0$ there exist $N(\theta,\rho) \in \mathbb{N}$ that satisfy $\Gamma_{\nu_n,\nu_m}(\rho) > 1 - \theta$ for any $n, m \geq N$.

A b-Menger space $(\Omega, \Gamma, \mathfrak{H}, s)$ is complete if each Cauchy sequence in Ω is convergent to some point in Ω.

Definition 2.6 ([11])**.** A fuzzy b-metric space is a quadruple $(\Omega, \mathcal{M}, \mathfrak{H}, s)$ in which Ω is a nonempty set, $\mathfrak{H}$ is a t-norm and $\mathcal{M}$ is a fuzzy set on $\Omega \times \Omega \times (0, +\infty)$ verifying the requirements listed below:

(1) $\mathcal{M}(p, q, 0) = 0$,
(2) $\mathcal{M}(p, q, t) = 1$ for all $t > 0$ iff $p = q$,
(3) $\mathcal{M}(p, q, t) = \mathcal{M}(q, p, t)$,
(4) $\mathcal{M}(p, r, s(t + v)) \geq \mathfrak{H}(\mathcal{M}(p, q, t), \mathcal{M}(q, r, v))$,
(5) $\mathcal{M}(p, q, .) : [0, +\infty) \longrightarrow [0, 1]$ is left-continuous and nondecreasing
for all $p, q, r \in \Omega$ and $t, v > 0$.

When $s = 1$ then $(\Omega, \mathcal{M}, \mathfrak{H}, 1)$ is a fuzzy metric space according to Michalek and Kramosil [9].

In the following, we suppose that b-Menger space $(\Omega, \Gamma, \mathfrak{H}, s)$ that $\mathfrak{H}$ is a continuous t-norm, and we denotes by $C(\Omega)$ the class of all nonempty closed subsets of Ω, where for all $\mathcal{A}, \mathcal{B} \in C(\Omega)$ and $x \in \Omega$ the functions $\Gamma_{x,\mathcal{A}}(.)$ and $\Gamma_{\mathcal{A},\mathcal{B}}(.)$ are defined as follows

$$\Gamma_{x,\mathcal{A}}(t) = \sup_{y\in\mathcal{A}} \Gamma_{x,y}(t) \quad for\ each\ t \in [0, +\infty),$$
$$\Gamma_{\mathcal{A},\mathcal{B}}(t) = \inf_{x\in\mathcal{A}} \sup_{y\in\mathcal{B}} \Gamma_{x,y}(t) \quad for\ each\ t \in [0, +\infty).$$

Lemma 2.7. We consider $(\Omega, \Gamma, \mathfrak{H}, s)$ a b-Menger space where $\mathfrak{H}$ is continuous, then for all $\mathcal{A} \in C(\Omega)$ and $x, y \in \Omega$ we have

(1) $\Gamma_{x,\mathcal{A}}(t_0) = 1$ for any $t_0 > 0$ if and only if $x \in \mathcal{A}$.
(2) $\Gamma_{x,\mathcal{A}}(s(t_0 + t_1)) \geq \mathfrak{H}\left(\Gamma_{x,y}(t_0), \Gamma_{y,\mathcal{A}}(t_1)\right)$ for all $t_0, t_1 > 0$.

Proof. (1) Suppose that $\Gamma_{x,\mathcal{A}}(t_0) = 1$ for all $t_0 > 0$.
So, for any $\theta > 0$ and $\rho \in (0, 1)$ there exists $x_0 \in \mathcal{A}$ such that $\Gamma_{x,x_0}(\theta) > 1 - \rho$.
As $\mathcal{A} \in C(\Omega)$ then $x \in A$.
Conversely, if $x \in \mathcal{A}$, then for all $t > 0$

$$\begin{aligned}\Gamma_{x,\mathcal{A}}(t_0) &= \sup_{y\in\mathcal{A}} \Gamma_{x,y}(t_0)\\ &\geq \Gamma_{x,x}(t_0) = 1.\end{aligned}$$

Hence, $\Gamma_{x,\mathcal{A}}(t_0) = 1$ for all $t_0 > 0$.

(2) Let $t_0, t_1 > 0$.
Case 1
If $\Gamma_{y,\mathcal{A}}(t_1) = 0$, the inequality it's verified.
Case 2
Suppose that $\Gamma_{y,\mathcal{A}}(t_1) > 0$.
Let $\theta > 0$ such that $\theta < \Gamma_{y,\mathcal{A}}(t_1)$.

Since $\Gamma_{y,\mathcal{A}}(t_1) = \sup_{z\in\mathcal{A}} \Gamma_{y,z}(t_1) \geq 0$, there exists $v \in A$ such that $\Gamma_{y,v}(t_1) > \Gamma_{y,\mathcal{A}}(t_1) - \theta$.
The b-Menger triangle inequality gives us the following

$$\begin{aligned}\Gamma_{x,\mathcal{A}}\left(s(t_0+t_1)\right) &= \sup_{z\in\mathcal{A}} \Gamma_{x,z}\left(s(t_0+t_1)\right)\\ &\geq \sup_{z\in\mathcal{A}} \mathfrak{H}\Big(\Gamma_{x,y}(t_0), \Gamma_{y,z}(t_1)\Big)\\ &\geq \mathfrak{H}\Big(\Gamma_{x,y}(t_0), \Gamma_{y,v}(t_1)\Big)\\ &\geq \mathfrak{H}\Big(\Gamma_{x,y}(t_0), \Gamma_{y,\mathcal{A}}(t_1) - \theta\Big).\end{aligned}$$

Letting $\theta \to 0$, then the inequality is proved.

□

3. Main result

We will consider the class Φ_r of all function $\omega : [0,+\infty) \longrightarrow [0,+\infty)$ satisfying

$$0 < \omega(t) < t \ \ \forall t > 0;$$

$$\forall t > 0, \ \ \exists r \geq t \ \ \lim_{n\to\infty} \omega^n(r) = 0. \tag{3.1}$$

To obtain our primary result, we will require the next lemma.

Lemma 3.1. Let $(\Omega, \Gamma, \mathfrak{H}, s)$ be a b-Menger space under a t-norm of H-type and $\{v_n\}$ is a sequence in Ω. If there's a function $\omega \in \Phi_r$ that satisfy

$$\Gamma_{v_n,v_m}(\omega(t)) \geq \Gamma_{v_{n-1},v_{m-1}}(st) \quad for\ all\ n,m > 0\ and\ t > 0, \tag{3.2}$$

then $\{v_n\}$ is a Cauchy sequence in Ω.

Proof. We consider $\{v_n\}$ a sequence in Ω that satisfy the condition (3.2), so we have that

$$\Gamma_{v_n,v_{n+1}}(\omega(t)) \geq \Gamma_{v_{n-1},v_n}(st) \quad \forall n \in \mathbb{N}, \ \forall t > 0.$$

Then, by induction we get:

$$\begin{aligned}\Gamma_{v_n,v_{n+1}}(\omega^n(t)) &\geq \Gamma_{v_0,v_1}(st)\\ &\geq \Gamma_{v_0,v_1}(t) \quad \forall n \in \mathbb{N}\ \forall t > 0.\end{aligned}$$

Next, we demonstrate that $\lim_{n\to\infty} \Gamma_{v_n,v_{n+1}}(t) = 1$ for all $t > 0$.
Since $\lim_{t\to+\infty} \Gamma_{v_0,v_1}(t) = 1$.
Then, for $\theta \in (0,1]$ there exists $t_0 > 0$ satisfying $\Gamma_{v_0,v_1}(t_0) > 1 - \theta$.
As $\omega \in \Phi_r$, then there exists $t_1 \geq t_0$ such that $\lim_{n\to\infty} \omega^n(t_1) = 0$. Hence, for all $t > 0$ there's $n_0 \in \mathbb{N}$ where $\omega^n(t_1) < t$ for all $n \geq n_0$.
Due to Γ's monotonicity, we obtain

$$\Gamma_{v_n,v_{n+1}}(t) \geq \Gamma_{v_n,v_{n+1}}(\omega^n(t_1)) \geq \Gamma_{v_0,v_1}(t_1) > \Gamma_{v_0,v_1}(t_0) > 1 - \theta \ \ \forall n \geq n_0$$

which is implied that

$$\lim_{n\to+\infty} \Gamma_{v_n,v_{n+1}}(t) = 1 \quad for\ all\ t > 0. \tag{3.3}$$

Since $\omega \in \Phi_r$, then for any $t > 0$ an $r \geq t$ can be found where $\omega(r) < t$.
In fact, Assuming that there's some $r_1 > 0$ such that for any $r \geq r_1$, $\omega(r) \geq r_1$, we can infer by induction that

$$\omega^n(r) \geq r_1 \text{ for all } n \in \mathbb{N}.$$

Therefore, $\lim_{n\to\infty} \omega^n(r) \neq 0$ for $r \geq r_1$, which is a contradiction with (3.1).
Using induction, we demonstrate that for all $k > 1$,

$$\Gamma_{v_n,v_{n+k}}(st) \geq \mathfrak{H}^{k-1}\left(\Gamma_{v_n,v_{n+1}}(t - \omega(r))\right). \tag{3.4}$$

The inequality (3.4) is satisfied for $k = 2$. Now, suppose that (3.4) holds for $k > 1$. Using (3.2) and the monotonicity of $\mathfrak{H}$, we obtain

$$\begin{aligned}
\Gamma_{\nu_n,\nu_{n+k+1}}(st) &= \Gamma_{\nu_n,\nu_{n+k+1}}\,(s(t-\omega(r)) + s\omega(r)) \\
&\geq \mathfrak{H}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(r)), \Gamma_{\nu_{n+1},\nu_{n+1+k}}(\omega(r)) \\
&\geq \mathfrak{H}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(r)), \Gamma_{\nu_n,\nu_{n+k}}(sr)) \\
&\geq \mathfrak{H}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(r)), \Gamma_{\nu_{n+1},\nu_{n+k}}(st)) \\
&\geq \mathfrak{H}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(r)), \mathfrak{H}^{k-1}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(t))) \\
&= \mathfrak{H}^{k}(\Gamma_{\nu_n,\nu_{n+1}}(t-\omega(r))).
\end{aligned}$$

Hence, (3.4) is proved for all $k > 1$.

Finally, we demonstrate that $\{\nu_n\}$ is a Cauchy. Let $\theta \in (0, 1)$, since $\mathfrak{H}$ is of H-type, there exists $\delta > 0$ that satisfy

$$\mathfrak{H}^n(t) > 1-\theta \ \ for\ all\ t \in (1-\delta, 1]\ and\ n \in \mathbb{N}. \tag{3.5}$$

As $\frac{t-\omega(r)}{s} > 0$, then by (3.3) we have that $\lim_{n\to\infty} \Gamma_{\nu_n,\nu_{n+1}}(\frac{t-\omega(r)}{s}) = 1$. So, there's $N \in \mathbb{N}$ satisfying $\Gamma_{\nu_n,\nu_{n+1}}(\frac{t-\omega(r)}{s}) > 1-\delta$ for all $n \geq N$. From (3.4) and (3.5) we get:

$$\Gamma_{\nu_n,\nu_{n+k}}(t) \geq \mathfrak{H}^{k-1}(\Gamma_{\nu_n,\nu_{n+1}}(\frac{t-\omega(r)}{s})) > 1-\theta \ \ for\ all\ n \geq N,\ and\ k > 1.$$

As a result, the sequence $\{\nu_n\}$ is a Cauchy. □

Now, we will generalize the definition of ω-contraction to multi-valued case in b-Menger spaces.

Definition 3.2. Let $(\Omega, \Gamma, \mathfrak{H}, s)$ be a b-Menger space and $\omega : [0, \infty) \to [0, \infty)$. A mapping $f : \Omega \to C(\Omega)$ is called a multi-valued probabilistic ω-contraction if for every $x, y \in \Omega$ and every $p \in fx$ there exists $q \in fy$ such that

$$\Gamma_{p,q}(\omega(t)) \geq \Gamma_{x,y}(st) \quad \forall t > 0. \tag{3.6}$$

Remark 3.3. Note that if f is a multi-valued ω-contraction, we have

$$\Gamma_{fx,fy}(\omega(t)) \geq \Gamma_{x,y}(st) \quad \forall x, y \in \Omega,\ \forall t > 0.$$

Theorem 3.4. Let $(\Omega, \Gamma, \mathfrak{H}, s)$ be a complete b-Menger space under a continuous t-norm $\mathfrak{H}$ of H-type and $f : \Omega \to C(\Omega)$ be a probabilistic ω-contraction multi-valued mapping where $\omega \in \Phi_r$. Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

Proof. We consider $\nu_0 \in \Omega$ and $\nu_1 \in f\nu_0$, then there exists $\nu_2 \in f\nu_1$ such that by (3.6) we get

$$\Gamma_{\nu_1,\nu_2}(\omega(t)) \geq \Gamma_{\nu_0,\nu_1}(st) \ \ \forall t > 0.$$

Thus, we may create a sequence $\{\nu_n\}$ that satisfies the following requirements:

$$\nu_{n+1} \in f\nu_n \ \ and \ \ \Gamma_{\nu_n,\nu_{n+1}}(\omega(t)) \geq \Gamma_{\nu_{n-1},\nu_n}(st) \ \ for\ each\ n \in \mathbb{N},\ and\ t > 0.$$

We show first that for any $t > 0$,

$$\Gamma_{\nu_n,\nu_{n+m}}(\omega(t)) \geq \Gamma_{\nu_{n-1},\nu_{n+m-1}}(st) \quad \forall m > 0. \tag{3.7}$$

It's obvious that (3.7) holds for $m = 1$. Suppose that (3.7) holds for $m > 0$. Since $\nu_n \in f\nu_{n-1}$ and $\nu_{n+m+1} \in f\nu_{n+m}$, using Remark 3.3, we obtain

$$\begin{aligned}
\Gamma_{\nu_n,\nu_{n+m+1}}(\omega(t)) &\geq \Gamma_{f\nu_{n-1},\nu_{n+m+1}}(\omega(t)) \\
&\geq \Gamma_{f\nu_{n-1},f\nu_{n+m}}(\omega(t)) \\
&\geq \Gamma_{\nu_{n-1},\nu_{n+m}}(st).
\end{aligned}$$

So by induction we proved that (3.7) holds for all $m > 0$. Using Lemma 3.1, so $\{\nu_n\}$ is a Cauchy sequence. Since $(\Omega, \Gamma, \mathfrak{H}, s)$ is complete, then $\nu_n \to \sigma \in \Omega$. So to complete the demonstration, we prove that $\sigma \in f\sigma$. Let $\theta > 0$, using the b-Menger triangle inequality we get

$$\Gamma_{\sigma,\nu_{n+1}}(\theta) \geq \mathfrak{H}\Big(\Gamma_{\sigma,\nu_n}(\frac{\theta}{2s}), \Gamma_{\nu_n,\nu_{n+1}}(\frac{\theta}{2s})\Big). \tag{3.8}$$

From (3.3) we have that $\lim_{n\to\infty} \Gamma_{\nu_n,\nu_{n+1}}(\frac{\theta}{2s}) = 1$. Letting $n \to +\infty$ in (3.8), we obtain that

$$\lim_{n\to\infty} \Gamma_{\sigma,\nu_{n+1}}(\theta) = 1.$$

On the other hand, by the hypothesis we have that $\mathfrak{H}$ is of H-type, we concluded that for any $\rho \in (0,1)$ we can found $\delta \in (0,1)$ satisfying

$$\mathfrak{H}(1-\delta, 1-\delta) > 1-\rho.$$

And since that

$$\begin{aligned}\Gamma_{\nu_{n+1},f\sigma}(\frac{t}{2s}) &\geq \Gamma_{\nu_{n+1},f\sigma}\left(\omega(\frac{t}{2s})\right)\\ &\geq \Gamma_{f\nu_n,f\sigma}\left(\omega(\frac{t}{2s})\right)\\ &\geq \Gamma_{\nu_n,\sigma}(\frac{t}{2}).\end{aligned}$$

Then, there exists $N \in \mathbb{N}$ such that

$$\Gamma_{\sigma,\nu_{n+1}}(\frac{t}{2s}) > 1-\delta \ \textit{ and } \ \Gamma_{\nu_{n+1},f\sigma}(\frac{t}{2s}) > 1-\delta \ \ \forall n > N.$$

Which implies that

$$\begin{aligned}\Gamma_{\sigma,f\sigma}(t) &\geq \mathfrak{H}\left(\Gamma_{\sigma,\nu_{n+1}}(\frac{t}{2s}), \Gamma_{\nu_{n+1},f\sigma}(\frac{t}{2s})\right)\\ &\geq T(1-\delta, 1-\delta)\\ &\geq 1-\rho.\end{aligned}$$

Finally, we conclude that $\Gamma_{\sigma,f\sigma}(t) = 1$. Therefore, by Lemma 2.7, we get that $\sigma \in f\sigma$. □

Example 3.5. Let $\Omega = [0,\infty)$. We define $\Gamma : \Omega \times \Omega \to \triangle^+$ as follows

$$\Gamma_{x,y}(t) = \theta_0(t - |x-y|^2).$$

From Lemma 3.1 and Example 3.3 in [11], we have that $(\Omega, \Gamma, \mathfrak{H}_M, 2)$ is a complete b-Menger space. And we consider the function $f : \Omega \to C(\Omega)$ given by $f(x) = \left\{1, \frac{x}{2}, \frac{x}{3}\right\}$.
Then, for any $x, y \in \Omega$ and $p \in fx$, we have three cases:

Case 1: If $p = 1 \in fx$, we put $q = 1$.
Then, for all $t > 0$, $\Gamma_{p,q}(\frac{1}{2}t) = 1 \geq \Gamma_{x,y}(2t)$.
Case 2: If $p = \frac{x}{2} \in fx$, we choose $q = \frac{y}{2}$.
So, for all $t > 0$,

$$\Gamma_{p,q}(\frac{1}{2}t) = \theta_0(\frac{1}{2}t - \left|\frac{x}{2} - \frac{y}{2}\right|^2) = \theta_0(2t - |x-y|^2) = \Gamma_{x,y}(2t).$$

Case 3: If $p = \frac{x}{3} \in fx$, we choose $q = \frac{y}{3}$.
Then, for all $t > 0$, we get

$$\Gamma_{p,q}(\frac{1}{2}t) = \theta_0(\frac{1}{2}t - \left|\frac{x}{3} - \frac{y}{3}\right|^2) = \theta_0(\frac{9}{2}t - |x-y|^2),$$

therefore

$$\begin{aligned}\Gamma_{p,q}(\frac{1}{2}t) &\geq \theta_0(2t - |x-y|^2)\\ &= \Gamma_{x,y}(2t).\end{aligned}$$

Hence, we concluded that f is a probabilistic ω-contraction multi-valued mapping with $\omega(t) = \frac{1}{2}t$. Consequently, every need stated in Theorem 3.4 is met, hence f have two fixed points which are 0 and 1.

Example 3.6. Let $\Omega = [0, \infty)$. We define $\Gamma : \Omega \times \Omega \to \Lambda^+$ by

$$\Gamma_{x,y}(t) = \theta_0(t - |x - y|^2).$$

Then, $(\Omega, \Gamma, \mathfrak{H}_M, 2)$ is a complete b-Menger space. And we consider the function $f : \Omega \to C(\Omega)$ given by $f(x) = [0, \frac{x}{2}]$. We will prove that for all $x, y \in \Omega$ and $p \in fx$ there exists $q \in fy$ such that

$$\Gamma_{p,q}(\omega(t)) \geq \Gamma_{x,y}(2t).$$

With $\omega(t) = \frac{3}{4}t$. It suffices to show that for all $x, y \in \Omega$ and $p \in fx$ there exists $q \in fy$ such that

$$|p - q| \leq \left|\frac{x}{2} - \frac{y}{2}\right|.$$

Indeed, if $p < \frac{y}{2}$ then $p \in [0, \frac{y}{2}]$. So, there exists $q \in [0, \frac{y}{2}]$ such that

$$|p - q| \leq \left|\frac{x}{2} - \frac{y}{2}\right|.$$

If $p \geq \frac{y}{2}$, since $0 \leq p \leq \frac{x}{2}$, we get

$$0 \leq p - \frac{y}{2} \leq \frac{x}{2} - \frac{y}{2}.$$

Then, if we take $q = \frac{y}{2}$, we obtain

$$|p - q| \leq \left|\frac{x}{2} - \frac{y}{2}\right|.$$

Therefore,

$$\begin{aligned}\Gamma_{p,q}(\omega(t)) &= \theta_0(\frac{3}{4}t - |p - q|^2)\\ &\leq \theta_0(\frac{3}{4}t - \left|\frac{x}{2} - \frac{y}{2}\right|^2)\\ &= \theta_0(\frac{3}{4}t - \frac{1}{4}|x - y|^2).\end{aligned}$$

Now, suppose that $\Gamma_{x,y}(st) = \theta_0(2t - |x - y|^2) > 0$, this implies that $2t > |x - y|^2$.
Then,

$$t > \frac{1}{2}|x - y|^2.$$

It follows

$$\frac{3}{4}t > \frac{3}{8}|x - y|^2.$$

Hence,

$$\omega(t) > \frac{1}{4}|x - y|^2.$$

From the previous inequality, we obtain

$$\Gamma_{p,q}(\omega(t)) = \theta_0(\frac{3}{4}t - |p - q|^2) > 0.$$

Therefore,

$$\Gamma_{p,q}(\omega(t)) = 1.$$

Then, all the conditions of Theorem 3.4 are verified, which implies that f have a fixed point which is 0.

Remark 3.7. Note that the condition of $\mathfrak{H}$ is a t-norm of H-type is a necessary condition of the existence of a fixed point as shown in [14] and confirmed by the following example.

Example 3.8 ([16])**.** Define $\mathcal{L}$ as the distribution function as follows:

$$\mathcal{L}(t) = \begin{cases} 0 & \text{if} \quad t \leq 4, \\ 1 - \frac{1}{n} & \text{Si} \quad 2^n < t < 2^{n+1}, \quad n > 1. \end{cases}$$

We consider $\Omega = \{1, 2, ...\}$ and we define $\Gamma : \Omega \times \Omega \longrightarrow \triangle^+$ as follows :

$$\Gamma_{n,n+m}(t) = \begin{cases} 0 & \text{if} \quad t = 0, \\ \mathfrak{H}_L^m \left(\mathcal{L}(2^n t), \mathcal{L}(2^{n+1} t), ..., \mathcal{L}(2^{n+m} t) \right) & \text{if} \quad t > 0. \end{cases}$$

Then, we obtain that $(\Omega, \Gamma, \mathfrak{H}_L, 1)$ is a complet b-Menger space, and since that every single-valued mapping is a multi-valued mapping we put $f(n) = n + 1$, which is ω-contractive with $\omega(t) = \frac{1}{2}t$. However, f have no fixed point, since that $\mathfrak{H}_L$ is not of H-type.

In the following we present some consequences of this result. By taking $\omega(t) = kt$ with $k \in (0, 1)$, as a first consequence of Theorem 3.4 we get:

Corollary 3.9. Let $(\Omega, \Gamma, \mathfrak{H}, s)$ be a complete b-Menger space under a continuous t-norm $\mathfrak{H}$ of H-type and $f : \Omega \to C(\Omega)$ be a multi-valued probabilistic k-contraction mapping where $k \in (0, 1)$, i.e., for any $x, y \in \Omega$ and every $p \in fx$ there exist $q \in fy$ that satisfy

$$\Gamma_{p,q}(kt) \geq \Gamma_{x,y}(st) \quad for\ all\ t > 0.$$

Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

The following Corollary is a special instance of Theorem 3.4.

Corollary 3.10 ([5]). Let $(\Omega, \Gamma, \mathfrak{H})$ be a complete Menger space under a continuous t-norm $\mathfrak{H}$ of H-type and $f : \Omega \to C(\Omega)$ be a multi-valued probabilistic ω-contraction mapping in the sense of Fang. Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

Given that the condition $\Gamma_{p,q}(0) = 0$ was not required in the proof of Theorem 3.4, we now present the following result in fuzzy b-metric spaces, where $\Gamma_{x,y}(t) = \mathcal{M}(x, y, t)$ for all $t > 0$.

Corollary 3.11. Let $(\Omega, \mathcal{M}, \mathfrak{H}, s)$ be a complete fuzzy b-metric space where $\lim\limits_{t\to\infty} \mathcal{M}(x, y, t) = 1$ for all $x, y \in \mathcal{X}$, $\mathfrak{H}$ is a continuous t-norm $\mathfrak{H}$ of H-type and $f : \Omega \to C(\Omega)$ is a fuzzy ω-contraction mapping with $\omega \in \Phi_r$, i.e., for any $x, y \in \Omega$ and every $p \in fx$ there exist $q \in fy$ such that

$$\mathcal{M}(p, q, \omega(t)) \geq \mathcal{M}(x, y, st) \quad for\ all\ t > 0.$$

Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

4. Application

We prove the analogous fixed point theorems in ordinary b-metric spaces as an application of the main result.

Corollary 4.1. Let (Ω, d) be a complete b-metric space with a constant $s \geq 1$ and $\omega \in \Phi_r$ a non-decreasing function. If a multi-valued mapping $f : \Omega \to C(\Omega)$ satisfy that for any $x, y \in \Omega$ and $p \in fx$ there exists $q \in fy$ such that

$$d(p, q) \leq \omega\left(\frac{d(x, y)}{s}\right). \tag{4.1}$$

Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

Proof. Let define a mapping $\Gamma : \Omega \times \Omega \to \triangle^+$ by:

$$\Gamma_{x,y} = \theta_{d(x,y)}.$$

Mbarki et al. in [11] proved that $(\Omega, \Gamma, \mathfrak{H}_M, s)$ is a complete b-Menger space. It is not hard to show that if a non empty set A is closed in (Ω, d), then it is in $(\Omega, \Gamma, \mathfrak{H}_M, s)$. So, to complete the proof, it's enough to show that the condition (4.1) implies that f is a multi-valued probabilistic ω-contraction mapping in $(\Omega, \Gamma, \mathfrak{H}_M, s)$. Indeed, let $f : \Omega \to C(\Omega)$ be a multi-valued mapping satisfying the condition (4.1). Then, for any $x, y \in \Omega$ and $p \in fx$ there exists $q \in fy$ such that

$$d(p, q) \leq \omega\left(\frac{d(x, y)}{s}\right).$$

Let $t \geq 0$, then $\Gamma_{pq}(\omega(t)) = \theta_{d(p,q)}(\omega(t)) = \begin{cases} 0, & \omega(t) \leq d(p, q); \\ 1, & \omega(t) > d(p, q). \end{cases}$

Case 1: If $\Gamma_{xy}(st) = 0$, then $\Gamma_{pq}(\omega(t)) \geq \Gamma_{xy}(st)$.

Case 2: If $\Gamma_{xy}(st) = 1$ then $t > \frac{d(p,q)}{s}$.

So, $\omega(t) > \omega(\frac{d(p,q)}{s}) \geq d(p,q)$. Then, $\Gamma_{pq}(\omega(t)) = 1$, which implies that $\Gamma_{pq}(\omega(t)) \geq \Gamma_{xy}(st)$.

So, from case 1 and case 2 we conclude that f is a multi-valued probabilistic ω-contraction. Thus, Theorem 3.4 implies the conclusion of Corollary 4.1. □

In the above corollary, if we put $\omega(t) = kt$ where $k \in (0,1)$, so we have the next corollary.

Corollary 4.2. Let (Ω, d) be a complete b-metric space with a constant $s \geq 1$ and $k \in (0,1)$. If a multi-valued mapping $f : \Omega \to C(\Omega)$ satisfy that for any $x, y \in X$ and $p \in fx$ there exists $q \in fy$ that satisfy

$$d(p,q) \leq k\frac{d(x,y)}{s}.$$

Then, there exists $\sigma \in \Omega$ such that $\sigma \in f\sigma$.

5. Conclusion

In this work, we have established the existence of fixed points for a multi-valued type of contraction, specifically ω-contractions, within the context of b-Menger spaces, a relatively recent addition to the mathematical literature. Our research advances the understanding of fixed point theory by not only proving the existence of such points in b-Menger spaces but also extending these results to b-fuzzy metric spaces under certain specialized conditions. Furthermore, we have demonstrated the practical applicability of our theoretical findings through a constructed application in ordinary b-metric spaces. This application underscores the relevance and utility of our results in real-world scenarios, particularly in areas where probabilistic and fuzzy structures are prevalent.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Each author equally contributed to this paper, read and approved the final manuscript.

References

[1] Achtoun, Y., Radenović, S., Tahiri, I., Sefian, M. L., *Exploring multivalued probabilistic ψ-contractions with orbits in b-Menger spaces*, Vojnotehnički Glasnik, **72**(2)(2024), 563–582.
[2] Achtoun, Y., Lamarti, M. S., Tahiri, I., *Multi-valued Hicks contractions in b-Menger spaces*, Nonlinear Functional Analysis and Applications, **29**(2)(2024), 477–485.
[3] Achtoun, Y., Radenović, S., Tahiri, I., Sefian, M. L., *The nonlinear contraction in probabilistic cone b-metric spaces with application to integral equation*, Nonlinear Analysis: Modelling and Control, **2024**, 1–12.
[4] Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fundamenta Mathematicae, **3**(1)(1922), 133–181.
[5] Fang, J.-x., *A note on fixed point theorems of Hadžic*, Fuzzy Sets and Systems, **48**(3)(1992), 391–395.
[6] Hadžic, O., *Fixed point theorems for multivalued mappings in some classes of fuzzy metric spaces*, Fuzzy Sets and Systems, **29**(1)(1989), 115–125.
[7] Hadžić, O., *Fixed point theorems for multivalued mappings in probabilistic metric spaces*, Matematički Vesnik, **3**(66)(1979), 125–134.
[8] Pap, E., Hadžić, O., Mesiar, R., *A fixed point theorem in probabilistic metric spaces and an application*, Journal of Mathematical Analysis and Applications, **202**(2)(1996), 433–449.
[9] Kramosil, I., Michálek, J., *Fuzzy metrics and statistical metric spaces*, Kybernetika, **11**(5)(1975), 336–344.
[10] Menger, K., *Statistical metrics*, Selecta Mathematica: Volume 2, Springer, 2003, 433–435.
[11] Mbarki, A., Oubrahim, R., *Probabilistic b-metric spaces and nonlinear contractions*, Fixed Point Theory and Applications, **2017**(1), 1–15.
[12] Mbarki, A., Oubrahim, R., *Common fixed point in b-Menger spaces with a fully convex structure*, International Journal of Applied Mathematics, **32**(2)(2019), 219.
[13] Mbarki, A., Oubrahim, R., *Fixed point theorem satisfying cyclical conditions in-Menger spaces*, Moroccan Journal of Pure and Applied Analysis, **5**(1)(2019), 31–36.
[14] Mbarki, A., Ouahab, A., Tahiri, I., *(f,g)-Boyd-Wong Contraction Mappings in Probabilistic Metric Space*, Applied Mathematical Sciences, **7**(13)(2013), 623–632.
[15] Schweizer, B., Sklar, A., *Probabilistic metric spaces*, North Holland, Amsterdam, 1982.
[16] Sherwood, H., *Complete probabilistic metric spaces*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, **20**(2)(1971), 117–128.