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Research Article

G-convergence and G-sequential spaces

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Abstract

In this paper, for a method G on a set X, we consider G-convergence and G-sequential convergence; we analyse the distinction between them with some counterexamples. Then, G-sequential continuity is investigated, and the relations between G-continuous and G-sequentially continuous functions are determined. Moreover, G-sequential spaces are introduced and studied. Finally, examples of G-sequential spaces associated with G-methods are given.

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1. Introduction

Convergent sequences enable us to define sequential versions of some topological notions that agree with the usual ones when X is the first countable topological space. Several forms of convergence are important in pure mathematics and other mathematically based fields.

In the case with a unique limit of any convergent sequence in a topological space, there is a function, denoted by lim, mapping any convergent sequence to its limit. Motivated by this, many mathematicians have recently been able to define some topological definitions associated with different convergence: A-continuity was studied in [20,21], statistical convergence in topological spaces defined in [12] and G-method presented in [3]. A G-method enables one to define G-closed and G-open subsets. Çakallı examined G-continuity in [5] (we refer to [9] and [6] for more different types of continuities), the study of sequential compactness in the idea of G-method was done in [4]. Similarly, the sequential definition of connectedness was extended and discussed in terms of G-connectedness in [7] (also see [8]). The authors of the paper [18] examined the definition of G-open subsets, G-neighbourhoods, and a few more features of G-continuities (see also [13,19]). Lin and Liu in [10] introduced some different topological properties in G-methods. The idea of the G-method for topological groups with operations was studied and enhanced in terms of G-connectedness in [16], and then the same authors extended the idea to G-compactness in [17]. Some G-convergently separation axioms are recently considered in [15]. We refer

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the readers to [14] for a few G-methods with some counterexamples and [2] for various G-convergence. The concepts are extended to neutrosophic topological spaces by the authors in [1].

For a set X and a method G on it, we define the concept of *G-sequential convergence* of a sequence $\mathbf{a} = (a_n)$ to a point x whenever eventually all terms of the sequence lie in every G-open neighbourhood of x. A subset $A \subseteq X$ is said to be *G-sequentially closed* if, whenever any sequence $\mathbf{a} = (a_n)$ in A G-sequentially converges to x, then $x \in A$. Accordingly, *G-sequentially open* subsets are the complements of G-sequentially closed subsets. Although G-open subsets do not form a topology, in this paper, we show that G-sequentially open subsets lead to a topology which is a G-sequential space.

We acknowledge that the results and examples of this paper will be taken into account in the PhD thesis of the first author, which is under preparation.

2. Preliminaries

In this paper, the boldface letters **a b**,... represent sequences not only in a topological space but also in a set; $s(X)$ represents all sequences in a set X, and $c(X)$ convergent sequences in a topological space X. We now give two types of convergence and preliminaries for a given G-method on X.

2.1. G-convergence

A topological space with a unique limit of any convergent sequence, such as Hausdorff space, defines a function $\lim\colon c(X) \to X$ mapping any convergent sequence to its unique limit. We are motivated by this idea to define and discuss the sequential convergence for a given method G defined on a set $c_G(X)$ of some sequences in X and shortly write (X, G) for such a G-method. A G-method is defined not only for a topological space but also for a set. A sequence $\mathbf{a} = (a_n)$ in the domain of G is said to have *G-convergence* to a point ℓ whenever $G(\mathbf{a}) = \ell$ which is called G-convergent limit. In the case where any convergent sequence $\mathbf{a}$ belongs to the domain $c_G(X)$ and satisfies $G(\mathbf{a}) = \lim \mathbf{a}$, the method G is referred to be *regular*. The regularity is restated not only for topological spaces but also for sets and called *pointwise method* in the sense that for any $x \in X$, the constant sequence $\mathbf{x} = (x, x, x, \cdots)$ is G-convergent to x [11, p.279].

It is called *subsequential* if for a G-convergent sequence $\mathbf{a}$ to ℓ there exists a subsequence $\mathbf{b}$ such that $\lim \mathbf{b} = \ell$ [3]. The set of G-convergent limits of all sequences in $c_G(A) = s(A) \cap c_G(X)$ is said to be G-hull of a subset A and denoted by $[A]^G$ [10]; and A is considered as *G-closed* when it includes G-hull $[A]^G$. Hence if there is a sequence $\mathbf{a}$ in $c_G(A)$ with $G(\mathbf{a}) = x \notin A$, then A is not considered to be G-closed.

The complements of G-closed subsets are defined to be *G-open*, and the openness is shared by X and $\emptyset$. Ultimately, the union of G-open subsets keeps G-openness, but this is not necessarily true for the intersection. The *G-interior* of A, denoted by A^{0G}, is defined as the largest G-open subset of A. Obviously $A^{0G} \subseteq A$, and if A is G-open, then $A \subseteq A^{0G}$. Thus, A is G-open when $A^{0G} = A$.

2.2. G-sequential convergence

It is known that whenever every open neighbourhood of a point $x \in X$ includes eventually all terms, the sequence $\mathbf{a} = (a_n)$ converges to x. Analogously, in the context of the G-method, we say that the sequence $\mathbf{a}$ is G-sequentially convergent to a point $x \in X$ and write $G_s(\mathbf{a}) = x$ if eventually all terms lie in every G-open subset containing x. The point x is the G-sequentially convergent limit of $\mathbf{a}$. It is clear that if any sequence is G-sequentially convergent to a point x, then any subsequence is also G- sequentially convergent to x. If there exists a sequence $\mathbf{a} = (a_n)$ in A that G-sequentially converges to x, then we say that x belongs to *G-sequentially hull* of A and write $x \in [A]_G$. If $[A]_G \subseteq A$, then the set A is called *G-sequentially closed.* Since the constant sequence $\mathbf{a} = (a, a, \dots)$ is G-sequentially convergent to x, one has $A \subseteq [A]_G$ and therefore A is G-sequentially closed iff $[A]_G = A$.

We note that in a first countable space X, for a point $a \in X$, we have $a \in A^0$ if and only if any sequence $\mathbf{a} = (a_n)$ converging to a is almost in A. In the case of G-method, we define $(A^0)_G$ as the collection of points a's such that any sequence $\mathbf{a} = (a_n)$, which is G-sequentially convergent to a, is eventually included in A. We say that A is *G-sequentially open* if $A \subseteq (A^0)_G$. The G-sequential convergence of the sequence $\mathbf{a} = (a, a, \dots)$ to a implies that $(A^0)_G \subseteq A$. Thus, G-openness of A is equivalent to $(A^0)_G = A$, and its complement is characterized by the existence of an element $a \in A$ and a sequence $\mathbf{a} = (a_n)$ which is G-sequentially convergent to a but is not eventually contained in A. Clearly, it can be stated that the sets X and $\emptyset$ are G-sequentially open.

Theorem 2.1. *Let X be a set with a method G and A a subset of X. Then the following are equivalent:*

(1) *The subset A is G-sequentially open.*
(2) *The subset $X \setminus A$ is G-sequentially closed.*

Proof. (1) $\Rightarrow$ (2): Assuming $A \subseteq A^0_G$ to prove $[X \setminus A]_G \subseteq X \setminus A$, choose $x \in [X \setminus A]_G$ which implies the existence of a sequence $\mathbf{x} = (x_n)$ in $X \setminus A$ with G-sequential convergence to x. Then $x \in X \setminus A$, otherwise if $x \in A$, the sequence $\mathbf{x}$ is eventually in A because of $A \subseteq A^0_G$ but it is given that $\mathbf{x} = (x_n)$ belongs to the subset $X \setminus A$, which is a contradiction.

(2) $\Rightarrow$ (1): Assume $[X \setminus A]_G \subseteq X \setminus A$, and prove that $A \subseteq A^0_G$. Let $a \in A$ and $\mathbf{x} = (x_n)$ be a sequence G-sequentially converging to a. We claim that the sequence $\mathbf{x}$ is eventually included in A. Otherwise $\mathbf{x}$ has a subsequence $\mathbf{y}$ with the terms in $X \setminus A$, which is also G-sequentially convergent to a and since $X \setminus A$ is G-sequentially closed $a \in X \setminus A$ which contradicts with $a \in A$. $\square$

Remark 2.2. If A is G-open, then any sequence $\mathbf{a} = (a_n)$ that G-sequentially converges to $a \in A$ is eventually in A and therefore A is G-sequentially open. A G-closed subset is G-sequentially closed. However, G-sequentially open subsets are not necessarily G-open. For example, in Example 3.1, all cofinite subsets are G-sequentially open but not G-open.

3. Counterexamples of G-methods

The following counterexamples indicate the differences between G-convergence and G-sequential convergence and some related topological concepts.

Example 3.1. Consider a method G defined by $G(\mathbf{x}) = \lim \dfrac{x_n + x_{n+1}}{2}$ on $\mathbb{R}$. Then the subsets $\mathbb{R}$, $\emptyset$, $(-\infty, a]$, $[a, b]$, $\{a\}$, $[b, \infty)$ are G-closed and their complements are G-open subsets.

If a subset B is finite, then any sequence in B is in the form $(a_n) = (.., a_{n_0}, \dots, a_{n_0}, ...)$ which is not eventually in the G-open neighbourhood $\mathbb{R} \setminus \{a_{n_0}\}$ of a point x when $x \notin B$. Thus, any point $x \in B^c$ is not in $[B]_G$ and therefore $[B]_G = B$. Hence, finite subsets are G-sequentially closed, and co-finite subsets are G-sequentially open.

This example indicates that G-hull and G-sequentially hull of a subset are different. For instance, if $A = \{0, 1\}$, then G-hull of A is $[A]^G = \{0, \frac{1}{2}, 1\}$ but the G-sequentially hull $[A]_G = \{0, 1\}$.

Example 3.2. Let a method G on a set X be defined by $G(\mathbf{x}) = x_1$ for any sequence $\mathbf{x} = (x_n)$. For a subset A, $x \in [A]^G$ means the existence of a sequence $\mathbf{a} = (a_n)$ in A with $G(\mathbf{a}) = a_1 = x \in A$, which implies that $[A]^G \subseteq A$ and therefore A is G-closed. Hence, all subsets are G-closed and therefore G-open. By Remark 2.2, all subsets are also G-sequentially closed and G-sequentially open.

Example 3.3. For a set X and a constant $x_0 \in X$, let a method G be defined by $G(\mathbf{x}) = x_0$ for any sequence $\mathbf{x} = (x_n)$. Non-empty G-closed subsets include x_0 and G-open subsets are X and those not including x_0. The G-hull $[A]^G$ of a non-empty subset A is the singleton $\{x_0\}$.

The G-sequential hull of A is

$$[A]_G = \begin{cases} A, & x_0 \in A \\ A \cup \{x_0\}, & x_0 \notin A \end{cases}$$

The proof is as follows:

If $x_0 \in A$, then it is G-closed and thefore G-sequentially closed which implies that $[A]_G = A$.

Let $x_0 \notin A$ and $x \in [A]_G$. Then there exists a sequence $\mathbf{a} = (a_n)$ in A which G-sequentially converges to x. If $x \neq x_0$, then $\{x\}$ is a G-open neighbourhood of x and the terms of the sequence $\mathbf{a}$ are almost x that means $x \in A$. If $x = x_0$, then G-open neighbourhood of x is only X and so $x \in [A]_G$. Hence $[A]_G = A \cup \{x_0\}$. The G-sequentially closed subsets are empty set and those containing x_0; therefore, the G-sequentially open subsets are X and those not including x_0.

Example 3.4. Let G-method be defined by $G(\mathbf{x}) = \lim x_n x_{n+1}$ for various sequences $\mathbf{x} = (x_n)$ in $\mathbb{R}$. Then the G-closed subsets are $\{0\}$, $\{1\}$, $\{0,1\}$, $[0,1]$, $[a,\infty)$, $(a > 1)$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\emptyset$; therefore G-open subsets are the complements of them.

4. (G,H)-continuous functions

To compare and highlight the differences between (G,H)-continuity and (G,H)-sequentially continuity, we briefly summarize the concept of (G,H)-continuity between two methods (X,G) and (Y,H) as follows:

Definition 4.1. A function $f\colon (X,G) \to (Y,H)$ between any two methods is (G,H)-*continuous* if the G-convergence of any sequence $\mathbf{x} = (x_n)$ in X to x implies the H-convergence of the sequence $\mathbf{y} = (f(x_n))$ to $f(x)$.

The proofs of the following theorems can be performed as similar to [18, Theorem 2.23] and [18, Theorem 2.24].

Theorem 4.2. *For a (G,H)-continuous map $f\colon (X,G) \to (Y,H)$, the H-closedness of a subset in Y implies G-closedness of the inverse image in X.*

Theorem 4.3. *For a (G,H)-continuous map $f\colon (X,G) \to (Y,H)$, H-openness of a subset in Y implies G-openness of the inverse image in X.*

The following counterexample shows that the opposite sides of Theorem 4.2 and Theorem 4.3 do not hold.

Example 4.4. For the method (X,G) in Example 3.2, any subset of X is G-closed and G-open. If the method H on Y is defined by $H(\mathbf{y}) = y_0$ for a constant $y_0 \in Y$ as in Example 3.3, then for any function $f\colon (X,G) \to (Y,H)$ the inverse image of a H-closed (resp. H-open) subset of Y is G-closed (resp. G-open) in X. However any sequence $\mathbf{x} = (x_n)$ in X is G-convergent to x_1 and the sequence $\mathbf{y} = (f(x_n))$ is H-convergent to y_0 and therefore f is not (G,H)-continuous when $f(x_1) \neq y_0$.

5. (G,H)-sequentially continuous functions

We can define (G,H)-sequentially continuity of a function between two methods as follows.

Definition 5.1. Let (X, G) and (Y, H) be two methods, and $f\colon (X, G) \to (Y, H)$ a map between these methods. The map f is (G, H)-*sequentially continuous* if the sequence $\mathbf{b} = (f(a_n))$ is H-sequentially convergent to $f(x)$ whenever the sequence (a_n) is G-sequentially convergent to x.

Theorem 5.2. *The following are equivalent for a function $f\colon (X, G) \to (Y, H)$.*

(1) *(G, H)-sequentially continuity of f*
(2) *H-sequentially openness of $V \subseteq H$ implies G-sequentially openness of $f^{-1}(V)$.*

Proof. (1) ⇒ (2): Suppose f is (G, H)-sequentially continuous and V is an H-sequentially open subset of Y. We will prove that $f^{-1}(V)$ is G-sequentially open. For $x \in f^{-1}(V)$ consider a sequence (a_n) that G-sequentially converges to the point x. Then $f(x)$ is in V and as f is (G, H)-sequentially continuous, the sequence $(f(a_n))$ is H-sequentially convergent to $f(x)$. Since V is H-sequentially open, $(f(a_n))$ is almost in V, and therefore (a_n) is almost in the inverse image $f^{-1}(V)$ of V. Thus, $f^{-1}(V)$ is G-sequentially open.

(2) ⇒ (1): Assuming that the inverse images of H-sequentially open subsets in Y are G-sequentially open in X, we aim to prove the (G, H)-sequentially continuity of f. Let $\mathbf{a} = (a_n)$ be a sequence in X that G-sequentially converges to $x \in X$. We need to show that the sequence $(f(a_n))$ is H-sequentially convergent to $f(x)$. Let V be an H-open neighbourhood of $f(x)$. By assumption $f^{-1}(V)$ is G-sequentially open and $x \in f^{-1}(V)$. Therefore the sequence (a_n) is almost in $f^{-1}(V)$. Hence, the sequence $(f(a_n))$ eventually lies in V and H-sequentially converges to $f(x)$. □

Theorem 5.3. *We have the equivalence below for a function $f\colon (X, G) \to (Y, H)$.*

(1) *f is (G, H)-sequentially continuous.*
(2) *The inverse image of any H-sequentially closed subset $F \subseteq Y$ is G-sequentially closed.*

Proof. (1) ⇒ (2): Consider f to be a (G, H)-sequentially continuous, and an H-sequentially closed subset $F \subseteq Y$. Then by Theorem 2.1, $Y \setminus F$ is H-sequentially open and by Theorem 5.2, $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is G-sequentially open. Hence $f^{-1}(F)$ is G-sequentially open subset of X.

(2) ⇒ (1): Let the inverse images of H-sequentially closed subsets be G-sequentially closed. For the proof of (G, H)-sequentially continuity of f, we use Theorem 5.2. The H-sequentially openness of a subset $U \subseteq Y$ implies H-sequentially closedness of $Y \setminus U$. By assumption $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ is G-sequentially closed and so $f^{-1}(U)$ is G-sequentially open. Hence, by Theorem 5.2, this shows that the map f is G-sequentially continuous. □

Proposition 5.4. *Let assume that $f\colon (X, G) \to (Y, H)$ is (G, H)-sequentially continuous. Then (H, K)-sequentially continuity of $g\colon (Y, H) \to (Z, K)$ implies that $gf\colon (X, G) \to (Z, K)$ is (G, K)-sequentially continuous.*

Proof. Suppose a sequence $\mathbf{a} = (a_n)$ in X is G-sequentially convergent to x. Then by (G, H)-sequentially continuity of f, the sequence $\mathbf{b} = (f(a_n))$ is H-sequentially convergent to $f(x)$ and by the (H, K)-sequentially continuity of g, the sequence $(g(f(a_n)))$ is K-sequentially convergent to $g(f(x))$, which completes the proof. □

If G is a method on a set X with domain $c_G(X)$ and A is a subset of X, then there is an induced method G_A on A with domain

$$c_G(A) = \{\mathbf{a} \in c_G(X) \cap s(A) \colon G(\mathbf{a}) \in A\}$$

Proposition 5.5. *For a subset $A \subseteq X$, the (G, H)-sequentially continuity of $f\colon (X, G) \to (Y, H)$ implies (G_A, H)-sequentially continuity of the restriction map $f_A\colon A \to Y$.*

Proof. The G_A-sequential convergence of a sequence $\mathbf{a} = (a_n)$ in $c_G(A)$ to $a \in A$ implies G-sequential convergence to a. By the (G,H)-sequential continuity of $f\colon (X,G) \to (Y,H)$ it follows that the sequence $\mathbf{b} = (f(a_n))$ is H-sequentially convergent to $f(a) = f_A(a)$. Therefore $f_A\colon (A,G_A) \to (Y,H)$ is (G_A,H)-sequentially continuous. □

A well-known conclusion is that a continuous function is sequentially continuous in classical topology, but the converse is not true. The following theorem proves this result for (G,H)-continuous functions.

Theorem 5.6. *The (G,H)-continuity of $f\colon (X,G) \to (Y,H)$ implies the (G,H)-sequentially continuity.*

Proof. Assuming the (G,H)-continuity of f, consider a sequence (a_n) in X which is G-sequentially convergent to x and prove that the sequence $(f(a_n))$ is H-sequentially convergent to $f(x)$. Suppose $V \subseteq Y$ is an H-open subset and contains $f(x)$. By the (G,H)-continuity of f, Theorem 4.3 implies that $f^{-1}(V)$ becomes G-open including x. Thus the sequence (a_n) is almost in $f^{-1}(V)$ which implies $(f(a_n))$ is almost in V. Hence, $(f(a_n))$ is H-sequentially convergent to $f(x)$, which completes the (G,H)-sequentially continuity of f. □

As we can see in the following counterexample, the converse of Theorem 5.6 is not valid.

Example 5.7. Consider the methods (X,G) and (Y,H); and a map $f\colon (X,G) \to (Y,H)$ as in Example 4.4. The reason that all subsets of X are G-closed and G-open implies that any subset is both G-sequentially closed and open. Then, by Theorem 5.2, any map $f\colon (X,G) \to (Y,H)$ becomes G-sequentially continuous. However, the function f is not (G,H)-continuous. For any sequence $\mathbf{x} = (x_n)$, we have $G(\mathbf{x}) = x_1$ and $H(\mathbf{y}) = y_0$ for $\mathbf{y} = (f(a_n))$; and therefore, $f(G(\mathbf{x})) \neq H(\mathbf{y})$ when $f(x_1) \neq y_0$.

Theorem 5.8. *(G,H)-sequentially continuity of a function $f\colon (X,G) \to (Y,H)$ is equivalent to that $B \subseteq X$ implies $f([B]_G) \subseteq [f(B)]_H$.*

Proof. The proof of the necessity is a consequence of that, if $x \in [B]_G$ is a G-sequentially convergent limit of a sequence $\mathbf{a} = (a_n)$ in $c_G(X) \cap s(B)$, then by (G,H)-continuity of f, the sequence $\mathbf{b} = (f(x_n))$ becomes H-sequentially convergent to $f(x)$.

For the proof of sufficiency, let $F \subseteq Y$ be an H-sequentially closed subset and (a_n) a sequence in $f^{-1}(F) = B$ which is G-sequentially convergent to x, meaning that $x \in [B]_G$. The assumption

$$f(x) \in f([B]_G) \subseteq [f(B)]_H \subseteq [F]_H \subseteq F$$

implies that $f(x) \in F$ and $x \in f^{-1}(F)$. Thus $f^{-1}(F)$ is G-sequentially closed. Hence by Theorem 5.3, f is (G,H)-sequentially continuous. □

Theorem 5.9. *The assumption $f(A)^0_H \subseteq f({A^0}_G)$ for any subset A implies that the function $f\colon (X,G) \to (Y,H)$ is (G,H)-sequentially open.*

Proof. Consider $f\colon (X,G) \to (Y,H)$ with $f(A)^0_H \subseteq f(A^0_G)$ for any subset A. If A is G-sequentially open, it implies that $A^0_G = A$, and therefore, by assumption, we have $f(A)^0_H \subseteq f(A)$. Hence, $f(A)$ is H-sequentially open; thus, the function f is G-sequentially open. □

6. G-sequential spaces

In a topological space X, a subset A is *sequentially open* if any sequence $\mathbf{x} = (x_n)$ converging to $a \in A$ is eventually in A. An open subset is sequentially open, but the converse is not true. For instance, when X is an uncountable set endowed with co-countable topology, any subset is sequentially open but not necessarily open. A topological space in which any sequentially open subset is open is referred to be as *sequential.* The set of sequentially open subsets in a topological space X acquires a finer topology than the initial one. In this section, we extend the definition of sequential space to G-methods and define G-sequential space.

For a G-method on X, the intersection of G-open subsets does not preserve the G-openness. For example in Example 3.1, the subsets $\mathbb{R} \setminus \{a\}$ and $\mathbb{R} \setminus \{b\}$ are G-open however $\mathbb{R} \setminus \{a\} \cap \mathbb{R} \setminus \{b\} = \mathbb{R} \setminus \{a, b\}$ is not G-open. Hence, G-open subsets do not form a topology. As we prove below, unlike G-open subsets, G-sequentially open subsets give a topology in which G-sequentially open subsets are open.

Theorem 6.1. *The following properties hold for a method G on a set X and a class of subsets $\{A_i : i \in I\}$ in X.*

(1) $(\bigcap_{i=1}^{n} A_i)_G^0 = \bigcap_{i=1}^{n}(A_i)_G^0$.
(2) $\bigcup_{i\in I}(A_i)_G^0 \subseteq (\bigcup_{i\in I} A_i)_G^0$.

Proof. (1) Assume that $a \in (\bigcap_{i=1}^{n} A_i)_G^0$ and $\mathbf{x} = (x_n)$ is a sequence which is G-sequentially convergent to a. It implies that the sequence $\mathbf{x}$ is almost in $\bigcap_{i=I}^{n} A_i$. Hence $\mathbf{x}$ is almost in A_i for $1 \leq i \leq n$ and therefore $a \in (A_i)_G^0$, which means that $a \in \bigcap_{i=I}^{n}(A_i)_G^0$.

If $a \in \bigcap_{i=1}^{n}(A_i)_G^0$, and $\mathbf{x} = (x_n)$ is G-sequentially convergent to a, then by $a \in (A_i)_G^0$, the sequence $\mathbf{x}$ is almost in A_i for $1 \leq i \leq n$, which implies that $\mathbf{x}$ is almost included in $\bigcap_{i=I}^{n} A_i$. Hence $a \in (\bigcap_{i=I}^{n} A_i)_G^0$.

(2) If $a \in \bigcup_{i\in I}(A_i)_G^0$ and $\mathbf{x} = (x_n)$ is G-sequentially convergent to a, then by $a \in (A_{i_0})_G^0$ for an $i_0 \in I$, $\mathbf{x}$ is almost in A_{i_0}, and therefore almost in $\bigcup_{i\in I} A_i$. Hence $x \in (\bigcup_{i\in I} A_i)_G^0$. □

Corollary 6.2. *For a method G on a set X, G-sequentially open subsets constitute a topology on X.*

Proof. Consider a class $\{A_i : i \in I\}$ of G-sequentially open subsets in X. Since $A_i \subseteq (A_i)_G^0$ for every $i \in I$, by Theorem 6.1 we have

$$\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{n}(A_i)_G^0 = (\bigcap_{i=1}^{n} A_i)_G^0$$

that means $\bigcap_{i\in I}^{n} A_i$ is G-sequentially open. Moreover by

$$\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I}(A_i)_G^0 \subseteq (\bigcup_{i\in I} A_i)_G^0$$

we have $\bigcup_{i\in I} A_i \subseteq (\bigcup_{i\in I} A_i)_G^0$ and therefore $\bigcup_{i\in I} A_i$ is G-sequentially open.

Immediately, X is G-sequentially open. The empty subset $\emptyset$ is G-sequentially open. Otherwise, $\mathbf{x} = (x_n)$ would be an inconsistent sequence that G-sequentially converges to $a \in \emptyset$ which is a contradiction. □

As an implication, we can phrase that finite union and arbitrary intersection of G-sequentially closed subsets are G-sequentially closed. Note that arbitrary unions of G-sequentially closed subsets are not necessarily G-sequentially closed. For instance, in Example 3.1 for each $a \in A = [0, 1)$, the subset $\{a\}$ is G-sequentially closed, but $\bigcup_{a\in A}\{a\} = A$ is not G-sequentially closed since $[A]_G = [0, 2]$.

For a G-method on a topological space X, the topology on X associated with G-method as in Corollary 6.2 is denoted by τ_G whose open subsets are G-sequentially open. Motivated by this, we will use the following definition:

Definition 6.3. *Let G be a method on a topological space X. In the case where any G-sequentially open subset X is open, we say the given topology on X is* G-sequential*.*

Specifically, the definition of G-sequential space agrees with sequential space when X is a topological space in which any convergent sequence has a unique limit, and the method G is lim.

Theorem 6.4. *For a method (X, G), the topology τ_G associated with G is G-sequential.*

Proof. The proof is straightforward from Definition 6.3 and Corollary 6.2. □

We can formulate the following corollary from Theorem 5.2.

Corollary 6.5. *For the methods (X, G) and (Y, H), the (G, H)-sequentially continuity of a function $f\colon (X, G) \to (Y, H)$ is equivalent to the continuity of $f\colon (X, \tau_G) \to (Y, \tau_H)$.*

In the following example, we obtain G-sequentially spaces associated with different G-methods.

We note that in some G-methods, such as Examples 3.1 and 3.4, the characterization of G-sequentially open subsets is unclear. However, the subsets generated by G-open subsets as the union and intersections of G-open subsets are G-sequentially open.

Example 6.6.
(1) For the method G of Example 3.1, the subsets of $\mathbb{R}$ generated by G-open subsets are G-sequentially open and therefore open subsets of G-sequential topological space.
(2) Since for the G-method in Example 3.2, all subsets are G-sequentially open, the G-sequentially topology τ_G associated with the method is discrete topology.
(3) If G-method is given as in Example 3.3, then G-sequentially topology τ_G on X associated with the method G is
$$\tau_G = \{A \subseteq X : x_0 \notin A\} \cup \{X\}$$
(4) According to the G-method in Example 3.4, the subsets of $\mathbb{R}$ generated by G-open subsets that are G-sequentially open; therefore open subsets of G-sequential topological space.

7. Conclusion

By examining G-convergence and G-sequentially convergence, we gain a deeper understanding of relationships between them. As illustrated in particular examples and results, some topological properties behave differently under these two types of convergence. This paper shows that the class of all G-open subsets does not form a topology. However, the collection of G-sequentially open subsets form a topology in which G-sequentially open subsets are open. This observation motivated us to define G-sequential spaces and provide some counter-examples.

It would be interesting to compare the topological concepts associated with G-sequential convergence and those arising in G-sequential topology induced by the G-method.

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