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Statistical Convergence in L-Fuzzy Metric Spaces

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Research Article

Abstract — Statistical convergence, defined in terms of the natural density of positive integers, has been studied in many different spaces, such as intuitionistic fuzzy metric spaces, partial metric spaces, and L-fuzzy normed spaces. The main goal of this study is to define statistical convergence in L-fuzzy metric spaces (L-FMSs), one of the essential tools for modeling uncertainty in everyday life. Furthermore, this paper introduces the concept of statistical Cauchy sequences and investigates its relation with statistical convergence. Then, it defines statistically complete L-FMSs and analyzes some of their basic properties. Finally, the paper inquires the need for further research.

Keywords *Statistical convergence, statistical Cauchy sequences, L-fuzzy metric spaces, completeness*

Mathematics Subject Classification (2020) 40A35, 40A05

1. Introduction

Zadeh [1] put forward the concept of fuzzy sets, which have been used in many fields, such as decision-making, artificial intelligence, weather forecasting, and probability theory, and can model problems involving uncertainty. One of the applications of these fields is fuzzy metric spaces, presented by Kramosil and Michalek [2] and Kaleva and Seikkala [3]. George and Veeramani [4] reformulated it with the help of triangular norms since this space is not Hausdorff. Gregori et al. [5] investigated convergence in fuzzy metric spaces. Atanassov [6] introduced the concept of intuitionistic fuzzy sets, a generalization of fuzzy sets. Later, Park [7] defined intuitionistic fuzzy metric spaces using fuzzy metric spaces as derived from George and Veremaani and proved some known results, such as Baire's theorem and the uniform limit theorem in the mentioned space. L-fuzzy metric spaces (L-FMSs) based on specific logical algebraic structures have been characterized by Saadati et al. [8] as a natural generalization of intuitionistic fuzzy metric spaces. Saadati [9] studied L-fuzzy topological spaces and proved that L-FMSs have many properties, such as being a normal, separable, and metrizable space. Many researchers have generalized the classical concepts of topology and functional analysis to fuzzy metric spaces. L-FMSs provided a more general framework for generalizing the classical concepts to a fuzzy setting. Motivated by this fact, in this present study, we propose statistical convergence in L-FMSs.

In 1951, the concept of statistical convergence was introduced by Fast [10] and Steinhaus [11] as a generalization of the classical convergence. This concept is dependent on the theory of natural

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densities [12]. The concept of statistical convergence was further analyzed by the authors, such as Salat [13], Fridy [14], Connor [15], and Mursaalen and Edely [16], along with many fields, such as summability theory [17], operator theory [18], and approximation theory [19]. In 2020, Li et al. [20] investigated the notion of statistical convergence in fuzzy metric spaces, Savaş [21] researched statistical convergence of double sequences, and Varol [22] defined statistical convergence in intuitionistic fuzzy metric spaces. In 2023, Özcan et al. [23, 24] researched statistical convergence of double sequences and λ-statistical convergence in these spaces, respectively.

The remainder of the present paper is organized as follows: In Section 2, we present some basic definitions and properties to be needed in the following section. In Section 3, we analyze statistical convergence in L-FMSs and then study the statistical Cauchy sequences for complete metric spaces. Furthermore, we research the relationship between these notions and obtain some results and findings. Finally, we argue that they are essential for future study in this space.

2. Preliminaries

This section presents some basic notions and properties to be required for the following sections. Throughout this study, the notations $\mathbb{N}$ and $\mathbb{R}$ represent the set of all positive integers and the set of all real numbers, respectively.

Definition 2.1. [11] A sequence (ξ_k) is referred to as statistically convergent to ξ, denoted by $st\text{-}\lim \xi_k = \xi$, if, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |\xi_k - \xi| \geq \varepsilon\}| = 0$$

where the notation $|\ \ |$ represents the cardinality of a set.

Definition 2.2. [12] The natural density of a set $A \subseteq \mathbb{N}$, denoted by $\delta(A)$, is defined by

$$\delta(A) := \lim_{n\to\infty} \frac{|\{k \leq n : k \in A\}|}{n}$$

where the notation $|\ \ |$ represents the cardinality of a set.

Definition 2.3. Let $\mathbb{L}$ be a lattice equipped with a partial order $\preceq_{\mathbb{L}}$. Then, $(\mathbb{L}, \preceq_{\mathbb{L}})$ is referred to as a complete lattice if $\sup S \in \mathbb{L}$ and $\inf S \in \mathbb{L}$, for all subsets S of $\mathbb{L}$.

Across this study, L represents the pair $(\mathbb{L}, \preceq_{\mathbb{L}})$. Moreover, the notations 1_L and 0_L denote $\sup \mathbb{L}$ and $\inf \mathbb{L}$, respectively.

Definition 2.4. [1, 25] Let $\mathbb{L}$ be a complete lattice, E be a non-empty universal set, and $\mu : E \to \mathbb{L}$ be a mapping. Then, the mapping μ is called an $\mathbb{L}$-fuzzy set on E, which for all $e \in E$, $\mu(e)$ specifies the grade of belonging of e to the $\mathbb{L}$-fuzzy set μ.

Lemma 2.5. [27] The partially ordered set $(\mathbb{L}^* \preceq_{\mathbb{L}^*})$ defined by

$$\mathbb{L}^* = \{(\alpha, \beta) : \alpha, \beta \in [0, 1] \text{ and } \alpha + \beta \leq 1\} \quad \text{and} \quad (\alpha, \beta) \preceq_{\mathbb{L}^*} (\gamma, \omega) \Leftrightarrow \alpha \leq \gamma \text{ and } \beta \geq \omega$$

is a complete lattice.

A triangular norm $T : [0, 1]^2 \to [0, 1]$ on the complete lattice $([0, 1], \leq)$ is a function that is commutative, increasing, and associative and satisfies the condition $T(1, \alpha) = \alpha$, for all $\alpha \in [0, 1]$. Using a complete lattice L, this concept have been generalized as follows:

Definition 2.6. [27] Let L be a complete lattice and $\varphi : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ be a function. Then, φ is called a triangular norm (t-norm) on L if the following are satisfied: For all $\alpha, \beta, \gamma, \omega \in \mathbb{L}$,

i. $\varphi(\beta, \alpha) = \varphi(\alpha, \beta)$

ii. $\varphi(\alpha, \varphi(\beta, \gamma)) = \varphi(\varphi(\alpha, \beta), \gamma)$

iii. $\varphi(\alpha, 1_L) = \varphi(1_L, \alpha) = \alpha$

iv. If $\gamma \preceq_{\mathbb{L}} \omega$ and $\alpha \preceq_{\mathbb{L}} \beta$, then $\varphi(\alpha, \gamma) \preceq_{\mathbb{L}} \varphi(\beta, \omega)$

Definition 2.7. [27] Let L be a complete lattice. Then, the function $\mathcal{N} : \mathbb{L} \to \mathbb{L}$ is called a negator on L if it satisfies the following conditions:

i. $\mathcal{N}$ is a decreasing function

ii. $\mathcal{N}(1_L) = 0_L$ and $\mathcal{N}(0_L) = 1_L$

In addition, $\mathcal{N}$ is referred to as an involutive negator if it provides the condition $\mathcal{N}(\mathcal{N}(\alpha)) = \alpha$, for all $\alpha \in \mathbb{L}$.

One of the well-known examples of involutive negators is the function $\mathcal{N} : [0, 1] \to [0, 1]$ defined by $\mathcal{N}(\alpha) = 1 - \alpha$ where $([0, 1], \leq)$ is a complete lattice.

Definition 2.8. [8] Let $\mathbb{X}$ be a non-empty set and φ be a continuous t-norm on L. An L-fuzzy metric is a mapping $\mu : \mathbb{X}^2 \times (0, \infty) \to \mathbb{L}$ satisfying the following conditions: For all $t, s > 0$ and for all $\alpha, \beta, \gamma \in \mathbb{X}$,

i. $\mu(\alpha, \beta, t) \succ_{\mathbb{L}} 0_L$

ii. $\mu(\alpha, \beta, t) = 1_L$ if and only if $\alpha = \beta$

iii. $\mu(\alpha, \beta, t) = \mu(\beta, \alpha, t)$

iv. $\varphi(\mu(\alpha, \beta, t), \mu(\beta, \gamma, s)) \preceq_{\mathbb{L}} \mu(\alpha, \gamma, t + s)$

v. $\mu_{\alpha\beta} : (0, \infty) \to \mathbb{L}$ is continuous

Moreover, the triple $(\mathbb{X}, \mu, \varphi)$ is called an L-fuzzy metric space (L-FMS).

Definition 2.9. [8] Let $\mathcal{N}$ be a negator on L, the triple $(\mathbb{X}, \mu, \varphi)$ be an L-FMS, $\alpha \in \mathbb{X}$, $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$, and $t > 0$. Then, the set

$$B(\alpha, \mathcal{N}(\varepsilon), t) := \{\beta \in \mathbb{X} : \mu(\alpha, \beta, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

is called the open ball with centre α and radius $\mathcal{N}(\varepsilon)$.

Example 2.10. [8] Let $(\mathbb{X}, d)$ be a metric space, μ be an $\mathbb{L}$-fuzzy set on $\mathbb{X}^2 \times (0, \infty)$ defined by

$$\mu(\xi, \eta, t) = \frac{ht^n}{ht^n + md(\xi, \eta)}$$

where $h, n, m > 0$, and φ be a continuous t-norm described by $\varphi(\alpha, \beta) = \alpha\beta$, for all $\alpha, \beta \in \mathbb{L}$. Then, the function μ satisfies the conditions in Definition 2.8. Thus, $(\mathbb{X}, \mu, \varphi)$ is an L-FMS.

Definition 2.11. [8] Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$.

i. (ξ_k) is called convergence to $\xi \in \mathbb{X}$, denoted by $\xi_n \xrightarrow{L} \xi$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k \geq k_\varepsilon$, $\mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

ii. (ξ_k) is called a Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists a $k_\varepsilon \in \mathbb{N}$ such that for all $k, n \geq k_\varepsilon$, $\mu(\xi_k, \xi_n, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

iii. $(\mathbb{X}, \mu, \varphi)$ is complete if and only if every Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$ is convergent.

Note 2.12. Let φ be a continuous t-norm on L and $\mathcal{N}$ be an involutive negator on L. Then, for all $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\alpha \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\alpha), \mathcal{N}(\alpha)) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$.

3. Statistical Convergence in L-Fuzzy Metric Spaces

This section analyzes relations between statistical convergence and classical convergence in L-FMSs. In addition, it presents a characterization of statistical convergence with subsequences.

Definition 3.1. Let (ξ_k) be sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, (ξ_k) is referred to as statistical convergent to $\xi \in \mathbb{X}$, denoted by $\xi_k \xrightarrow{st_L} \xi$, if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$,

$$\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$$

Here, the element ξ is called a statistical limit point of the sequence (ξ_k).

Example 3.2. Let μ be an $\mathbb{L}$-fuzzy set on $\mathbb{R}^2 \times (0, \infty)$ and φ be a continuous t-norm on L defined by

$$\mu(\xi, \eta, t) = \frac{t}{t + |\xi - \eta|} \quad \text{and} \quad \varphi(\alpha, \beta) = \alpha\beta$$

Then, $(\mathbb{R}, \mu, \varphi)$ is L-FMS. Consider the sequence (ξ_k) defined by

$$\xi_k = \begin{cases} 5, & \exists n \in \mathbb{N} \ni k = n^2 \\ 0, & \text{otherwise} \end{cases}$$

and the set

$$A = \{k \in \mathbb{N} : \mu(\xi_k, 0, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$. Then, (ξ_k) is statistical convergent to 0 as $\delta(A) = 0$, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$.

Lemma 3.3. Let $(\mathbb{X}, \mu, \varphi)$ be an L-FMS. Then, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, the following conditions are equivalent:

i. $\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$

ii. $\delta\left(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 1$

The proof can be readily observed from Definition 3.1 and density properties.

Theorem 3.4. Let (ξ_k) be a sequence in a L-FMS $(\mathbb{X}, \mu, \varphi)$. If (ξ_k) is statistical convergent, then its statistical limit point is unique.

Proof. Let $\xi_k \xrightarrow{st_L} \ell_1$ and $\xi_k \xrightarrow{st_L} \ell_2$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi\left(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)\right) \succ_{\mathbb{L}} \mathcal{N}(r)$. Assume that

$$\ell_3 \in B\left(\ell_1, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \cap B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right)$$

Then,

$$\begin{aligned} \mu(\ell_1, \ell_2, t) &\succeq_{\mathbb{L}} \varphi\left(\mu\left(\ell_1, \ell_3, \tfrac{t}{2}\right), \mu\left(\ell_2, \ell_3, \tfrac{t}{2}\right)\right) \\ &\succeq_{\mathbb{L}} \varphi\left(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)\right) \\ &\succ_{\mathbb{L}} \mathcal{N}(r) \end{aligned}$$

which is a contradiction. Thus,

$$B\left(\ell_1, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \cap B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right) = \emptyset$$

Hence,

$$B\left(\ell_2, \mathcal{N}(\varepsilon), \frac{t}{2}\right) \subseteq \left\{\ell_3 \in \mathbb{X} : \mu\left(\ell_1, \ell_3, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}$$

Then,

$$\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_2, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\} \subseteq \left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_1, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}$$

Since

$$1 = \delta\left(\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_2, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}\right) \leq \delta\left(\left\{k \in \mathbb{N} : \mu\left(\xi_k, \ell_1, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\right\}\right) = 0$$

a contradiction occurs. Consequently, $\ell_1 = \ell_2$. □

Theorem 3.5. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$ and $\xi \in \mathbb{X}$. If (ξ_k) convergent to ξ, then (ξ_k) is statistically convergent to ξ.

Proof. Let (ξ_k) is convergent to $\xi \in \mathbb{X}$. Then, there exists a $k_\varepsilon \in \mathbb{N}$ such that $\mu(\xi_k, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k > k_\varepsilon$, $\varepsilon \in \mathbb{L} - \{0_L\}$, and $t > 0$. Therefore, there are just a finite number of terms in the set

$$A = \{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

Since the property of natural density "finite subsets of natural numbers has density zero", $\delta(A) = 0$. Therefore, (ξ_k) is statistical convergent to ξ. □

The converse of the Theorem 3.5 is not always true (see Example 3.6).

Example 3.6. Consider the L-FMS in Example 3.2 and the sequence (ξ_k) defined as follows:

$$\xi_k = \begin{cases} 9, & \exists n \in \mathbb{N} \ni k = n^2 \\ 6, & \text{otherwise} \end{cases}$$

It can be observed that (ξ_k) is not convergent to 6 but statistical convergent to 6 because

$$\delta(\{k \in \mathbb{N} : \mu(\xi_k, 6, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 0$$

Theorem 3.7. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, $\xi_k \xrightarrow{st_L} \xi$ if and only if $\xi_{k_j} \xrightarrow{L} \xi$ such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$.

Proof. Assume that $\xi_k \xrightarrow{st_L} \xi$. Let

$$S_j(q) = \begin{cases} \mathbb{N}, & j = 1 \\ \{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}, & j \geq 2 \end{cases}$$

for all $q > 0$ and $j \in \mathbb{N}$. Thus, $S_{j+1}(q) \subset S_j(q)$, for all $q > 0$ and $j \in \mathbb{N}$. Since (ξ_k) is statistical convergent to ξ, then

$$\delta(S_j(q)) = 1 \tag{3.1}$$

Let $t_1 \in S_1(q)$. Since $\delta(S_2(q)) = 1$, then there is a number $t_2 \in S_2(q)$ such that $t_2 > t_1$ and

$$\frac{1}{n}|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_2)\}| > \frac{1}{2}$$

for all $n \geq t_2$. By (3.1), $\delta(S_3(q)) = 1$. Thus, there exists a $t_3 \in S_3(q)$ such that $t_3 > t_2$ and

$$\frac{1}{n}|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_3)\}| > \frac{2}{3}$$

for all $n \geq t_3$ and the procedure is continued similarly. Then, by induction, we can construct a sequence of increasing indexes of positive integers (t_j) such that $t_j \in S_j(q)$. Besides,

$$\frac{1}{n}|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}| > \frac{j-1}{j} \tag{3.2}$$

for all $n \geq t_j$ and $j \in \mathbb{N}$. Moreover, (w_j) is a decreasing sequence in $\mathbb{L} - \{0_L\}$ such that $\mathcal{N}(w_j) \to 1_L$. Suppose that

$$S := \{k \leq n : 1 < k < t_1\} \cup \left[\bigcup_{j \in \mathbb{N}} \{k \in S_j(q) : t_j \leq k < t_{j+1}\} \right]$$

Since $S_{j+1}(q) \subset S_j(q)$ and due to (3.2), $S = \{k_j : j \in \mathbb{N}\}$. Let $k > t_2$. Then, there exists a $j \in \mathbb{N}$ such that $t_j \leq k < t_{j+1}$. Hence,

$$\begin{aligned} \frac{|\{k \leq n : k \in S\}|}{n} &\geq \frac{|\{k \leq n : k \in S_j(q)\}|}{n} \\ &= \frac{|\{k \leq n : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(w_j)\}|}{n} \\ &> \frac{j-1}{j} \end{aligned}$$

for all $n \geq t_j$. As $n, j \to \infty$,

$$\lim_{n \to \infty} \frac{|\{k \leq n : k \in S\}|}{n} = 1$$

i.e., $\delta(S) = 1$. Let $w \in L - \{0_L\}$ and $j \in \mathbb{N}$ such that $w \succ_{\mathbb{L}} w_j$. Such a number j always exists since $w_j \to 0_{\mathbb{L}}$. Let $k \geq t_j$ and $k \in S$. Then, according to the definition of S, a number $t \geq j$ exists such that $t_j \leq k_m < t_{j+1}$ and $k_m \in S_j(q)$. Thus, for all $w \in L - \{0_L\}$ and for all $k_m \geq t_j$ such that $k_m \in S_j(q)$,

$$\mu(\xi_{k_m}, \xi, t) \succ_{\mathbb{L}} \mathcal{N}(w_j) \succ_{\mathbb{L}} \mathcal{N}(w)$$

Consequently, the sequence (ξ_{k_j}) is convergent to ξ in the L-FMS.

Conversely, suppose that the subsequence (ξ_{k_j}) is convergent to ξ in the L-FMS such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $\mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k_j \geq n_0$. Therefore, $T = \{k_j \in A : \mu(\xi_{k_j}, \xi, q) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$ is a finite set, which implies that $\delta(T) = 0$. Since $\delta(A) = 1$, $\delta(\{k_j \in A : \mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 1$. Because

$$\{k_j \in A : \mu(\xi_{k_j}, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\} \subseteq \{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}$$

then

$$\delta(\{k \in \mathbb{N} : \mu(\xi_k, \xi, q) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 1$$

Consequently, the sequence (ξ_k) is statistical convergent to ξ in the L-FMS. □

4. Completeness in L-Fuzzy Metric Spaces

This section defines statistical Cauchy sequences in an L-FMS and complete L-FMSs by them. Then, it provides the crucial relations.

Definition 4.1. Let (ξ_k) be sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, (ξ_k) is referred to as a statistically Cauchy sequence if, for all $\varepsilon \in \mathbb{L} - \{0_L\}$ and $t > 0$, there exists an $n \in \mathbb{N}$ such that

$$\delta\left(\{k \in \mathbb{N} : \mu\left(\xi_k, \xi_n, t\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}\right) = 0$$

Proposition 4.2. Every Cauchy sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$ is a statistical Cauchy sequence. However, the converse is not always true.

The proof is similar to the proof of Theorem 3.5.

Theorem 4.3. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X}, \mu, \varphi)$. Then, the following conditions are equivalent:

i. (ξ_k) is a statistical Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$

ii. $\left(\xi_{k_j}\right)$ is a Cauchy sequence in $(\mathbb{X}, \mu, \varphi)$ such that $\delta(A) = 1$ where $A = \{k_j : j \in \mathbb{N}\}$

The proof is similar to the proof of Theorem 3.7.

Theorem 4.4. Let (ξ_k) be a sequence in an L-FMS $(\mathbb{X},\mu,\varphi)$. If (ξ_k) is statistically convergent, then (ξ_k) is a statistically Cauchy sequence in $(\mathbb{X},\mu,\varphi)$.

PROOF. Let $\xi_k \xrightarrow{st_L} \xi$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$. Since $\delta(\{k \in \mathbb{N} : \mu(\xi_k, \xi, t) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)\}) = 0$, $\left(\xi_{k_j}\right)$ is convergent to ξ from Theorem 3.7. Hence, there exists a $k_{j_0} \in \{k_j : j \in \mathbb{N}\}$ such that $\mu\left(\xi_{k_j}, \xi, \frac{t}{2}\right) \not\succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$, for all $k_j \geq k_{j_0}$. Then,

$$\mu\left(\xi_k, \xi_{k_{j_0}}, t\right) \succeq_{\mathbb{L}} \varphi\left(\mu\left(\xi_k, \xi, \frac{t}{2}\right), \mu\left(\xi, \xi_{k_{j_0}}, \frac{t}{2}\right)\right) \succeq_{\mathbb{L}} \varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$$

Thus, $\delta(\{k \in \mathbb{N} : \mu\left(\xi_k, \xi_{k_{j_0}}, t\right) \not\succ_{\mathbb{L}} \mathcal{N}(r)\}) = 0$. Consequently, (ξ_k) is a statistically Cauchy sequence in $(\mathbb{X},\mu,\varphi)$. □

Definition 4.5. Let $(\mathbb{X},\mu,\varphi)$ be an L-FMS. If every statistical Cauchy sequence in $(\mathbb{X},\mu,\varphi)$ is statistical convergent, then $(\mathbb{X},\mu,\varphi)$ is referred to as a statistically complete L-FMS.

Theorem 4.6. Let $(\mathbb{X},\mu,\varphi)$ be an L-FMS. If $(\mathbb{X},\mu,\varphi)$ is a statistically complete L-FMS, then it is a complete L-FMS.

PROOF. Let (ξ_k) be a Cauchy sequence in L-FMS $(\mathbb{X},\mu,\varphi)$. From Note 2.12, for all $r \in \mathbb{L} - \{0_L, 1_L\}$, there exists an $\varepsilon \in \mathbb{L} - \{0_L, 1_L\}$ such that $\varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \succ_{\mathbb{L}} \mathcal{N}(r)$. Thus, there exists a $K_0 \in \mathbb{N}$ such that

$$\mu\left(\xi_k, \xi_n, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$$

for all $k, n \geq K_0$. Since Proposition 4.2, it can be observed that (ξ_k) is a statistical Cauchy sequence in $(\mathbb{X},\mu,\varphi)$. Since $(\mathbb{X},\mu,\varphi)$ is a statistically complete L-FSM, (ξ_k) is statistical convergence to a $\xi \in \mathbb{X}$. Therefore, by Theorem 3.7, there exists a subsequence (ξ_{k_j}) of (ξ_k) such that $\xi_{k_j} \to \xi$. Hence, there exists a $k_{j_0} \in \{k_j : j \in \mathbb{X}\}$ with $k_{j_0} \geq K_0$ such that

$$\mu\left(\xi_{k_j}, \xi, \frac{t}{2}\right) \succ_{\mathbb{L}} \mathcal{N}(\varepsilon)$$

for all $k_j \geq k_{j_0}$. Therefore,

$$\begin{aligned} \mu(\xi_k, \xi, t) &\succeq_{\mathbb{L}} \varphi\left(\mu\left(\xi_k, \xi_{k_j}, \tfrac{t}{2}\right), \mu\left(\xi_{k_j}, \xi, \tfrac{t}{2}\right)\right) \\ &\succeq_{\mathbb{L}} \varphi(\mathcal{N}(\varepsilon), \mathcal{N}(\varepsilon)) \\ &\succ_{\mathbb{L}} \mathcal{N}(r) \end{aligned}$$

for all $k \geq k_{j_0} \geq K_0$. Therefore, (ξ_k) is convergent to ξ. Consequently, $(\mathbb{X},\mu,\varphi)$ is complete. □

5. Conclusion

We were motivated to write this paper in light of Fast, Steinhaus, Zadeh, and the articles that followed these studies. In this paper, we introduced statistical convergence in L-FMSs, a generalization of convergence in L-FMSs, a mathematical tool for modeling uncertainty, and investigated the relations of essential notions. Then, we proposed the statistical Cauchy sequences and the concept of complete metric spaces with their help. Through this paper, our most significant target is stimulating authors' motivation in this critical space. In future studies, concepts such as ideal convergence, lacunary ideal convergence, and the other generalizations of statistical convergence can be analyzed in the space in question. Besides, with the help of this study, the relationship between statistical convergence and summability theory can be discussed.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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