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THE GENERALIZED DRAZIN INVERSE OF ANTI-TRIANGULAR BLOCK-OPERATOR MATRICES

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Abstract. In this paper we investigate the g-Drazin invertibility of an anti-triangular block-operator matrix $\begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$ with $F^{\pi}EF^{d} = 0$ and $F^{\pi}EF^{i}E = 0$ for all $i \in \mathbb{N}$. This generalizes the main results of [Guo, Zou and Chen, Hacet. J. Math. Stat., 49(3)(2020), 1134-1149] and [Chen and Sheibani, Appl. Math. Comput., 463(2024), 128368 (12 pp)] to a wider case.

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1. Introduction

Let X be a Banach space and $\mathcal{B}(X)$ be a Banach algebra of all bounded linear operators over X. The generalized Drazin inverse (g-Drazin inverse in short) X of $A \in \mathcal{B}(X)$ is the solution to satisfy the following three equations:

$$AX = XA, X = XAX, A - A^2X \in \mathcal{B}(X)^{qnil}.$$

If such X exists, it is unique, and we denote it by A^d. Here, $T \in \mathcal{B}(X)^{qnil}$ if and only if $\lim\limits_{n\to\infty} \parallel T^n \parallel^{\frac{1}{n}} = 0$ if and only if $I - \lambda T \in \mathcal{B}(X)^{-1}$ for any $\lambda \in \mathbb{C}$. If replace $\mathcal{B}(X)^{qnil}$ by the set $\mathcal{B}(X)^{nil}$ (i.e., the set of nilpotent operators on X), X is called the Drazin inverse of A, and denote it by A^D. Evidently, the Drazin inverse and g-Drazin inverse exist and coincide for an arbitrary complex matrix.

Let $E, F \in \mathcal{B}(X)$ and I be the identity operator on X. It is attractive to investigate the Drazin and g-Drazin invertibility of the operator matrix $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. It was firstly posed by Campbell that the solutions to singular systems of differential equations are determined by the Drazin invertibility of the preceding special complex matrix M (see [1]). The g-Drazin (Drazin) inverse of M has been extensively studied by many authors from different point of views, e.g., [2,3,7,8,9,10,11,12].

In [4, Theorem 3.1], Deng and Wei gave a formula for the g-Drazin inverse of M under the condition $EF = 0$. In [11, Theorem 2.3], Zhang and Mosić studied the g-Drazin inverse of M under the condition $F^{\pi}EF = 0$. In [3, Theorem 2.6], Chen and Sheibani presented the g-Drazin inverses of M under the condition $F^{\pi}EF^2 = 0$ and $F^{\pi}EFE = 0$.

The aim of this paper is to establish the g-Drazin invertibility of M with $F^{\pi}EF^d = 0$ and $F^{\pi}EF^iE = 0$ for all $i \in \mathbb{N}$. This extends the results mentioned above to a wide case. For a complex matrix, the Drazin and g-Drazin inverses coincide with each others. Thus our results also provide new kind of singular different equations which could be solved by using algebraic methods.

Throughout the paper, all operators are bounded linear operators over a Banach space X. Let $T \in \mathcal{B}(X)$ and $p^2 = p \in \mathcal{B}(X)$. Then T has the Pierce decomposition relative to p, and we denote it by $\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}_p$. Let $T \in \mathcal{B}(X)^d$. We use T^{π} to stand for the spectral idempotent operator $I - TT^d$. Let $\mathbb{C}$ and $\mathbb{C}^{n\times n}$ be the field of all complex numbers and the Banach algebra of all $n \times n$ complex matrices. $\mathbb{N}$ stands for the set of all natural numbers.

2. Key lemmas

In this section, we provide several lemmas which will be used in the sequel. We begin with the following.

Lemma 2.1. *Let $P, Q \in \mathcal{B}(X)^d$. If $PQ^d = 0$ and $PQ^iP = 0$ for all $i \in \mathbb{N}$, then*

$$\begin{array}{rcl} (P+Q)^d & = & Q^{\pi}\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty} Q^i(P^d)^{i+j+1}Q^j + \sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^{\pi} \\ & - & \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(Q^d)^{i+1}P^i(P^d)^{j+1}Q^{j+1} \\ & + & \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{i}(Q^d)^{i+3}P^{j+1}Q^{i-j+1}. \end{array}$$

Proof. See [7, Theorem 4.2]. $\square$

Lemma 2.2. *Let $A, B \in \mathcal{B}(X)^d$, and let $M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$. Then*

$$M^d = \begin{pmatrix} A^d & 0 \\ X & B^d \end{pmatrix},$$

where

$$X = \sum_{i=0}^{\infty}(B^d)^{i+2}CA^iA^{\pi} + \sum_{i=0}^{\infty}B^iB^{\pi}C(A^d)^{i+2} - B^dCA^d.$$

Proof. See [5, Theorem 5.1]. $\square$

Lemma 2.3. *Let $A \in \mathcal{B}(X)^d, D \in \mathcal{B}(X)^{qnil}$, and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. If $BD^iC = 0$ for any nonnegative integer i, then M has g-Drazin inverse and*

$$M^d = \begin{pmatrix} A^d & \Gamma \\ \Delta & \Delta A \Gamma \end{pmatrix},$$

where

$$\Gamma = \sum_{n=0}^{\infty} (A^d)^{n+2} B D^n, \Delta = \sum_{n=0}^{\infty} D^n C (A^d)^{n+2}.$$

Proof. See [7, Lemma 3.1]. $\square$

In [7, Theorem 4.2], Guo, Zou and Chen studied the g-Drazin invertibility for the sum of two bounded linear operators P and Q with $PQ^d = 0$ and $PQ^iP = 0$ $(i = 1, 2, \cdots)$. We extend this result to an anti-triangular block-operator matrix.

Lemma 2.4. *Let $E, F, EF^{\pi} \in \mathcal{B}(X)^d$, and let $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$. If $EF^d = 0$ and $EF^iE = 0$ for all $i \in \mathbb{N}$, then M has g-Drazin inverse and*

$$M^d = \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix},$$

where

$$\begin{array}{rcl}
\Gamma & = & E(E^2+F)^d + \sum\limits_{i=0}^{\infty} F^i F^{\pi} F E[(E^2+F)^d]^{i+2} \\
& + & \sum\limits_{i=0}^{\infty} (F^d)^{i+2} F E (E^2+F)^i (E^2+F)^{\pi} - F^d F E (E^2+F)^d, \\
\Delta & = & \sum\limits_{i=0}^{\infty} E[(E^2+F)^d]^{i+2} E F^i \\
& + & \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} (F^d)^i F E [(E^2+F)^d]^{i+j+3} E F^j \\
& - & \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} (F^d)^{i+1} F E (E^2+F)^i [(E^2+F)^d]^{j+2} E F^j \\
& + & \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{\infty} (F^d)^{i+3} F E (E^2+F)^j E F^{i-j}, \\
\Lambda & = & F(E^2+F)^d, \\
\Xi & = & \sum\limits_{i=0}^{\infty} F[(E^2+F)^d]^{i+2} E F^i.
\end{array} \qquad (*)$$

Proof. Clearly, we have

$$M^2 = \begin{pmatrix} E^2+F & E \\ FE & F \end{pmatrix} = P + Q,$$

where

$$P = \begin{pmatrix} E^2+F & E \\ FE & FF^{\pi} \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & F^2F^d \end{pmatrix}.$$

By hypothesis, we check that $E^2F^d=0$ and $E^2F^iE^2=0$ for all $i\in\mathbb{N}$. In light of Lemma 2.1, $E^2+F\in\mathcal{B}(X)^d$ and

$$\begin{array}{ll} & (E^2+F)^d \\ = & \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}F^i(E^d)^{2(i+j+1)}F^j+\sum\limits_{i=0}^{\infty}(F^d)^{i+1}E^{2i}E^{\pi} \\ - & \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(F^d)^{i+1}E^{2i}(E^d)^{2(j+1)}F^{j+1}+\sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{i}(F^d)^{i+3}E^{2j+2}F^{i-j+1}. \end{array}$$

Clearly, Q has g-Drazin inverse and

$$Q^d=\begin{pmatrix} 0 & 0 \\ 0 & F^d \end{pmatrix}, Q^{\pi}=\begin{pmatrix} I & 0 \\ 0 & F^{\pi} \end{pmatrix}.$$

Hence, $Q^iQ^{\pi}=0$ for all $i\in\mathbb{N}$. Since $EF^d=0$ and $EF^iE=0$, $E(FF^{\pi})^i(FE)=EF^{i+1}F^{\pi}E=EF^iE=0$ for all $i\in\mathbb{N}$. As $FF^{\pi}=F-F^2F^d\in\mathcal{B}(X)^{qnil}$, it follows by Lemma 2.3 that P has g-Drazin inverse and

$$P^d=\begin{pmatrix} (E^2+F)^d & \Gamma' \\ \Delta' & \Delta'(E^2+F)\Gamma' \end{pmatrix},$$

where

$$\Gamma'=\sum_{n=0}^{\infty}[(E^2+F)^d]^{n+2}EF^nF^{\pi}, \Delta'=\sum_{n=0}^{\infty}F^{n+1}F^{\pi}E[(E^2+F)^d]^{n+2}.$$

It follows from $EF^d=0$ that $PQ=0$. In light of [6, Theorem 2.3], M^2 has g-Drazin inverse. Explicitly, we directly compute that

$$\begin{array}{lll} (M^2)^d & = & (P+Q)^d \\ & = & \sum\limits_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+1}+\sum\limits_{i=0}^{\infty}(Q^d)^{i+1}P^iP^{\pi} \\ & = & Q^{\pi}P^d+Q^dP^{\pi}+\sum\limits_{i=1}^{\infty}(Q^d)^{i+1}P^iP^{\pi} \\ & = & \begin{pmatrix} \gamma & \delta \\ \lambda & \xi+F^d \end{pmatrix}, \end{array}$$

where

$$\begin{array}{rcl}\gamma &=& (E^2+F)^d,\\ \delta &=& \sum\limits_{i=0}^{\infty}[(E^2+F)^d]^{i+2}EF^i,\\ \lambda &=& \sum\limits_{i=0}^{\infty}F^iF^{\pi}FE[(E^2+F)^d]^{i+2}+\sum\limits_{i=0}^{\infty}(F^d)^{i+2}FE(E^2+F)^i(E^2+F)^{\pi}\\ &-& F^dFE(E^2+F)^d,\\ \xi &=& \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(F^d)^iFE[(E^2+F)^d]^{i+j+3}EF^j\\ &-& \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(F^d)^{i+1}FE(E^2+F)^i[(E^2+F)^d]^{j+2}EF^j\\ &+& \sum\limits_{i=0}^{\infty}\sum\limits_{j=0}^{\infty}(F^d)^{i+3}FE(E^2+F)^jEF^{i-j}.\end{array}$$

Therefore

$$\begin{array}{rcl}M^d &=& M(M^d)^2=M(M^2)^d\\ &=& \begin{pmatrix} E & I\\ F & 0\end{pmatrix}\begin{pmatrix}\gamma & \delta\\ \lambda & \xi\end{pmatrix}\\ &=& \begin{pmatrix} E\gamma+\lambda & E\delta+\xi\\ F\gamma & F\delta\end{pmatrix}\\ &=& \begin{pmatrix}\Gamma & \Delta\\ \Lambda & \Xi\end{pmatrix},\end{array}$$

where Γ, Δ, Λ and Ξ are given as in $(*)$ by direct computation. $\square$

3. Main results

In this section, we are focusing on investigating the g-Drazin invertibility of M under multiplicative perturbations. We omit the detailed formula of the g-Drazin inverse M^d as it can be derived by some straightforward computation according to our proof. We now derive the following.

Theorem 3.1. *Let $E, F, EF^{\pi} \in \mathcal{B}(X)^d$. If $F^{\pi}EF^d = 0, F^{\pi}EF^iE = 0$ for all $i \in \mathbb{N}$, then M has g-Drazin inverse.*

Proof. Let $p = \begin{pmatrix} F^{\pi} & 0\\ 0 & 0\end{pmatrix}$. Since $F^{\pi}EF^d = 0$, we have

$$M = \begin{pmatrix} a & b\\ c & d\end{pmatrix}_p,$$

where

$$a=\begin{pmatrix} F^{\pi}E & 0\\ 0 & 0\end{pmatrix}, b=\begin{pmatrix} 0 & F^{\pi}\\ 0 & 0\end{pmatrix},$$
$$c=\begin{pmatrix} FF^dEF^{\pi} & 0\\ FF^{\pi} & 0\end{pmatrix}, d=\begin{pmatrix} EFF^d & FF^d\\ F^2F^d & 0\end{pmatrix}.$$

Then $M=P+Q$, where

$$P=\begin{pmatrix} a & b\\ c & 0\end{pmatrix}, Q=\begin{pmatrix} 0 & 0\\ 0 & d\end{pmatrix}.$$

We directly check that $d^d=\begin{pmatrix} 0 & F^d\\ FF^d & -EF^d\end{pmatrix}$. Since $bd=0$, we have $PQ=0$. It will suffice to prove that P has g-Drazin inverse. Obviously, we have

$$\begin{aligned} bc &= \begin{pmatrix} 0 & F^{\pi}\\ 0 & 0\end{pmatrix}\begin{pmatrix} FF^dEF^{\pi} & 0\\ FF^{\pi} & 0\end{pmatrix}\\ &= \begin{pmatrix} FF^{\pi} & 0\\ 0 & 0\end{pmatrix}\\ &\in \mathcal{B}(X)^d. \end{aligned}$$

Clearly, $(bc)^d=0$, and so $a(bc)^d=0$. For every $i\in\mathbb{N}$, we have

$$\begin{aligned} a(bc)^ia &= \begin{pmatrix} F^{\pi}E & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^iF^{\pi} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} F^{\pi}E & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix} F^{\pi}EF^iF^{\pi}E & 0\\ 0 & 0\end{pmatrix}\\ &= \begin{pmatrix} F^{\pi}EF^iE-(F^{\pi}EF^d)F^{i+1}E & 0\\ 0 & 0\end{pmatrix}\\ &= 0. \end{aligned}$$

According to Lemma 2.4, the 2×2 operator matrix $\begin{pmatrix} a & 1\\ bc & 0\end{pmatrix}\in M_2(p\mathcal{B}(X)p)$ has g-Drazin inverse. Observing that

$$\begin{aligned} \begin{pmatrix} a & 1\\ bc & 0\end{pmatrix} &= \begin{pmatrix} 1 & 0\\ 0 & b\end{pmatrix}\begin{pmatrix} a & 1\\ c & 0\end{pmatrix},\\ \begin{pmatrix} a & b\\ c & 0\end{pmatrix} &= \begin{pmatrix} a & 1\\ c & 0\end{pmatrix}\begin{pmatrix} 1 & 0\\ 0 & b\end{pmatrix}. \end{aligned}$$

By using Cline's formula, we prove that $\begin{pmatrix} a & b\\ c & 0\end{pmatrix}$ has g-Drazin inverse. Since $PQ=0$, by virtue of [6, Theorem 2.3], M has g-Drazin inverse, as asserted. $\square$

As an immediate consequence, we derive

Corollary 3.2. (See [11, Theorem 2.3]) *Let* $E, F, EF^{\pi} \in \mathcal{B}(X)^d$*, and let* $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$*. If* $F^{\pi}EF = 0$*, then* M *has g-Drazin inverse.*

Proof. Since $F^{\pi}EF = 0$, we have $F^{\pi}EF^d = F^{\pi}EF(F^d)^2 = 0, F^{\pi}EF^iE = (F^{\pi}EF)F^{i-1}E = 0$ for all $i \in \mathbb{N}$. This completes the proof by Theorem 3.1. □

Corollary 3.3. (See [3, Theorem 2.6]) *Let* $E, F, EF^{\pi} \in \mathcal{B}(X)^d$*, and let* $M = \begin{pmatrix} E & I \\ F & 0 \end{pmatrix}$*. If* $F^{\pi}EF^2 = 0$ *and* $F^{\pi}EFE = 0$*, then* M *has g-Drazin inverse.*

Proof. Since $F^{\pi}EF^2 = 0$, we see that $F^{\pi}EF^d = (F^{\pi}EF^2)(F^d)^3 = 0$. Moreover, we check that $F^{\pi}EFE = 0$ and $F^{\pi}EF^iE = (F^{\pi}EF^2)\ (F^{i-2}E) = 0$. Then $F^{\pi}EF^iE = 0$ for all $i \in \mathbb{N}$. This completes the proof by Theorem 3.1. □

The following example illustrates that Theorem 3.1 is a nontrivial generalization of [3, Theorem 2.6].

Example 3.4. Let $M = \begin{pmatrix} E & I_3 \\ F & 0 \end{pmatrix} \in \mathbb{C}^{6\times 6}$, where

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $F^{\pi}EF^d = 0, F^{\pi}EF^iE = 0$ for all $i \in \mathbb{N}$, while $EF^2 \neq 0$. Construct Γ, Δ, Λ and Ξ as in Lemma 2.4, we have $(E^2 + F)^d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and therefore

$$\begin{aligned} M^D &= \begin{pmatrix} \Gamma & \Delta \\ \Lambda & \Xi \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

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References

[1] C. Bu, K. Zhang and J. Zhao, *Representation of the Drazin inverse on solution of a class singular differential equations*, Linear Multilinear Algebra, 59(8) (2011), 863-877.

[2] H. Chen and M. Sheibani, *The g-Drazin inverses of special operator matrices*, Oper. Matrices, 15(1) (2021), 151-162.

[3] H. Chen and M. Sheibani, *The g-Drazin inverses of anti-triangular block operator matrices*, Appl. Math. Comput., 463 (2024), 128368 (12 pp).

[4] C. Deng and Y. Wei, *A note on the Drazin inverse of an anti-triangular matrix*, Linear Algebra Appl., 431(10) (2009), 1910-1922.

[5] D. S. Djordjević and P. S. Stanimirović, *On the generalized Drazin inverse and generalized resolvent*, Czechoslovak Math. J., 51 (2001), 617-634.

[6] D. S. Djordjević and Y. Wei, *Additive results for the generalized Drazin inverse*, J. Aust. Math. Soc., 73(1) (2002), 115-125.

[7] L. Guo, H. Zou and J. Chen, *The generalized Drazin inverse of operator matrices*, Hacet. J. Math. Stat., 49(3) (2020), 1134-1149.

[8] Y. Liao, J. Chen and J. Cui, *Cline's formula for the generalized Drazin inverse*, Bull. Malays. Math. Sci. Soc. (2), 37(1) (2014), 37-42.

[9] M. Mouçouf, *$\mathcal{P}$-canonical forms and Drazin inverses of matrices*, Filomat, 38(7) (2024), 2399-2417.

[10] D. Zhang, *Representations for generalized Drazin inverse of operator matrices over a Banach space*, Turkish J. Math., 40(2) (2016), 428-437.

[11] D. Zhang and D. Mosić, *Explicit formulae for the generalized Drazin inverse of block matrices over a Banach algebra*, Filomat, 32(17) (2018), 5907-5917.

[12] S. Zriaa and M. Mouçouf, *The explicit formulas of the Drazin inverse of matrices and its nth powers*, Rend. Circ. Mat. Palermo (2), 73(2) (2024), 603-612.

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