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# A Result on the 2-Distance Coloring of Planar Graphs with Girth Five

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**Abstract**

A vertex coloring of a graph $G$ is said to be a 2-distance coloring if any two vertices at distance at most 2 from each other receive different colors, and the least number of colors for which $G$ admits a 2-distance coloring is known as the 2-distance chromatic number of $G$, and denoted by $\chi_2(G)$. We prove that if $G$ is a planar graph with girth 5 and maximum degree $\Delta \geq 12$, then $\chi_2(G) \leq \Delta(G)+5$.

***Keywords:*** *2-distance coloring; coloring; girth; planar graph.*

***2010 Mathematics Subject Classification:*** *05C15; 05C10; 05C12.*

## 1. Preliminaries

All graphs in this paper are assumed to be simple, i.e., finite and undirected, with no loops or multiple edges. We refer to [11] for terminology and notation not defined here. Let $G$ be a graph; we use $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $g(G)$ to denote the vertex set, edge set, face set, maximum degree, and girth of $G$, respectively. When the context is clear, we abbreviate $\Delta(G)$ and $g(G)$ as $\Delta$ and $g$. A *2-distance coloring* is a vertex coloring in which two vertices that are either adjacent or share a common neighbor receive different colors. The smallest number of colors required for $G$ to admit a 2-distance coloring is called the *2-distance chromatic number* and is denoted by $\chi_2(G)$.
In 1977, Wegner [10] made the following conjecture:

**Conjecture 1.1.** *For every planar graph G,*

$$\chi_2(G) \leq \begin{cases} 7, & \Delta = 3, \\ \Delta+5, & 4 \leq \Delta \leq 7, \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \Delta \geq 8. \end{cases}$$

Thomassen [9] proved that this conjecture holds for planar graphs with $\Delta = 3$. However, it remains open for planar graphs with $\Delta \geq 4$. For upper bounds, van den Heuvel and McGuinness [5] established that $\chi_2(G) \leq 2\Delta + 25$, and Molloy and Salavatipour [8] proved that $\chi_2(G) \leq \lceil \frac{5}{3}\Delta \rceil + 78$.
For planar graphs without short cycles, Bu and Zhu [1] demonstrated that $\chi_2(G) \leq \Delta + 5$ when $g \geq 6$, this confirms Conjecture 1.1 for planar graphs with girth at least six. Recently, Deniz [3] improved this result by showing that $\chi_2(G) \leq \Delta + 4$ when $g \geq 6$ and $\Delta \geq 6$. On the other hand, La [7] proved that $\chi_2(G) \leq \Delta + 3$ if either $g \geq 7$ and $\Delta \geq 6$ or $g \geq 8$ and $\Delta \geq 4$. For planar graphs with girth 5, Deniz [4] showed that $\chi_2(G) \leq \Delta + 7$, further proving that $\chi_2(G) \leq \Delta + 6$ when $\Delta \geq 10$. On the other hand, Bu and Zhu [2] established that $\chi_2(G) \leq \Delta + 5$ for $\Delta \geq 15$.
In this paper, we improve the result of Bu and Zhu [2] as follows.

**Theorem 1.2.** *If $G$ is a planar graph with $g \geq 5$ and $\Delta \geq 12$, then $\chi_2(G) \leq \Delta + 5$.*

Given a planar graph $G$, we denote by $\ell(f)$ the length of a face $f$ and by $d(v)$ the degree of a vertex $v$. A $k$-vertex is a vertex of degree $k$. A $k^-$-vertex is a vertex of degree at most $k$, and a $k^+$-vertex is a vertex of degree at least $k$. We similarly define a $k$-face (or $k^-$ or $k^+$-face) and $k$-neighbor (or $k^-$ or $k^+$-neighbor). A $k(d)$-vertex is a $k$-vertex adjacent to $d$ 2-vertices. When $r \leq d(v) \leq s$, the vertex $v$ is called an $(r\text{-}s)$-vertex.
For a vertex $v \in V(G)$, we use $n_i(v)$ (resp. $n_2^k(v)$) to denote the number of $i$-vertices (resp. 2-vertices with a $k$-neighbor) adjacent to $v$. Let $v \in V(G)$; we define

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$$D(v) = \sum_{v_i \in N(v)} d(v_i).$$

We denote by $d(u,v)$ the distance between any pair $u,v \in V(G)$. Additionally, we set $N_i(v) = \{u \in V(G) \mid 1 \leq d(u,v) \leq i\}$, for $i \geq 1$, and clearly $N_1(v) = N(v)$.
If there is a path $uvw$ in $G$ with $d_G(v) = 2$ such that $uw \notin E(G)$, then we say that $u$ and $w$ are *weak-adjacent*. A pair of weak-adjacent vertices is said to be *weak neighbors* of each other. Neighbors of a $3(1)$-vertex that are not 2-vertices are called *star-adjacent*. Clearly, every 2-vertex $v$ with a $k^-$-neighbor satisfies $D(v) \leq \Delta + k$.
A vertex $v$ is called *expendable* if

$$D(v) < \Delta + 5 + \sum_{i=3}^{4} n_2^i(v).$$

We denote by $e(v)$ the number of expendable vertices in $N_2(v)$. If $v$ satisfies $D(v) < \Delta + 5 + e(v)$, it is called *light*; otherwise, it is called *heavy*. Note that every 2-vertex adjacent to a $4^-$-vertex is expendable, so

$$\sum_{i=3}^{4} n_2^i(v) \leq e(v)$$

for any vertex $v$. Thus, every expendable vertex is a light vertex.

## 2. The Proof of Theorem 1.2

### 2.1. The Structure of Minimum Counterexample

Let $G$ be a minimal counterexample to Theorem 1.2 such that $G$ does not admit any 2 distance coloring with $\Delta + 5$ colors, but any proper subgraph of $G$ does. By the minimality, $G - x$ has a 2-distance coloring with $\Delta + 5$ colors for every $x \in V(G) \cup E(G)$. Clearly, $G$ is connected and satisfies $\delta(G) \geq 2$.
We first present some preliminary results from the paper [4], adapting our graph $G$ to their framework.

**Lemma 2.1.** *[4] If $v$ is a light vertex, then each neighbour of $v$ is a heavy vertex.*

**Lemma 2.2.** *[4] Every $5^-$-vertex having a 2-neighbour is heavy. In particular, every 4-vertex having a $3(1)$-neighbour is heavy as well.*

**Lemma 2.3.** *[4] For $v \in V(G)$, let $S$ be the set of light vertices in $N_2(v)$. If $v$ has a light neighbour, then $D(v) \geq \Delta + 5 + |S|$.*

The following can be easily obtained from Lemmas 2.1, 2.2 and 2.3.

**Corollary 2.4.** *(a) Every 2-vertex having a $4^-$-neighbour is light.*
*(b) $G$ has no adjacent 2-vertices.*
*(c) $G$ has no $3(2)$-, $3(3)$-, $4(3)$- $4(4)$-, and $5(5)$-vertices.*
*(d) A $3(1)$-vertex cannot be adjacent to any 3-, $4(1)$-, $4(2)$- or light 4-vertex.*

We call a path $xyz$ as a *poor path* if $5 \leq d(y) \leq 9$, $2 \leq d(x) \leq 3$ and $2 \leq d(z) \leq 3$. If a poor path $xyz$ lies on the boundary of a face $f$, then the vertex $y$ is called *$f$-poor vertex*.
It follows from the definition of the poor path that we can bound the number of those paths in a face.

**Corollary 2.5.** *Each face $f$ has at most $\lfloor \frac{\ell(f)}{2} \rfloor$ $f$-poor vertices.*

In the rest of this section, we will apply discharging to show that $G$ does not exist. We assign to each vertex $v$ a charge $\mu(v) = \frac{3d(v)}{2} - 5$ and to each face $f$ a charge $\mu(f) = \ell(f) - 5$. By Euler's formula, we have

$$\sum_{v \in V} \left( \frac{3d(v)}{2} - 5 \right) + \sum_{f \in F} (\ell(f) - 5) = -10$$

We next present some rules and redistribute accordingly. Once the discharging finishes, we check the final charge $\mu^*(v)$ and $\mu^*(f)$. If $\mu^*(v) \geq 0$ and $\mu^*(f) \geq 0$, we get a contradiction that no such counterexample graph $G$ can exist.

### 2.2. Discharging Rules

We apply the following discharging rules.

**R1:** Every 2-vertex receives 1 from each neighbour.
**R2:** Every $3(1)$-vertex receives $\frac{1}{2}$ from each (4-6)-neighbour; $\frac{2}{3}$ from each 7-neighbour; $\frac{3}{4}$ from each 8-neighbour; $\frac{5}{6}$ from each 9-neighbour.
**R3:** Every light $3(0)$-vertex receives $\frac{1}{6}$ from each $3^+$-neighbour.
**R4:** Let $v$ be a heavy $3(0)$-vertex, then $v$ receives $\frac{1}{6}$ from each (5-6)-neighbour; $\frac{1}{3}$ from each (7-8)-neighbour, $\frac{1}{2}$ from each 9-neighbour.
**R5:** Let $v$ be a $4(1)$-vertex.

- $(a)$ If $v$ is adjacent to $3(1)$-vertex, then $v$ receives $\frac{1}{6}$ from each 6-neighbour; $\frac{1}{4}$ from each 7-neighbour; $\frac{1}{3}$ from each 8-neighbour.
- $(b)$ If $v$ is adjacent to light $3(0)$-vertex, then $v$ receives $\frac{1}{12}$ from each (6-8)-neighbour.

**R6:** Every $4(2)$-vertex receives $\frac{1}{4}$ from each 6-neighbour; $\frac{1}{2}$ from each (7-8)-neighbour; $\frac{3}{4}$ from each 9-neighbour.

**R7:** Every $5(2^+)$-vertex receives $\frac{1}{4}$ from each 7-neighbour, and $\frac{1}{2}$ from each 8-neighbour.
**R8:** Every 9-vertex gives $\frac{1}{2}$ to each (4-8)-neighbour other than $4(2)$-vertex.
**R9:** Every $10^+$-vertex gives 1 to each $8^-$-neighbour. In particular, if $v$ is a $12^+$-vertex, then $v$ also sends $\frac{1}{12}$ to each of its weak neighbour.
**R10:** If an 8-vertex $v$ is adjacent to two $8^+$-vertices $u$ and $w$, then $v$ gives $\frac{1}{4}$ to each face containing one of $uv$, $wv$.
**R11:** If a 9-vertex $v$ is adjacent to a $9^+$-vertex $u$, then $v$ gives $\frac{1}{4}$ to each face containing $uv$.
**R12:** If a $10^+$-vertex $v$ is adjacent to a $9^+$-vertex $u$, then $v$ gives $\frac{1}{2}$ to each face containing $uv$.
**R13:** After applying R10-R12, every face $f$ transfers its positive charge equally to its incident $f$-poor vertices.

**Remark 2.6.** *Let $f$ be a $6^+$-face. It follows from Corollary 2.5 that $f$ has at most $\lfloor \frac{\ell(f)}{2} \rfloor$ $f$-poor vertices. Suppose that $f$ is incident to $k$ non-poor $8^+$-vertices.*

- *If $k=0$, then $f$ has at most $\lfloor \frac{\ell(f)}{2} \rfloor$ $f$-poor vertices, and so $f$ sends at least $\frac{1}{3}$ to each of its incident $f$-poor vertices by R13.*
- *If $k=1$, then $f$ has at most $\lfloor \frac{\ell(f)-2}{2} \rfloor$ $f$-poor vertices, and so $f$ sends at least $\frac{1}{2}$ to each of its incident $f$-poor vertices by R13.*
- *If $k\geq 2$, then $f$ has at most $\lfloor \frac{\ell(f)-3}{2} \rfloor$ $f$-poor vertices, and so $f$ sends at least 1 to each of its incident $f$-poor vertices by R13.*

**Checking** $\mu^*(v), \mu^*(f) \geq 0$, for $v \in V(G), f \in F(G)$

Clearly $\mu^*(f) \geq 0$ for each $f \in F(G)$, since every face transfers its positive charge equally to its incident $f$-poor vertices by R13.

We pick a vertex $v \in V(G)$ with $d(v)=k$. We denote by $x_1, x_2, \ldots, x_t$ the 2-neighbours of $v$ for $0 \leq n_2(v) = t \leq k$, and let $y_i$ be the other neighbour of $x_i$ different from $v$. Let $f_1, f_2 \ldots, f_k$ be faces incident to $v$.

**(1).** Let $k=2$. Then $\mu(v)=-2$. By Corollary 2.4(b), $v$ is adjacent to two $3^+$-vertices. It follows from applying R1 that $v$ receives 1 from each of its neighbours, and so $\mu^*(v) \geq 0$.

**(2).** Let $k=3$. Then $\mu(v) = -\frac{1}{2}$. By Corollary 2.4(c), $v$ has at most one 2-neighbour. First assume that $v$ has a 2-neighbour, say $x_1$. Denote by $w$ and $z$ the other neighbour of $v$ with $d(w) \leq d(z)$. Clearly, $x_1$ is a light vertex, and so $v$ must be heavy by Lemma 2.1. It then follows from Lemma 2.3 that $d(w)+d(z) \geq \Delta + 4 \geq 16$.

- If $4 \leq d(w) \leq 6$, then $z$ is a $10^+$-vertex. Thus, $v$ receives 1 from $z$ by R9 and $\frac{1}{2}$ from $w$ by R2. Therefore, $\mu^*(v) \geq -\frac{1}{2}+1+\frac{1}{2}-1=0$ after $v$ transfers 1 to $x_1$ by R1.
- If $w$ is a 7-vertex, then $z$ would be a $9^+$-vertex. By applying R2 and R9, $v$ receives $\frac{2}{3}$ from $w$ and at least $\frac{5}{6}$ from $z$. Therefore, $\mu^*(v) \geq -\frac{1}{2}+\frac{2}{3}+\frac{5}{6}-1=0$ after $v$ transfers 1 to $x_1$ by R1.
- If $w$ is an 8-vertex, then $z$ would also be an $8^+$-vertex. By applying R2 and R9, $v$ receives at least $\frac{3}{4}$ from each of $w, z$. Therefore, $\mu^*(v) \geq -\frac{1}{2}+2\times\frac{3}{4}-1=0$ after $v$ transfers 1 to $x_1$ by R1.

Let us now assume that $v$ has no 2-neighbour. If $v$ is a light vertex, then each neighbour of $v$ must be heavy by Lemma 2.1, and so $v$ receives at least $\frac{1}{6}$ from each of it neighbours by R3; hence $\mu^*(v) \geq -\frac{1}{2}+3\times\frac{1}{6}=0$. We further suppose that $v$ is heavy. Denote by $u, w$, and $z$ the neighbours of $v$ with $d(u) \leq d(w) \leq d(z)$. Notice that $v$ cannot have two light 3-neighbours by Lemma 2.3. If $v$ has no light 3-neighbour, then we deduce that $v$ has either a $9^+$-neighbour or a $7^+$-neighbour and a $5^+$-neighbour or three $5^+$-neighbours, since $d(u)+d(w)+d(z) \geq \Delta+5 \geq 17$. In each case, $v$ receives totally at least $\frac{1}{2}$ from its neighbours by applying R4. Thus, $\mu^*(v) \geq -\frac{1}{2}+\frac{1}{2}=0$. If $v$ has a light 3-neighbour, say $u$, then we have $d(w)+d(z) \geq 15$ by Lemma 2.3. This clearly implies that $v$ has either a $10^+$-neighbour or a $6^+$- and a $9^+$-neighbour or two $7^+$-neighbours. By applying R4 and R9, $v$ receives totally at least $\frac{2}{3}$ from its neighbours. Thus, $\mu^*(v) \geq -\frac{1}{2}+\frac{2}{3}-\frac{1}{6}=0$ after $v$ transfers $\frac{1}{6}$ to $u$ by R3.

**(3).** Let $k=4$. The initial charge of $v$ is $\mu(v)=1$. By Corollary 2.4(c), $v$ has at most two 2-neighbours. Note that $v$ is heavy whenever it has a 2-neighbour by Lemma 2.2.

Let $n_2(v)=0$. Recall that $v$ has at most three $3(1)$-neighbours by Corollary 2.4(d); moreover, by the same reason, $v$ is heavy whenever it has a $3(1)$-neighbour. If $v$ has at most one $3(1)$-neighbour, then $\mu^*(v) \geq 1-\frac{1}{2}-3\times\frac{1}{6} \geq 0$ after $v$ sends $\frac{1}{2}$ to its $3(1)$-neighbour by R2, and $\frac{1}{6}$ to each of its light $3(0)$-neighbours by R3. Suppose now that $v$ has exactly two $3(1)$-neighbours. If $v$ has no light $3(0)$-neighbour, then $\mu^*(v) \geq 1-2\times\frac{1}{2} \geq 0$ after $v$ sends $\frac{1}{2}$ to each of its $3(1)$-neighbour by R2. If $v$ has a light $3(0)$-neighbour, then $v$ would also have a $11^+$-neighbour by Lemma 2.3. Thus, $v$ receives 1 from its $11^+$-neighbour by R9 and sends at most $\frac{1}{2}$ to each of its 3-neighbours by R2-R3, and so $\mu^*(v) \geq 1+1-3\times\frac{1}{2} > 0$. Finally suppose that $v$ has exactly three $3(1)$-neighbours. Since $v$ is heavy and there exist three expendable vertices (2-neighbours of $3(1)$-vertices) in $N_2(v)$, we deduce that $v$ has a $11^+$-neighbour. Thus, $v$ receives 1 from its $11^+$-neighbour by R9 and sends at most $\frac{1}{2}$ to each of its 3-neighbours by R2-R3, and so $\mu^*(v) \geq 1+1-3\times\frac{1}{2} > 0$.

Let $n_2(v)=1$. Suppose first that $v$ has no $3(1)$-neighbours. If $v$ has no light $3(0)$-neighbour, then $\mu^*(v) \geq 1-1=0$ after $v$ sends 1 to $x_1$ by R1. Thus we assume that $v$ has a light $3(0)$-neighbour. Since $v$ is heavy by Lemma 2.2, $v$ can have at most two light $3(0)$-neighbours. If $v$ has exactly one light $3(0)$-neighbour, then $v$ has either a $9^+$-neighbour, or two $6^+$-neighbours. In each case, $v$ receives totally at least $\frac{1}{6}$ from its $6^+$-neighbours by R5(b), R8 and R9. Thus $\mu^*(v) \geq 1+\frac{1}{6}-1-\frac{1}{6}=0$ after $v$ sends 1 to $x_1$ by R1, and $\frac{1}{6}$ to its light $3(0)$-neighbour by R3. On the other hand, if $v$ is adjacent to two light $3(0)$-vertices, then $v$ has also a $12^+$-neighbour $z$ by Lemma 2.3. It follows that $v$ receives 1 from $z$ by R9, and so $\mu^*(v) \geq 1+1-1-2\times\frac{1}{6} > 0$ after $v$ sends 1 to $x_1$ by R1 and $\frac{1}{6}$ to each of its light $3(0)$-neighbours by R3.
Now suppose that $v$ has exactly one $3(1)$-neighbour. Denote by $w$ and $z$ the neighbours of $v$ other than 2- and $3(1)$-vertices. Since $v$ is heavy by Lemma 2.2, we have $d(w)+d(z) \geq \Delta+2 \geq 14$. If $v$ has no light $3(0)$-neighbour, then $v$ has either one $6^+$-neighbour and one $8^+$-neighbour or two $7^+$-neighbours or a $9^+$-neighbour by Lemma 2.3, where we recall that $x_1$ is a light neighbour of $v$. In each case, $v$ receives totally at least $\frac{1}{2}$ from its $6^+$-neighbours by R5(a), R8, and R9. Thus $\mu^*(v) \geq 1+\frac{1}{2}-1-\frac{1}{2}=0$ after $v$ sends 1 to $x_1$ by R1, and $\frac{1}{2}$ to its $3(1)$-neighbour by R2. If $v$ has a light $3(0)$-neighbour, then $v$ would have a $12^+$-neighbour by Lemma 2.3. It follows that $v$

receives 1 from its $12^+$-neighbours by R9, and sends 1 to $x_1$ by R1, $\frac{1}{2}$ to its $3(1)$-neighbour by R2, $\frac{1}{6}$ to its light $3(0)$-neighbour by R3. Thus $\mu^*(v) \geq 1+1-1-\frac{1}{2}-\frac{1}{6} > 0$.
Finally suppose that $v$ has two $3(1)$-neighbours. By Lemma 2.3, $v$ has a $12^+$-neighbour $z$. It then follows that $v$ receives 1 from $z$ by R9, and sends 1 to $x_1$ by R1, $\frac{1}{2}$ to each of its $3(1)$-neighbours by R2. Thus $\mu^*(v) \geq 1+1-1-2\times\frac{1}{2} = 0$.

Let $n_2(v) = 2$. By Corollary 2.4(d), $v$ has no $3(1)$-neighbour. Since $v$ is heavy, we deduce that $v$ has either a 6-neighbour and a $9^+$-neighbour or two $7^+$-neighbours or a $10^+$-neighbour by Lemma 2.3. In each case, $v$ receives totally at least 1 from its $6^+$-neighbours by R6 and R9. Thus $\mu^*(v) \geq 1+1-2\times 1 = 0$ after $v$ sends 1 to each of $x_1, x_2$ by R1.

**(4).** $k = 5$. The initial charge of $v$ is $\mu(v) = \frac{5}{2}$. Notice that if $v$ has no 2-neighbour, then $v$ gives at most $\frac{1}{2}$ to each of its neighbours by R2-R4, so $\mu^*(v) \geq \frac{5}{2} - 5\times\frac{1}{2} = 0$. Thus, we may assume that $1 \leq n_2(v) \leq 4$ where the last inequality comes from Corollary 2.4(c). Recall that $v$ is heavy by Lemma 2.2. In addition, if $v$ has a $3(1)$-neighbour $u$, then $u$ must have a $11^+$-neighbour, since any $3(1)$-vertex is heavy by Lemma 2.2. In such a case, $v$ is star-adjacent to a $11^+$-neighbour.

Let $n_2(v) = 1$. Notice that if $v$ has three 3-neighbours, then the last neighbour of $v$ must be a $6^+$ vertex, since $v$ is heavy. Therefore, $\mu^*(v) \geq \frac{5}{2} - 1 - 3\times\frac{1}{2} = 0$ after $v$ sends 1 to $x_1$ by R1, and at most $\frac{1}{2}$ to each of its 3-neighbours by R2-R4.

Let $n_2(v) = 2$. If one of $x_i$'s is expendable, then the other $x_i$ would be light vertex. In such a case, $v$ cannot have a $3(1)$-neighbour by Lemma 2.3. Then, since $v$ can have at most two $3(0)$-neighbours (as $v$ is heavy), we have $\mu^*(v) \geq \frac{5}{2} - 2\times 1 - 2\times\frac{1}{6} > 0$ after $v$ sends 1 to each of $x_i$ by R1 and at most $\frac{1}{6}$ to each of its $3(0)$-neighbours by R3-R4. Suppose now that none of the $x_i$'s is expendable. That is, each $y_i$ is a $12^+$-vertex. This means that $v$ is weak adjacent to two $12^+$-vertices. On the other hand, $v$ is adjacent to at most two $3(1)$-neighbours as it is heavy. If $v$ has no $3(1)$-neighbour, then $\mu^*(v) \geq \frac{5}{2} - 2\times 1 - 2\times\frac{1}{6} > 0$ after $v$ sends 1 to each $x_i$ by R1 and $\frac{1}{6}$ to each of its $3(0)$-neighbours by R3-R4.
If $v$ has exactly one $3(1)$-neighbour, then $v$ would have either a $8^+$-neighbour or no $3(0)$-neighbour. In the former, $v$ receives at least $\frac{1}{2}$ from its $8^+$-neighbour by R7-R9, and so $\mu^*(v) \geq \frac{5}{2} + \frac{1}{2} - 2\times 1 - \frac{1}{2} - \frac{1}{6} > 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{1}{2}$ to its $3(1)$-neighbour by R2, and $\frac{1}{6}$ to its $3(0)$-neighbour by R3-R4. In the latter, $v$ sends no charge to its neighbours other than 2- and $3(1)$-vertices. Thus, we have $\mu^*(v) \geq \frac{5}{2} - 2\times 1 - \frac{1}{2} = 0$ after $v$ sends 1 to each $x_i$ by R1 and $\frac{1}{2}$ to its $3(1)$-neighbour by R2.
If $v$ has two $3(1)$-neighbours, then $v$ would be star-adjacent to two $11^+$-vertices as stated above. Since $v$ is weak-adjacent to two $12^+$-vertices and star-adjacent to two $11^+$-vertices, there exists a face $f_i$ for $i \in [5]$ such that it is either a 5-face having two adjacent $11^+$-vertices $x$ and $y$ or a $6^+$-face. If $f_i$ is a $6^+$-face, then it transfers at least 1 to $v$ by R13 together with Remark 2.6. On the other hand, if $f_i$ is a 5-face having two adjacent $11^+$-vertices $x$ and $y$, then each of $x,y$ gives $\frac{1}{2}$ to the faces containing $xy$ by R12, so $f_i$ gets totally at least 1 from $x,y$ and transfers it to $v$ by R13. Consequently, $v$ receives at least 1 from $f_i$. Therefore, $\mu^*(v) \geq \frac{5}{2} + 1 - 2\times 1 - 2\times\frac{1}{2} > 0$ after $v$ transfers 1 to each $x_i$ by R1 and $\frac{1}{2}$ to each of its $3(1)$-neighbours by R2.

Let $n_2(v) = 3$. Suppose first that one of the $x_i$'s is expendable. Then the other 2-neighbours of $v$ would be light. Note that $v$ cannot have any $3(1)$-neighbour by Lemma 2.3. If $v$ has a $3(0)$-neighbour, then $v$ would have also $11^+$-neighbour by Lemma 2.3. By applying R9, $v$ receives 1 from its $11^+$-neighbour. Thus, $\mu^*(v) \geq \frac{5}{2} + 1 - 3\times 1 - \frac{1}{6} > 0$ after $v$ sends 1 to each of $x_i$ by R1 and $\frac{1}{6}$ to its $3(0)$-neighbour by R3-R4. Otherwise, if $v$ has no $3(0)$-neighbour, then $v$ has either an $8^+$-neighbour or two $7^+$-neighbours by Lemma 2.3. In each case, $v$ receives totally at least $\frac{1}{2}$ from its $6^+$-neighbours by R7. Then, $\mu^*(v) \geq \frac{5}{2} + \frac{1}{2} - 3\times 1 = 0$ after $v$ sends 1 to each $x_i$ by R1.
Next we suppose that none of the $x_i$'s is expendable, i.e., each $y_i$ is a $12^+$-vertex. Since $v$ is weak-adjacent to three $12^+$-vertices, we conclude that there exists a face $f_i$ for $i \in [5]$ such that it is either a 5-face having two adjacent $12^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{5}{2} + 1 - 3\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

Let $n_2(v) = 4$. If one of the $x_i$'s is expendable, then the other 2-neighbours of $v$ would be light vertices. In this case, we would have $D(v) \geq \Delta + 5 + |\{x_1, x_2, \ldots, x_4\}|$ by Lemma 2.3, however this is not possible as $D(v) \leq \Delta + 8$. We therefore suppose that none of the $x_i$'s is expendable, i.e., each $y_i$ is a $12^+$-vertex. Since $v$ is weak-adjacent to four $12^+$-vertices, we conclude that there exist two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [5]$ such that each of them is either a 5-face having two adjacent $12^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{5}{2} + 2\times 1 - 4\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

**(5).** Let $k = 6$. The initial charge of $v$ is $\mu(v) = 4$. Notice first that if $v$ has at most two 2-neighbours, then $v$ gives 1 to each of its 2-neighbours by R1 and at most $\frac{1}{2}$ to each of its other neighbours by R2-R6, so $\mu^*(v) \geq 4 - 2\times 1 - 4\times\frac{1}{2} = 0$. We may therefore assume that $n_2(v) \geq 3$.

Let $n_2(v) = 3$. We may assume that $v$ has a $3(1)$-neighbour, since otherwise, $\mu^*(v) \geq 4 - 3\times 1 - 3\times\frac{1}{4} > 0$ after $v$ transfers 1 to each $x_i$ by R1, and at most $\frac{1}{4}$ to each of its other neighbours by R3-R6. Suppose first that one of the $x_i$'s is light, and so $v$ is heavy by Lemma 2.1. Then $v$ would have either two 5-neighbours or a $6^+$-neighbour. In the former, $\mu^*(v) \geq 4 - 3\times 1 - \frac{1}{2} > 0$ after $v$ transfers 1 to each $x_i$ by R1, and $\frac{1}{2}$ to its $3(1)$-neighbour by R2. In the latter, $\mu^*(v) \geq 4 - 3\times 1 - 2\times\frac{1}{2} = 0$ after $v$ transfers 1 to each $x_i$ by R1, and at most $\frac{1}{2}$ to each of its $(3-4)$-neighbours by R2-R6. Next we suppose that all $x_i$'s are heavy. That is, each $y_i$ is a $11^+$-vertex. Recall that $v$ is star-adjacent to a $10^+$-vertex, since any $3(1)$-vertex is heavy. It then follows that $v$ is weak-adjacent to three $11^+$-vertices and star-adjacent to a $10^+$-vertex. We then conclude that there exists a face $f_i$ for $i \in [6]$ such that it is either a 5-face having two adjacent $10^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq 4 + 1 - 3\times 1 - 3\times\frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{2}$ to each of its $(3-4)$-neighbours by R2-R6.

Let $n_2(v) = 4$. Suppose first that two of the $x_i$'s are expendable. Then the other $x_i$'s would be light vertices. Clearly, $v$ is heavy by Lemma 2.1. If $v$ has a $3(1)$-neighbour, then $v$ would have a $11^+$-neighbour, and so $v$ receives 1 from its $11^+$-neighbour by R9. Thus, $\mu^*(v) \geq 4 + 1 - 4\times 1 - \frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{1}{2}$ to its $3(1)$-neighbour by R2. If $v$ has no $3(1)$-neighbour, then $v$ has either a $9^+$-neighbour or no $(3-4)$-neighbour by Lemma 2.3. In the former, $v$ receives at least $\frac{1}{2}$ from its $9^+$-neighbour by R8 and so $\mu^*(v) \geq 4 + \frac{1}{2} - 4\times 1 - \frac{1}{2} = 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{2}$ to its $(3-4)$-neighbour by R2-R6. In the latter, $\mu^*(v) \geq 4 - 4\times 1 = 0$ after $v$ sends 1 to each $x_i$ by R1.

Now we suppose that exactly one of the $x_i$'s is expendable, say $x_1$. Obviously, $v$ is heavy by Lemma 2.1. It then follows from Lemma 2.3 that if $v$ has a $(3-4)$-neighbour, then $v$ would have a $6^+$-neighbour as well. Notice that, for $i \in \{2,3,4\}$, each $y_i$ is a $11^+$-vertex since $x_i$ is not expendable.
We first consider the case that $v$ has a $3(1)$-neighbour. Then $v$ would be star-adjacent to $11^+$-vertex, as a $3(1)$-vertex is heavy by Lemma 2.2. Since $v$ is weak-adjacent to three $11^+$-vertices and star-adjacent to a $11^+$-vertex, we conclude that there exists a face $f_i$ for $i \in [6]$ such that it is either a 5-face having two adjacent $11^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq 4+1-4\times 1-\frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1 and $\frac{1}{2}$ to its $3(1)$-neighbours by R2.
Now we consider the case that $v$ has no $3(1)$-neighbour. If $y_2, y_3, y_4$ are $12^+$-vertices, then, by applying R9, each of $y_2, y_3, y_4$ sends $\frac{1}{12}$ to $v$, and so we have $\mu^*(v) \geq 4+3\times\frac{1}{12}-4\times 1-\frac{1}{4} = 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{4}$ to its $(3-4)$-neighbour by R3-R6. If exactly one of $y_2, y_3, y_4$ is a 11-vertex, say $y_2$, then, by applying R9, each of $y_3, y_4$ sends $\frac{1}{12}$ to $v$. Also we deduce that $v$ has either a $9^+$-neighbour or no $4(2)$-neighbour by Lemma 2.3. In the former, $v$ receives at least $\frac{1}{2}$ from its $9^+$-neighbour by R8 and so $\mu^*(v) \geq 4+2\times\frac{1}{12}+\frac{1}{2}-4\times 1-\frac{1}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{4}$ to its $(3-4)$-neighbour by R3-R6. In the latter, $\mu^*(v) \geq 4+2\times\frac{1}{12}-4\times 1-\frac{1}{6} = 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{6}$ to its $(3-4)$-neighbour by R2-R6. If two of $y_2, y_3, y_4$ are 11-vertices, then $v$ has either a $9^+$-neighbour or no $3(0)$-, $4(1)$- and $4(2)$-neighbour by Lemma 2.3. In the former, $v$ receives at least $\frac{1}{2}$ from its $9^+$-neighbour by R8 and so $\mu^*(v) \geq 4+\frac{1}{2}-4\times 1-\frac{1}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{4}$ to its $(3-4)$-neighbour by R3-R6. In the latter, $\mu^*(v) \geq 4-4\times 1 = 0$ after $v$ sends 1 to each $x_i$ by R1.
Finally suppose that none of the $x_i$' is expendable, i.e., each $y_i$ is a $11^+$-vertex. Since $v$ is weak-adjacent to four $11^+$-vertices, we conclude that there exists two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [6]$ such that each of them is either a 5-face having two adjacent $11^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq 4+2\times 1-4\times 1-2\times\frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{2}$ to each of its other neighbours by R2-R6.

Let $n_2(v) = 5$. If two of $x_i$'s are expendable, then the others would be light vertices. In such a case, $v$ would have a $12^+$-neighbour, since it is heavy by Lemma 2.1. It then follows that $v$ receives 1 from its $12^+$-neighbour by R9, and sends 1 to each of its 2-neighbour by R1. Thus $\mu^*(v) \geq 4+1-5\times 1 = 0$. Now, assume that at most one of the $x_i$'s is expendable. Since $v$ is weak-adjacent to four $11^+$-vertices, we conclude that there exist two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [6]$ such that each of them is either a 5-face having two adjacent $11^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq 4+2\times 1-5\times 1-\frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{2}$ to its other neighbour by R2-R6.

Let $n_2(v) = 6$. All the $x_i$'s are heavy, since otherwise, $v$ and one of its neighbour would be light, a contradiction by Lemma 2.1. Since $v$ is weak-adjacent to six $11^+$-vertices, we deduce that each $f_i$ is either a 5-face having two adjacent $11^+$-vertices or a $6^+$-face. By applying R12 and R13 together with Remark 2.6, each $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq 4+6\times 1-6\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

**(6).** Let $k=7$. The initial charge of $v$ is $\mu(v) = \frac{11}{2}$. Observe that a $3^+$-vertex may receive at most $\frac{2}{3}$ from $v$ by R2-R7. So, if $v$ has at most two 2-neighbours, then $\mu^*(v) \geq \frac{11}{2}-2\times 1-5\times\frac{2}{3} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{2}{3}$ to each of its other neighbours by R2-R7. Therefore we may assume that $n_2(v) \geq 3$.

Let $n_2(v) = 3$. If $v$ has at most three $3(1)$-neighbours, then $\mu^*(v) \geq \frac{11}{2}-3\times 1-3\times\frac{2}{3}-\frac{1}{2} = 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{2}{3}$ to each of its $3(1)$-neighbour by R2, and at most $\frac{1}{2}$ to each of its neighbours other than 2- and $3(1)$-vertices by R3-R7. If $v$ has four $3(1)$-neighbours, then $v$ would be light, and so each $x_i$ must be heavy by Lemma 2.1. This means that each $y_i$ is a $10^+$-vertex. Also each $3(1)$-neighbour of $v$ is adjacent to a $9^+$-vertex as a $3(1)$-vertex is heavy by Lemma 2.2. Since $v$ is weak-adjacent to three $10^+$-vertices and star-adjacent to four $9^+$-vertices, we conclude that there exists two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [7]$ such that each of them is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq \frac{11}{2}+2\times\frac{1}{3}-3\times 1-4\times\frac{2}{3} > 0$ after $v$ sends 1 to each $x_i$ by R1 and $\frac{2}{3}$ to each of its $3(1)$-neighbours by R2.

Let $n_2(v) = 4$. If all the $x_i$'s are heavy, then each $y_i$ must be a $10^+$-vertex. Since $v$ is weak-adjacent to four $10^+$-vertices, we conclude that there exists a face $f_i$ for $i \in [7]$ such that it is either a 5-face having two adjacent $10^+$-vertices or a $6^+$-face. By applying R12-R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{11}{2}+1-4\times 1-3\times\frac{2}{3} > 0$ after $v$ sends 1 to each $x_i$ by R1, and at most $\frac{2}{3}$ to each of its other neighbours by R2-R7.
Now we suppose that one of the $x_i$'s is light, then $v$ would be heavy by Lemma 2.1. If $v$ has no $3(1)$-neighbour, then $\mu^*(v) \geq \frac{11}{2}-4\times 1-3\times\frac{1}{2} = 0$ after $v$ sends 1 to each $x_i$ by R1, and at most $\frac{1}{2}$ to each of its other neighbours by R3-R7. If $v$ has exactly one $3(1)$-neighbour, then $v$ cannot have two $4(2)$-neighbour by Lemma 2.3. Thus, $\mu^*(v) \geq \frac{11}{2}-4\times 1-\frac{2}{3}-\frac{1}{2}-\frac{1}{3} = 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{2}{3}$ to its $3(1)$-neighbour by R2, $\frac{1}{2}$ to its $4(2)$-neighbour by R6, and at most $\frac{1}{3}$ to its neighbour other than 2-, $3(1)$-, and $4(2)$-vertex by R3-R5, R7. If $v$ has two $3(1)$-neighbours, then $v$ would have a $6^+$-neighbour by Lemma 2.3. Thus, $\mu^*(v) \geq \frac{11}{2}-4\times 1-2\times\frac{2}{3} > 0$ after $v$ sends 1 to each $x_i$ by R1 and $\frac{2}{3}$ to each of its $3(1)$-neighbours by R2.

Let $n_2(v) = 5$. If three of $x_i$'s are expendable, then the others would be light vertices. In such a case, $v$ would have either a $10^+$-neighbour or no $3(1)$-neighbour. In the former, $v$ receives 1 from its $10^+$-neighbour by R9, and sends 1 to each of its 2-neighbour, at most $\frac{2}{3}$ to its other $5^-$-neighbour by R2-R7. Thus $\mu^*(v) \geq \frac{11}{2}+1-5\times 1-\frac{2}{3} > 0$. In the latter, $v$ cannot have two $(3-5)$-neighbours by Lemma 2.3, and so $\mu^*(v) \geq \frac{11}{2}-5\times 1-\frac{1}{2} = 0$ after $v$ sends 1 to each of its 2-neighbour by R1 and at most $\frac{1}{2}$ to its $(3-5)$-neighbour by R3-R7.
Now, assume that exactly two of the $x_i$'s are expendable, say $x_1, x_2$. Then each of $y_3, y_4, y_5$ is a $10^+$-vertex since they are not expendable. Suppose first that exactly one of $y_3, y_4, y_5$ is a $11^-$-vertex, say $y_3$. Obviously, $y_3$ is light vertex. By R9, each of $y_4, y_5$ sends $\frac{1}{12}$ to $v$. Since $v$ is heavy by Lemma 2.1, we deduce that $v$ has either two $5^+$-neighbours or one $6^+$-neighbour. In each case, $v$ sends totally at most $\frac{2}{3}$ to its $(3-5)$-neighbours by R2-R7. Thus, $\mu^*(v) \geq \frac{11}{2}+2\times\frac{1}{12}-5\times 1-\frac{2}{3} = 0$ after $v$ sends 1 to each of its 2-neighbour by R1.
Next we suppose that at least two of $y_3, y_4, y_5$ are $11^-$-vertices. Since $v$ is heavy by Lemma 2.1, $v$ has a $6^+$-neighbour. Moreover, we deduce that $v$ has either $9^+$-neighbour or no $3(1)$-neighbour. In the former, $v$ receives at least $\frac{1}{2}$ from its $9^+$-neighbour by R8-R9, and so $\mu^*(v) \geq \frac{11}{2}+\frac{1}{2}-5\times 1-\frac{2}{3} > 0$ after $v$ sends 1 to each of its 2-neighbour by R1 and at most $\frac{2}{3}$ to its $(3-5)$-neighbour by R2-R7. In the

latter, we have $\mu^*(v) \geq \frac{11}{2} - 5\times 1 - \frac{1}{2} > 0$ after $v$ sends 1 to each of its 2-neighbour by R1 and at most $\frac{1}{2}$ to its $(3-5)$-neighbour by R3-R7. Now we suppose that each of $y_3, y_4, y_5$ is a $12^+$-vertex. So, each of $y_3, y_4, y_5$ sends $\frac{1}{12}$ to $v$. Since $v$ is heavy, $v$ has either one 4-neighbour and one $5^+$-neighbour or a $6^+$-neighbour. In each case, $v$ sends totally at most $\frac{3}{4}$ to its $(3-5)$-neighbours by R2-R7. Thus, $\mu^*(v) \geq \frac{11}{2} + 3\times\frac{1}{12} - 5\times 1 - \frac{3}{4} = 0$ after $v$ sends 1 to each of its 2-neighbour by R1.
Finally, assume that at most one of the $x_i$'s is expendable. Since $v$ is weak-adjacent to four $10^+$-vertices, we conclude that there exists a face $f_i$ for $i\in[7]$ such that it is either a 5-face having two adjacent $10^+$-vertices or a $6^+$-face. By applying R12-R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{11}{2} + 1 - 5\times 1 - 2\times\frac{2}{3} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{2}{3}$ to each of its other neighbours by R2-R7.

Let $n_2(v) = 6$. If three of the $x_i$'s are expendable, then the others would be light vertices. This implies that $v$ has a $11^+$-neighbour by Lemma 2.3. Then $v$ receives 1 from its $11^+$-neighbour by R9, and so $\mu^*(v) \geq \frac{11}{2} + 1 - 6\times 1 > 0$ after sends 1 to each of its 2-neighbour by R1. Suppose now that at most two of the $x_i$'s are expendable. Since $v$ is weak-adjacent to four $10^+$-vertices, we conclude that there exists a face $f_i$ for $i\in[7]$ such that it is either a 5-face having two adjacent $10^+$-vertices or a $6^+$-face. By applying R12-R13 together with Remark 2.6, $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{11}{2} + 1 - 6\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

Let $n_2(v) = 7$. All the $x_i$'s are heavy, since otherwise, $v$ and one of its neighbour would be light, a contradiction by Lemma 2.1. Since $v$ is weak-adjacent to seven $10^+$-vertices, we deduce that each $f_i$ is either a 5-face having two adjacent $10^+$-vertices or a $6^+$-face. By applying R12-R13 together with Remark 2.6, each $f_i$ transfers at least 1 to $v$. Thus, $\mu^*(v) \geq \frac{11}{2} + 6\times 1 - 7\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

**(7).** Let $k = 8$. The initial charge of $v$ is $\mu(v) = 7$. Note that if $v$ has at most four 2-neighbours, then $\mu^*(v) \geq 7 - 4\times 1 - 4\times\frac{3}{4} = 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to each of its other neighbours by R2-R7. So, we may assume that $n_2(v) \geq 5$.

Let $n_2(v) = 5$. If one of the $x_i$'s is light, then $v$ would be heavy by Lemma 2.1, and so $v$ cannot have three 3(1)-neighbours by Lemma 2.3. Thus, $\mu^*(v) \geq 7 - 5\times 1 - 2\times\frac{3}{4} - \frac{1}{2} = 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{3}{4}$ to each of its 3(1)-neighbour by R2, and at most $\frac{1}{2}$ to each of its other neighbours by R3-R7. If all the $x_i$'s are heavy, then each $y_i$ would be a $9^+$-vertex. Since $v$ is weak-adjacent to five $9^+$-vertices, we conclude that there exist two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [8]$ such that each of them is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + 2\times\frac{1}{3} - 5\times 1 - 3\times\frac{3}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to each of its other neighbours by R2-R7.

Let $n_2(v) = 6$. Observe that if three of the $x_i$'s are light, then $v$ would have either a $6^+$-neighbour or no 3(1)-neighbour by Lemma 2.3. In the former, $\mu^*(v) \geq 7 - 6\times 1 - \frac{3}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to its $(3-5)$-neighbour by R2-R7. In the latter, $\mu^*(v) \geq 7 - 6\times 1 - 2\times\frac{1}{2} = 0$ after $v$ sends 1 to each of its 2-neighbours by R1 and at most $\frac{1}{2}$ to each of its $(3-5)$-neighbours by R2-R7. Suppose now that exactly two of the $x_i$'s are light, say $x_1, x_2$. By Lemma 2.1, $v$ is heavy. It follows that $v$ cannot have two 3(1)-neighbours by Lemma 2.3. If $v$ has no 3(1)-neighbour, then $\mu^*(v) \geq 7 - 6\times 1 - 2\times\frac{1}{2} = 0$ after $v$ sends 1 to each of its 2-neighbours by R1 and at most $\frac{1}{2}$ to each of its $(3-5)$-neighbours by R2-R7. If $v$ has a 3(1)-neighbour, say $z$, then $z$ is adjacent to a $10^+$-vertex since a 3(1)-vertex is heavy by Lemma 2.2. That is, $v$ is star-adjacent to a $10^+$-vertex. Recall that $y_3, y_4, y_5, y_6$ are $9^+$-vertices. Since $v$ is weak-adjacent to four $9^+$-vertices and star-adjacent to a $10^+$-vertex, then we conclude that there exists a face $f_i$ for $i\in[8]$ such that it is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, $f_i$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + \frac{1}{3} - 6\times 1 - \frac{3}{4} - \frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1, $\frac{3}{4}$ to its 3(1)-neighbour by R2, and at most $\frac{1}{2}$ to each of its $3^+$-neighbours other than 3(1)-vertex by R3-R7. Next we suppose that at most one of the $x_i$'s is light, say $x_1$ (if exists). Clearly, $y_2, y_3, \ldots, y_6$ are $9^+$-vertices. Since $v$ is weak-adjacent to five $9^+$-vertices, we conclude that there exist two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [8]$ such that each of them is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + 2\times\frac{1}{3} - 6\times 1 - 2\times\frac{3}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to each of its $(3-5)$-neighbours by R2-R7.

Let $n_2(v) = 7$. If three of the $x_i$'s are light, then $v$ would have a $6^+$-neighbour by Lemmas 2.1 and 2.3. Thus, $\mu^*(v) \geq 7 - 7\times 1 = 0$ after $v$ sends 1 to each $x_i$ by R1. Suppose now that exactly two of the $x_i$'s are light, say $x_1, x_2$. Then $v$ would have a $5^+$-neighbour by Lemmas 2.1 and 2.3. On the other hand, $y_3, y_4, \ldots, y_7$ are $9^+$-vertices. Since $v$ is weak-adjacent to five $9^+$-vertices, we conclude that there exist two faces $f_{i_1}, f_{i_2}$ for $i_1, i_2 \in [8]$ such that each of them is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + 2\times\frac{1}{3} - 7\times 1 - \frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{1}{2}$ to its $5^+$-neighbour by R7. Finally suppose that at most one of the $x_i$'s is light, say $x_1$ (if exists). Then $y_2, y_3, \ldots, y_7$ are $9^+$-vertices. Since $v$ is weak-adjacent to six $9^+$-vertices, we conclude that there exists four faces $f_{i_1}, f_{i_2}, \ldots, f_{i_4}$ for $i_1, i_2, \ldots, i_4 \in [8]$ such that each of them is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each of $f_{i_1}, f_{i_2}, \ldots, f_{i_4}$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + 4\times\frac{1}{3} - 7\times 1 - \frac{3}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to its other neighbour by R2-R7.

Let $n_2(v) = 8$. All the $x_i$'s are heavy, since otherwise, $v$ and one of its neighbour would be light, a contradiction by Lemma 2.1. Since $v$ is weak-adjacent to eight $9^+$-vertices, we deduce that each $f_i$ is either a 5-face having two adjacent $9^+$-vertices or a $6^+$-face. By applying R11-R13 together with Remark 2.6, each $f_i$ transfers at least $\frac{1}{3}$ to $v$. Thus, $\mu^*(v) \geq 7 + 8\times\frac{1}{3} - 8\times 1 > 0$ after $v$ sends 1 to each $x_i$ by R1.

**(8).** Let $k = 9$. The initial charge of $v$ is $\mu(v) = \frac{17}{2}$. Note that if $v$ has at most six 2-neighbours, then $\mu^*(v) \geq \frac{17}{2} - 6\times 1 - 3\times\frac{5}{6} = 0$ after $v$ sends 1 to each $x_i$ by R1, at most $\frac{5}{6}$ to each of its other neighbours by R2-R7. So, we may assume that $n_2(v) \geq 7$.

Let $n_2(v) = 7$. If two of the $x_i$'s are light, then $v$ would have at most one neighbour forming a 3(1)- or 4(2)-vertex by Lemmas 2.1 and 2.3. In such a case, $\mu^*(v) \geq \frac{17}{2} - 7\times 1 - \frac{5}{6} - \frac{1}{2} > 0$ after $v$ sends 1 to each $x_i$ by R1, at most $\frac{5}{6}$ to its 3(1)- or 4(2)-neighbour (if exists) by R2,R6, and at most $\frac{1}{2}$ to each of its other neighbours by R3-R5,R7.
Suppose now that exactly one of the $x_i$'s is light, say $x_1$. Clearly, $y_2, y_3, \ldots, y_7$ are $8^+$-vertices. If $v$ has no 3(1)-neighbour, then $\mu^*(v) \geq \frac{17}{2} - 7\times 1 - 2\times\frac{3}{4} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{3}{4}$ to each of its other neighbour by R3-R7. If $v$ has a 3(1)-neighbour, say $z$, then $z$ is adjacent to a $8^+$-vertex other than $v$, since a 3(1)-vertex is heavy Lemma 2.2. That is, $v$ is star-adjacent to a $8^+$-vertex. Recall that $y_2, y_3, \ldots, y_7$ are $8^+$-vertices. Since $v$ is weak-adjacent to six $8^+$-vertices and star-adjacent to a $8^+$-vertex, we conclude that there exist

two consecutive faces $f_i, f_{i+1}$ for $i \in [9]$ such that each of them is either a 5-face having two adjacent $8^+$-vertices or a $6^+$-face. If both $f_i$ and $f_{i+1}$ are 5-faces, then there exists a common $8^+$-vertex $x_s$ on $f_i$ and $f_{i+1}$ adjacent to two $8^+$-vertices on those faces, and so $x_s$ sends at least $\frac{1}{4}$ to each of $f_i, f_{i+1}$ by R10-R12. If one of $f_i$ and $f_{i+1}$ is a $6^+$-face, then it sends at least $\frac{1}{3}$ to $v$ by R13 together with Remark 2.6. Consequently, $v$ receives totally at least $\frac{1}{3}$ from its incident faces by R13. Thus, $\mu^*(v) \geq \frac{17}{2} + \frac{1}{3} - 7 \times 1 - 2 \times \frac{5}{6} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{5}{6}$ to each of its other neighbour by R2-R7.
Finally we suppose that all the $x_i$'s are heavy. Clearly, $y_1, y_2, \ldots, y_7$ are $8^+$-vertices. Since $v$ is weak-adjacent to seven $8^+$-vertices, we conclude that there exist two consecutive faces $f_i, f_{i+1}$ for $i \in [8]$ such that each of them is either a 5-face having two adjacent $8^+$-vertices or a $6^+$-face. Similarly as above, $v$ receives totally at least $\frac{1}{3}$ from its incident faces by R10-R13. Thus, $\mu^*(v) \geq \frac{17}{2} + \frac{1}{3} - 7 \times 1 - 2 \times \frac{5}{6} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{5}{6}$ to each of its other neighbour by R2-R7.

Let $n_2(v) \geq 8$. If two of the $x_i$'s are light, then $v$ has neither 3(1)- nor 4(2)-neighbour by Lemmas 2.1 and 2.3. In this case, $\mu^*(v) \geq \frac{17}{2} - 8 \times 1 - \frac{1}{2} = 0$ after $v$ sends 1 to each of its 2-neighbour by R1 and at most $\frac{1}{2}$ to its other neighbour R3-R4, R8. Suppose now that at most one of the $x_i$'s is light, say $x_1$ (if exists). Clearly, $y_2, y_3, \ldots, y_8$ are $8^+$-vertices. Since $v$ is weak-adjacent to seven $8^+$-vertices, we conclude that there exist two consecutive faces $f_i, f_{i+1}$ for $i \in [8]$ such that each of them is either a 5-face having two adjacent $8^+$-vertices or a $6^+$-face. Similarly as above, $v$ receives totally at least $\frac{1}{3}$ from its incident faces by R10-R13. Thus, $\mu^*(v) \geq \frac{17}{2} + \frac{1}{3} - 7 \times 1 - 2 \times \frac{5}{6} > 0$ after $v$ sends 1 to each $x_i$ by R1 and at most $\frac{5}{6}$ to each of its other neighbour by R2-R7.

**(9).** Let $k \geq 10$. The initial charge of $v$ is $\mu(v) \geq \frac{3d(v)}{2} - 5 \geq 10$. By R9, $v$ sends 1 to each of its $8^-$-neighbour. Besides, if $v$ has a $9^+$-neighbour $u$, then $v$ sends $\frac{1}{2}$ to each face containing $vu$ by R12. Therefore, $v$ sends exactly 1 charge for each of its neighbours. On the other hand, if $v$ is a $12^+$-vertex, then $v$ sends $\frac{1}{12}$ to its each weak neighbour, and so $\mu^*(v) \geq \frac{3d(v)}{2} - 5 - d(v) \times 1 - d(v) \times \frac{1}{12} \geq 0$.

We obtain a non-negative charge on each vertex and face, which contradicts the fact that the total charge is negative. Thus, $G$ cannot exist. This concludes the proof of Theorem 1.2.

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