Dumlupınar Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Sayı 29, Aralık 2012 ISSN - 1302 - 3055

GENERALIZED QUATERNIONS SERRET-FRENET AND BISHOP FRAMES

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ABSTRACT

Serret-Frenet and Parallel-Transport frame are produced with the help of the generalized quaternions again by the method in [4].

Keywords: *Generalized quaternion*, *Serret-Frenet frame*, *Bishop frame*.

GENELLEŞTİRİLMİŞ KUATERNİYONLARIN SERRET-FRENET VE BISHOP ÇATILARI

ÖZET

Serret-Frenet ve Paralel taşıma çatıları, genelleştirilmiş kuaterniyonlar yardımıyla yine [4]te verilen metot ile oluşturulmuştur.

1. INTRODUCTION

The Frenet-Serret formulas describe the kinematic properties of a particle which moves along a continuous, differentiable curve in Euclidean space $\mathbb{R}^3$ or Minkowski space $\mathbb{R}_1^3$. These formulas have a common area of usage in mathematics, physics (especially in relative theory), medicine, computer graphics and such fields.

It is known by especially mathematicians and physicists that any unit (split) quaternion corresponds to rotation in Euclidean and Minkowski spaces. For detailed information it is referred to [1], [2] and [3]. The rotations are expressed by quaternions that is because the geodesic curves in unit (split) quaternion space S^3 can not be expressed by using Euler angles [4].

2. PRELIMINARIES

Our first goal is to define moving coordinate frames that are attached to a curve in 3D space.

2.1. Frenet-Serret frames:

The Frenet-Serret frame (see, [5], [6] and [7]) is defined as follows: Let $\vec{\alpha}(t)$ be any thrice-differentiable space curve with non-vanishing second derivative. We can choose this local coordinate system to be the Frenet-Serret frame consisting of the tangent $\vec{T}(t)$, the binormal $\vec{B}(t)$, and the principal normal $\vec{N}(t)$ vectors at a point on the curve given by

$$\vec{T}(t)=\frac{\vec{\alpha}'(t)}{\left\|\vec{\alpha}'(t)\right\|},\quad \vec{B}(t)=\frac{\vec{\alpha}'(t)\times\vec{\alpha}''(t)}{\left\|\vec{\alpha}'(t)\times\vec{\alpha}''(t)\right\|},\quad \vec{N}(t)=\vec{B}(t)\times\vec{T}(t). \tag{2.1}$$

The Frenet-Serret frame (also known as the Frenet frame) obeys the following differential equation in the parameter *t*:

$$\begin{bmatrix}\vec{T}'(t)\\ \vec{N}'(t)\\ \vec{B}'(t)\end{bmatrix}=v(t)\begin{bmatrix}0 & \kappa(t) & 0\\ -\kappa(t) & 0 & \tau(t)\\ 0 & -\tau(t) & 0\end{bmatrix}\begin{bmatrix}\vec{T}(t)\\ \vec{N}(t)\\ \vec{B}(t)\end{bmatrix} \tag{2.2}$$

where $v(t)=\left\|\alpha'(t)\right\|$ is scalar magnitude of the curve derivative (often reparametrized to be unity, so that *t* becomes the arc-length *s*), the intrinsic geometry of the curve is embodied in the scalar curvature $\kappa(t)$ and the torsion $\tau(t)$. In principle these quantities can be calculated in terms of the parametrized or numerical local values of $\vec{\alpha}(t)$ and its first three derivatives as follows:

$$\kappa(t)=\frac{\left\|\vec{\alpha}'(t)\times\vec{\alpha}''(t)\right\|}{\left\|\vec{\alpha}'(t)\right\|^{3}}$$

$$\tau(t)=\frac{\left(\vec{\alpha}'(t)\times\vec{\alpha}''(t)\right)\cdot\vec{\alpha}'''(t)}{\left\|\vec{\alpha}'(t)\times\vec{\alpha}''(t)\right\|^{2}} \tag{2.3}$$

If a non-vanishing curvature and torsion are given as smooth function of *t*, the system of equations can be integrated theoretically to find the unique numerical values of the corresponding space curve $\vec{\alpha}(t)$.

2.2. Parallel Transport Frames:

Intuitively, the Frenet frame's normal vector $\vec{N}$ always points toward the center of the osculating circle [8]. Thus, when the orientation of the osculating circle changes drastically or the second derivative of the curve becomes very small, The Frenet frame behaves erratically or may become undefined.
Parallel Transport Frames:

The Parallel Transport frame or Bishop frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative.

We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component to the frame. The parallel transport frame is based on the observation that, while $\vec{T}(t)$ for a given curve model is unique, we may choose any conventient arbitrary basis $\left(\vec{N_1}(t),\vec{N_2}(t)\right)$ for the remainder of the frame, as long as it is in the normal plane perpendicular to $\vec{T}(t)$ at each point. If the derivatives of $\left(\vec{N_1}(t),\vec{N_2}(t)\right)$ depend only on $\vec{T}(t)$ and not on each other, we can make $\vec{N_1}(t)$ and $\vec{N_2}(t)$

vary smoothly throughout the path regardless of the curvature. We therefore have the alternative frame equations

$$\begin{bmatrix} \vec{T}'(t) \\ \vec{N_1}'(t) \\ \vec{N_2}'(t) \end{bmatrix} = v(t) \begin{bmatrix} 0 & k_1(t) & k_2(t) \\ -k_1(t) & 0 & 0 \\ -k_2(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{T}(t) \\ \vec{N}(t) \\ \vec{B}(t) \end{bmatrix} \tag{2.4}$$

One can show (see, [10]) that

$$\kappa = \sqrt{k_1^{\,2} + k_2^{\,2}}, \tag{2.5}$$

$$\theta(t) = \arctan\left(\frac{k_2}{k_1}\right), \tag{2.6}$$

$$\tau(t) = \frac{d\theta(t)}{dt} \tag{2.7}$$

so that k_1 and k_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = \int \tau dt$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence the Frenet frame) due to the differentiation.

3. GENERALIZED QUATERNION FRAMES

Definition 3.1. The set $H_{\alpha,\beta} = \{q = a_0 1 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3, \alpha, \beta \in \mathbb{R}\}$ is a vector space over ℝ having basis $\{1, i, j, k\}$ with the following properties:

$$i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha\beta$$
$$ij = -ji = k$$
$$jk = -kj = \beta i$$
$$ki = -ik = \alpha j$$

Every element of the set $H_{\alpha\beta}$ is called a generalized quaternion [9].

Definition 3.2. A generalized quaternion frame is defined as a unit-lenght generalized quaternion

$$q = a_0 1 + a_1 i + a_2 j + a_3 k$$

and is characterized by the following properties:

Two generalized quaternions q and p obey the following multiplication rule,

$$\begin{aligned} qp = {} & \left(a_0 b_0 - \alpha a_1 b_1 - \beta a_2 b_2 - \alpha\beta a_3 b_3\right) + \left(a_0 b_1 + a_1 b_0 + \beta a_2 b_3 - \beta a_3 b_2\right) i \\ & + \left(a_0 b_2 + a_2 b_0 + \alpha a_3 b_1 - \alpha a_1 b_3\right) j + \left(a_0 b_3 + a_3 b_0 + a_1 b_2 - a_2 b_1\right) k. \end{aligned} \tag{3.1}$$

The conjugate of q is defined as

$$\overline{q} = a_0 1 - a_1 i - a_2 j - a_3 k \ .$$

A unit-length generalized quaternion's norm is defined as:

$$N_q = \overline{q}q = q\overline{q} = a_0^{\,2} + \alpha a_1^{\,2} + \beta a_2^{\,2} + \alpha\beta a_3^{\,2} = 1.$$

Every possible rotation R (a 3×3 special generalized orthogonal matrix) can be constructed from either of two related generalized quaternions, $q = a_0 1 + a_1 i + a_2 j + a_3 k$ or $-q = -a_0 1 - a_1 i - a_2 j - a_3 k$, using the transformation law:

$$q \cdot w \cdot \overline{q} = R \cdot w$$

$$\left[q \cdot w \cdot \overline{q} \right]_i = \sum_{j=1}^{3} R_{ij} \cdot w_j$$

where $w = v_1 i + v_2 j + v_3 k$ is a generalized pure quaternion. We compute R_{ij} directly from (3.1)

$$R = \begin{bmatrix} a_0^2 + \alpha a_1^2 - \beta a_2^2 - \alpha\beta a_3^2 & 2\beta a_1 a_2 - 2\beta a_0 a_3 & 2\beta a_0 a_2 + 2\alpha\beta a_1 a_3 \\ 2\alpha a_0 a_3 + 2\alpha a_1 a_2 & a_0^2 - \alpha a_1^2 + \beta a_2^2 - \alpha\beta a_3^2 & 2\alpha\beta a_2 a_3 - 2\alpha a_0 a_1 \\ 2\alpha a_1 a_3 - 2a_0 a_2 & 2a_0 a_1 + 2\beta a_2 a_3 & a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \end{bmatrix}$$

All rows of this matrix expressed in this form are orthogonal but not orthonormal. Dividing first, second and last column by α, β and $\alpha\beta$, respectively we get

$$R' = \begin{bmatrix} \frac{1}{\alpha} a_0^2 + a_1^2 - \frac{\beta}{\alpha} a_2^2 - \beta a_3^2 & 2a_1 a_2 - 2a_0 a_3 & \frac{2}{\alpha} a_0 a_2 + 2a_1 a_3 \\ 2\alpha a_0 a_3 + 2\alpha a_1 a_2 & \frac{1}{\beta} a_0^2 - \frac{\alpha}{\beta} a_1^2 + a_2^2 - \alpha a_3^2 & 2a_2 a_3 - \frac{2}{\beta} a_0 a_1 \\ 2a_1 a_3 - \frac{2}{\alpha} a_0 a_2 & \frac{2}{\beta} a_0 a_1 + 2a_2 a_3 & \frac{1}{\alpha\beta} a_0^2 - \frac{1}{\beta} a_1^2 - \frac{1}{\alpha} a_2^2 + a_3^2 \end{bmatrix} \tag{3.2}$$

All rows of this rotation matrix expressed in this form are orthonormal and create a roof. The quadratic form (3.2) for a general orthonormal frame coincides with Frenet and parallel transport frames.

Special cases:

(i) For $\alpha = \beta = 1$, the generalized quaternion algebra $H_{\alpha\beta}$ coincides with the real quaternion algebra H. In this case the rotation matrix R' becomes

$$R' = \begin{bmatrix} a_0^2 + a_1^2 - a_2^2 - a_3^2 & 2a_1 a_2 - 2a_0 a_3 & 2a_0 a_2 + 2a_1 a_3 \\ 2a_0 a_3 + 2\alpha a_1 a_2 & a_0^2 - a_1^2 + a_2^2 - a_3^2 & 2a_2 a_3 - 2a_0 a_1 \\ 2a_1 a_3 - 2a_0 a_2 & 2a_0 a_1 + 2a_2 a_3 & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{bmatrix}$$

These matrices form the three-dimensional special orthogonal group SO(3). Since the matrix R' can be obtained by the unit quaternions q and q', there are two unit quaternions for every rotation in Euclidean space $\mathbb{R}^3$.

(ii) For $\alpha = 1, \beta = -1$, the generalized quaternion algebra $H_{\alpha\beta}$ coincides with the split quaternion algebra H'. In this case the rotation matrix R' becomes

$$R' = \begin{bmatrix} {a_0}^2 + {a_1}^2 + {a_2}^2 + {a_3}^2 & 2a_1a_2 - 2a_0a_3 & 2a_0a_2 + 2a_1a_3 \\ 2a_0a_3 + 2\alpha a_1a_2 & -{a_0}^2 + {a_1}^2 + {a_2}^2 - {a_3}^2 & 2a_2a_3 + 2a_0a_1 \\ 2a_1a_3 - 2a_0a_2 & -2a_0a_1 + 2a_2a_3 & -{a_0}^2 + {a_1}^2 - {a_2}^2 + {a_3}^2 \end{bmatrix}$$

These matrices form the three-dimensional special orthogonal group SO(1,2). Similarly the matrix R' can be obtained by the unit split quaternions q and $-q$, there are two unit timelike quaternions for every rotation in Minkowski 3-space $\mathbb{R}_1^3$.

The equations obtained as a result of this coincidence are quaternion valued linear equations. If we derive the rows equation of (3.2) respectively, then we obtain following results;

$$d\vec{T} = 2\begin{bmatrix} \frac{1}{\alpha}a_0 & a_1 & -\frac{\beta}{\alpha}a_2 & -\beta a_3 \\ a_3 & a_2 & a_1 & a_0 \\ -\frac{1}{\alpha}a_2 & a_3 & -\frac{1}{\alpha}a_0 & a_1 \end{bmatrix}\begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = 2[A][q']$$

$$d\vec{N} = 2\begin{bmatrix} -a_3 & a_2 & a_1 & -a_0 \\ \frac{1}{\beta}a_0 & -\frac{\alpha}{\beta}a_1 & a_2 & -\alpha a_3 \\ \frac{1}{\beta}a_1 & \frac{1}{\beta}a_0 & a_3 & a_2 \end{bmatrix}\begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = 2[B][q'] \tag{3.3}$$

$$d\vec{B} = 2\begin{bmatrix} \frac{1}{\alpha}a_2 & a_3 & \frac{1}{\alpha}a_0 & a_1 \\ -\frac{1}{\beta}a_1 & -\frac{1}{\beta}a_0 & a_3 & a_2 \\ \frac{1}{\alpha\beta}a_0 & -\frac{1}{\beta}a_1 & -\frac{1}{\alpha}a_2 & a_3 \end{bmatrix}\begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = 2[C][q'].$$

3.1. Generalized Quaternion Frenet Frame Equation:

The Frenet equations themselves must take the form

$$2[A][q'] = \vec{T'} = v\kappa\vec{N} \tag{3.4}$$

$$2[B][q'] = \vec{N'} = -v\kappa\vec{T} + v\tau\vec{B} \tag{3.5}$$

$$2[C][q'] = \vec{B'} = -v\tau\vec{N} \tag{3.6}$$

where

$$[q'] = \begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \\ e_0 & e_1 & e_2 & e_3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Therefore; with the help of (3.4),(3.5) and (3.6) we obtain the following equations:

$$\begin{aligned}&b_0a_0a_3+b_1a_1a_3+b_2a_2a_3+b_3a_3^2+c_0a_0a_2+c_1a_1a_2+c_2a_2^2+c_3a_2a_3+d_0a_0a_1+d_1a_1^2+d_2a_1a_2\\&+d_3a_1a_3+e_0a_0^2+e_1a_0a_1+e_2a_0a_2+e_3a_0a_3=\frac{v}{2}\kappa\left(\frac{1}{\beta}a_0^2-\frac{\alpha}{\beta}a_1^2+a_2^2-\alpha a_3^2\right)\end{aligned} \tag{3.7}$$

$$\begin{aligned}&-b_0a_0a_3-b_1a_1a_3-b_2a_2a_3-b_3a_3^2+c_0a_0a_2+c_1a_1a_2+c_2a_2^2+c_3a_2a_3\\&+d_0a_0a_1+d_1a_1^2+d_2a_1a_2+d_3a_1a_3-e_0a_0^2-e_1a_0a_1-e_2a_0a_2-e_3a_0a_3\\&=-\frac{v}{2}\kappa\left(\frac{1}{\alpha}a_0^2+a_1^2-\frac{\beta}{\alpha}a_2^2-\beta a_3^2\right)+\frac{v}{2}\tau\left(\frac{2}{\alpha}a_0a_2+2a_1a_3\right)\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\frac{1}{\beta}b_0a_0a_1+\frac{1}{\beta}b_1a_1^2+\frac{1}{\beta}b_2a_1a_2+\frac{1}{\beta}b_3a_1a_3+\frac{1}{\beta}c_0a_0^2+\frac{1}{\beta}c_1a_0a_1+\frac{1}{\beta}c_2a_0a_2+\frac{1}{\beta}c_3a_1a_3\\&+d_0a_0a_3+d_1a_1a_3+d_2a_2a_3+d_3a_3^2+e_0a_0a_2+e_1a_1a_2+e_2a_2^2+e_3a_2a_3\\&=-\frac{v}{2}\kappa\left(2a_1a_3-\frac{2}{\alpha}a_0a_2\right)+\frac{v}{2}\tau\left(\frac{1}{\alpha\beta}a_0^2-\frac{1}{\beta}a_1^2-\frac{1}{\alpha}a_2^2+a_3^2\right)\end{aligned} \tag{3.9}$$

$$\begin{aligned}&-\frac{1}{\beta}b_0a_0a_1-\frac{1}{\beta}b_1a_1^2-\frac{1}{\beta}b_2a_1a_2-\frac{1}{\beta}b_3a_1a_3-\frac{1}{\beta}c_0a_0^2-\frac{1}{\beta}c_1a_0a_1-\frac{1}{\beta}c_2a_0a_2-\frac{1}{\beta}c_3a_1a_3\\&+d_0a_0a_3+d_1a_1a_3+d_2a_2a_3+d_3a_3^2+e_0a_0a_2+e_1a_1a_2+e_2a_2^2+e_3a_2a_3\\&=-\frac{v}{2}\tau\left(\frac{1}{\beta}a_0^2-\frac{\alpha}{\beta}a_1^2+a_2^2-\alpha a_3^2\right)\end{aligned} \tag{3.10}$$

Finally, we get

$$\begin{array}{llll} b_0=0, & b_1=-\frac{v}{2}\tau\alpha, & b_2=0, & b_3=-\frac{v}{2}\kappa\alpha,\\ c_0=\frac{v}{2}\frac{\tau}{\alpha}, & c_1=0, & c_2=\frac{v}{2}\kappa, & c_3=0,\\ d_0=0, & d_1=-\frac{v}{2}\frac{\kappa}{\beta}, & d_2=0, & d_3=\frac{v}{2}\tau\alpha,\\ e_0=\frac{v}{2}\frac{\kappa}{\beta}, & e_1=0, & e_2=-\frac{v}{2}\frac{\tau}{\alpha}, & e_3=0.\end{array}$$

Therefore, the generalized quaternion Frenet frame equation:

$$[q']=\begin{bmatrix}da_0\\da_1\\da_2\\da_3\end{bmatrix}=\frac{v}{2}\begin{bmatrix}0&-\tau\alpha&0&-\kappa\beta\\ \frac{\tau}{\alpha}&0&\kappa&0\\ 0&-\frac{\kappa}{\beta}&0&\tau\alpha\\ \frac{\kappa}{\beta}&0&-\frac{\tau}{\alpha}&0\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\\a_3\end{bmatrix}.$$

Special case:

(i) For $\alpha=\beta=1$ we get the real quaternion Frenet frame equation

$$[q'] = \begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = \frac{v}{2}\begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & -\kappa & 0 & \tau \\ \kappa & 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

for quaternion algebra H.

(ii) For $\alpha = 1$, $\beta = -1$ we get the split quaternion Frenet frame equation

$$[q'] = \begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = \frac{v}{2}\begin{bmatrix} 0 & -\tau & 0 & -\kappa \\ \tau & 0 & \kappa & 0 \\ 0 & \kappa & 0 & \tau \\ -\kappa & 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

with the split quaternion algebra H'.

3.2. Parallel-Transport Generalized Quaternion Frame Equation

Similarly, it can be easily shown that a parallel transport frame system with $\left(\overrightarrow{N_1}(t), \vec{T}(t), \vec{N}_2(t)\right)$ (in that order) corresponded to columns of equation (3.2) is completely equivalent to the following parallel-transport generalized quaternion frame equation:

$$2[B][q'] = \overrightarrow{T'} = vk_1\overrightarrow{N_1} + vk_2\overrightarrow{N_2} \tag{3.11}$$

$$2[A][q'] = \overrightarrow{N_1'} = -vk_1\vec{T} \tag{3.12}$$

$$2[C][q'] = \overrightarrow{N_2'} = -vk_2\vec{T} \tag{3.13}$$

where

$$[q'] = \begin{bmatrix} da_0 \\ da_1 \\ da_2 \\ da_3 \end{bmatrix} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \\ c_0 & c_1 & c_2 & c_3 \\ d_0 & d_1 & d_2 & d_3 \\ e_0 & e_1 & e_2 & e_3 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

Therefore; with the help of (3.11),(3.12) and (3.13) we obtain the following equations:

$$\begin{aligned} &-b_0a_0a_3 - b_1a_1a_3 - b_2a_2a_3 - b_3a_3^2 + c_0a_0a_2 + c_1a_1a_2 + c_2a_2^2 + c_3a_2a_3 \\ &+d_0a_0a_1 + d_1a_1^2 + d_2a_1a_2 + d_3a_1a_3 - e_0a_0^2 - e_1a_0a_1 - e_2a_0a_2 - e_3a_0a_3 \\ &= \frac{v}{2}k_1\left(\frac{1}{\alpha}a_0^2 + a_1^2 - \frac{\beta}{\alpha}a_2^2 - \beta a_3^2\right) + \frac{v}{2}k_2\left(\frac{2}{\alpha}a_0a_2 + 2a_1a_3\right) \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\frac{1}{\beta}b_0a_0a_1 + \frac{1}{\beta}b_1a_1^2 + \frac{1}{\beta}b_2a_1a_2 + \frac{1}{\beta}b_3a_1a_3 + \frac{1}{\beta}c_0a_0^2 + \frac{1}{\beta}c_1a_0a_1 + \frac{1}{\beta}c_2a_0a_2 + \frac{1}{\beta}c_3a_1a_3 \\ &+d_0a_0a_3 + d_1a_1a_3 + d_2a_2a_3 + d_3a_3^2 + e_0a_0a_2 + e_1a_1a_2 + e_2a_2^2 + e_3a_2a_3 \\ &= \frac{v}{2}k_1\left(2a_1a_3 - \frac{2}{\alpha}a_0a_2\right) + \frac{v}{2}k_2\left(\frac{1}{\alpha\beta}a_0^2 - \frac{1}{\beta}a_1^2 - \frac{1}{\alpha}a_2^2 + a_3^2\right) \end{aligned} \tag{3.15}$$

$$b_0a_0a_3+b_1a_1a_3+b_2a_2a_3+b_3a_3^{\ 2}+c_0a_0a_2+c_1a_1a_2+c_2a_2^{\ 2}+c_3a_2a_3+d_0a_0a_1+d_1a_1^{\ 2}+d_2a_1a_2$$
$$+d_3a_1a_3+e_0a_0^{\ 2}+e_1a_0a_1+e_2a_0a_2+e_3a_0a_3=-\frac{v}{2}k_1\left(\frac{1}{\beta}a_0^{\ 2}-\frac{\alpha}{\beta}a_1^{\ 2}+a_2^{\ 2}-\alpha a_3^{\ 2}\right) \tag{3.16}$$

$$-\frac{1}{\beta}b_0a_0a_1-\frac{1}{\beta}b_1a_1^{\ 2}-\frac{1}{\beta}b_2a_1a_2-\frac{1}{\beta}b_3a_1a_3-\frac{1}{\beta}c_0a_0^{\ 2}-\frac{1}{\beta}c_1a_0a_1-\frac{1}{\beta}c_2a_0a_2-\frac{1}{\beta}c_3a_1a_3$$
$$+d_0a_0a_3+d_1a_1a_3+d_2a_2a_3+d_3a_3^{\ 2}+e_0a_0a_2+e_1a_1a_2+e_2a_2^{\ 2}+e_3a_2a_3$$
$$=-\frac{v}{2}k_2\left(\frac{1}{\beta}a_0^{\ 2}-\frac{\alpha}{\beta}a_1^{\ 2}+a_2^{\ 2}-\alpha a_3^{\ 2}\right) \tag{3.17}$$

Finally, we get

$$b_0=0,\qquad b_1=-\frac{v}{2}k_2,\qquad b_2=0,\qquad b_3=\frac{v}{2}k_1\beta,$$
$$c_0=\frac{v}{2}\frac{k_2}{\alpha},\qquad c_1=0,\qquad c_2=-\frac{v}{2}\frac{k_1\beta}{\alpha},\qquad c_3=0,$$
$$d_0=0,\qquad d_1=\frac{v}{2}k_1,\qquad d_2=0,\qquad d_3=\frac{v}{2}k_2,$$
$$e_0=-\frac{v}{2}\frac{k_1}{\alpha},\qquad e_1=0,\qquad e_2=-\frac{v}{2}\frac{k_2}{\alpha},\qquad e_3=0.$$

Therefore, the generalized quaternion parallel-transport frame equation:

$$[q']=\begin{bmatrix} da_0\\ da_1\\ da_2\\ da_3\end{bmatrix}=\frac{v}{2}\begin{bmatrix} 0 & -k_2 & 0 & k_1\beta\\ \frac{k_2}{\alpha} & 0 & -\frac{k_1\beta}{\alpha} & 0\\ 0 & k_1 & 0 & k_2\\ -\frac{k_1}{\alpha} & 0 & -\frac{k_2}{\alpha} & 0\end{bmatrix}\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}.$$

Special case:

(i) For $\alpha=\beta=1$ we get the real quaternion parallel-transport frame equation

$$[q']=\begin{bmatrix} da_0\\ da_1\\ da_2\\ da_3\end{bmatrix}=\frac{v}{2}\begin{bmatrix} 0 & -k_2 & 0 & k_1\\ k_2 & 0 & -k_1 & 0\\ 0 & k_1 & 0 & k_2\\ -k_1 & 0 & -k_2 & 0\end{bmatrix}\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}.$$

for quaternion algebra H.

(ii) For $\alpha=1,\ \beta=-1$ we get the split quaternion Frenet frame equation

$$[q']=\begin{bmatrix} da_0\\ da_1\\ da_2\\ da_3\end{bmatrix}=\frac{v}{2}\begin{bmatrix} 0 & -k_2 & 0 & -k_1\\ k_2 & 0 & k_1 & 0\\ 0 & k_1 & 0 & k_2\\ -k_1 & 0 & -k_2 & 0\end{bmatrix}\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}.$$

with the split quaternion algebra H'.

3.3. Conclusion

While the rotations can be expressed by using the Euler angles, the rotations between the geodesic curves in the unit (split) quaternion space can not be obtained by the Euler angles. In addition, it is necessary to solve a nine-component equation for a rotation or a translation made by using the Euler angles. Whereas, instead of this, it can be made by a unit (split) quaternion.

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