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Inverse Sturm-Liouville Problem with Energy dependent potential

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Abstract: In this short note we analyze the asymptotics of eigenvalues, and Ambarzumyan type theorem for energy dependent potential problem with boundary condition including spectral parameter. We should mention that results are more general then the results given in [18].

Keywords: Energy dependent potential, spectrum, Ambarzumyan theorem.

1 Introduction

Consider the boundary-value problem generated by the quadratic (in the eigenvalue λ) differential equation

$$-y'' + [q(x) + 2\lambda p(x)]y = \lambda^2 y, \quad x \in (0,\pi) \tag{1}$$

with the homogeneous with the boundary conditions

$$y'(0) = 0, \tag{2}$$

$$(a_0 + a_1\lambda)y(\pi) + y'(\pi) = 0, \tag{3}$$

where a_0 and a_1 are any real numbers and $q(x) \in W_2^1[0,\pi]$, $p(x) \in W_2^2[0,\pi]$.

In the literature, equation (1) is called as quadratic of differential pencil and it is very important in quantum theory. For instance, this type equation come to light in Klein-Gordon equations by seperation of variables, which define the motion of particles. By the way, Sturm–Liouville energy-dependent potential is also used in viscous vibration of rope. (see [16]). We also emphasize that problems including the spectral parameter λ in boundary condition is related to the energy of the system. Inverse problems of quadratic pencil have been solved by many authors. Also, this eigenvalue problem arises in many fields such as mechanics, physics, electronics, geophysics, meteorology and other branches of sciences and there is a lot of literature on solving this problem [3],[2],[4],[10],[9],[12],[13],[14],[17],[24],[25].

Ambarzumyan's paper can be viewed as first and vital reference in the history of inverse spectral problems associated with Sturm-Liouville operators [1]. In 1929, he showed that for the Neumann boundary conditions ($\theta = \chi = \frac{\pi}{2}$), if the

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spectrum (collection of the eigenvalues) in (1) is $\{\lambda_n = n^2 : n = 0,1,2,...\}$, then the potential function $q(x)$ is zero almost everywhere on $[0,\pi]$. Ambarzumyan's theorem was extended to the second order differential systems of two dimensions in [7], to Sturm-Liouville differential systems of any dimension in [8], to the Sturm–Liouville problem with seperable conditions by adding more condition on the potential [22]. In addition, some different results of Ambarzumyan's theorem have been obtained in [6], [15], [20], [21], [23].

If $p(x) = 0$ the classical Sturm-Liouville operator is obtained. Some versions of the eigenvalue problem (1)-(4) were studied extensively in [5], [11], [22], [19].

In this study, by extending the results of classical Sturm-Liouville problem, we show that an explicit formula of eigenvalues can determine two functions in the quadratic pencil of Sturm-Lioville operator with spectral parameter in boundary condition.

We define

$$\Delta(\lambda) = (a_0 + a_1\lambda)y(\pi) + y'(\pi) \tag{4}$$

which is called the characteristic function. In the Sturm-Liouville theory, we known that if the λ is an eigenvalue of the problem (1)-(4) then $\Delta(\lambda) = 0$.

Theorem 1. [14] *Let $q(x) \in W_2^1[0,\pi]$, $p(x) \in W_2^2[0,\pi]$ and $y(x,\lambda)$ solution of (1) with the inital condition (2),*

$$y(x,\lambda) = \cos(\lambda x - \alpha(x)) + \int_0^x A(x,t)\cos\lambda t dt + \int_0^x B(x,t)\sin\lambda t dt, \tag{5}$$

where $A(x,t)$ and $B(x,t)$ satisfy the following equations

$$\frac{\partial^2 A(x,t)}{\partial x^2} - 2p(x)\frac{\partial B(x,t)}{\partial t} - q(x)A(x,t) = \frac{\partial^2 A(x,t)}{\partial t^2}$$

$$\frac{\partial^2 B(x,t)}{\partial x^2} + 2p(x)\frac{\partial A(x,t)}{\partial t} - q(x)B(x,t) = \frac{\partial^2 B(x,t)}{\partial t^2}$$

$$A(0,0) = 0, \quad B(x,0) = 0, \quad \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0,$$

$$q(x) + p^2(x) = 2\frac{d}{dx}[A(x,x)\cos\alpha(x) + B(x,x)\sin\alpha(x)],$$

$$A(0,0) = 0, \quad B(x,0) = 0, \quad \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0,$$

$$\alpha(x) = \int_0^x p(t)dt = p(0)x + 2\int_0^x [A(\zeta,\zeta)\sin\alpha(\zeta) - B(\zeta,\zeta)\cos\alpha(\zeta)]\,d\zeta.$$

2 Main results

In this section, some uniqueness theorems are given for the problem (1)-(4). It is shown that an explicit formula of eigenvalues can determine the function $q(x)$ be zero.

Let's considering a second quadratic Sturm -Liouville problem

$$-y'' + [\widetilde{q}(x) + 2\lambda p(x)]y = \lambda^2 y, \quad x \in (0,\pi) \tag{6}$$

$$y'(0) = 0, \tag{7}$$

$$(a_0 + a_1\lambda)y(\pi) + y'(\pi) = 0 \tag{8}$$

and showing this problem briefly $E(p,\widetilde{q},a_0,a_1)$. Also, we will show spectrums of the (1)-(4) and (6)-(8) as $\sigma(p,q,a_0,a_1)$ and $\sigma\left(p,\widetilde{q},a_0,a_1\right)$ respectively.

Theorem 2. *The eigenvalues of the problem satisfiying the* $\Delta(\lambda) = 0$ *are as following.*

(i) *If* $a_1 = 0$ *and* $a_0 \neq 0$ *and* $\alpha(\pi) = 0$,

$$\lambda_n = n - \frac{A(\pi,\pi)}{n} - \frac{a_0}{n} + \frac{a_0 B(\pi,\pi)}{n^2} + O\left(\frac{1}{n^2}\right),\ n \to \infty. \tag{9}$$

(ii) *If* $a_1, a_0 \neq 0$ *and* $\alpha(\pi) = 0$,

$$\lambda_n = n - \frac{\arctan a_1}{\pi} - \frac{A(\pi,\pi)}{n\pi(1+a_1^2)} + \frac{a_1 B(\pi,\pi)}{n\pi(1+a_1^2)} + O\left(\frac{1}{n^2}\right),\ n \to \infty. \tag{10}$$

Proof. From (8), we see that λ is an eigenvalue of the problem (6-8) if and only if

$$\Delta(\lambda) = (a_0 + a_1\lambda)y(\pi) + y'(\pi) = 0. \tag{11}$$

Applying integration by parts to (5), we obtain

$$y(x,\lambda) = \cos(\lambda x - \alpha(x)) + \frac{1}{\lambda}A(x,x)\sin\lambda x - \frac{1}{\lambda}B(x,x)\cos\lambda x - \frac{1}{\lambda}\int_0^x A_t(x,t)\sin\lambda t dt +$$

$$+\frac{1}{\lambda}\int_0^x B_t(x,t)\cos\lambda t dt,$$

or asymptotically

$$y(\pi,\lambda) = \cos(\lambda\pi - \alpha(\pi)) + \frac{1}{\lambda}A(\pi,\pi)\sin\lambda\pi - \frac{1}{\lambda}B(\pi,\pi)\cos\lambda\pi + O\left(\frac{1}{\lambda}\right) \tag{12}$$

and

$$y'(\pi,\lambda) = -\left(\lambda - \alpha'(\pi)\right)\sin(\lambda\pi - \alpha(\pi)) + A(\pi,\pi)\cos\lambda\pi + B(\pi,\pi)\sin\lambda\pi + O\left(\frac{1}{\lambda}\right). \tag{13}$$

Inserting (12) and (13) in (11), it is not difficult to obtain that

$$\left(\lambda - \alpha'(\pi)\right)\sin(\lambda\pi - \alpha(\pi)) + A(\pi,\pi)\cos\lambda\pi + B(\pi,\pi)\sin\lambda\pi \tag{14}$$

$$+(a_0 + a_1\lambda)\left[\cos(\lambda\pi - \alpha(\pi)) + \frac{1}{\lambda}A(\pi,\pi)\sin\lambda\pi - \frac{1}{\lambda}B(\pi,\pi)\cos\lambda\pi + O\left(\frac{1}{\lambda}\right)\right] = 0,$$

where $A(x,t)$, $B(x,t)$, $\frac{\partial}{\partial x,t}A(x,t)$ and $\frac{\partial}{\partial x,t}B(x,t)$ are bounded functions for $0 \leq x,t \leq \pi$.

If $a_1 = 0$ and $a_0 \neq 0$ and $\alpha(\pi) = 0$,we can easily see that from (14) for $\cos\lambda\pi \neq 0$

$$\tan\lambda\pi = -\frac{A(\pi,\pi)}{\lambda} - \frac{a_0}{\lambda} - \frac{B(\pi,\pi)\tan\lambda\pi}{\lambda} - \frac{a_0 A(\pi,\pi)\tan\lambda\pi}{\lambda^2} + \frac{a_0 B(\pi,\pi)}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right).$$

For $\lambda \to \infty$, $\tan\lambda\pi \to 0$. Then

$$\tan\lambda\pi = -\frac{A(\pi,\pi)}{\lambda} - \frac{a_0}{\lambda} + \frac{a_0 B(\pi,\pi)}{\lambda^2} + O\left(\frac{1}{\lambda^2}\right)$$

and we see that

$$\lambda_n\pi = n\pi - \frac{A(\pi,\pi)}{n} - \frac{a_0}{n} + \frac{a_0 B(\pi,\pi)}{n^2} + O\left(\frac{1}{n^2}\right).$$

If $a_1, a_0 \neq 0$ and $\alpha(\pi) = 0$, for $\cos\lambda\pi \neq 0$

$$\tan\lambda_n\pi + a_1 = -\frac{A(\pi,\pi)}{\lambda_n} + \frac{a_1 B(\pi,\pi)}{\lambda_n} + O\left(\frac{1}{\lambda_n}\right).$$

After some trigonometric identities, we obtain that

$$\lambda_n = n - \frac{\arctan a_1}{\pi} - \frac{A(\pi,\pi)}{n\pi(1+a_1^2)} + \frac{a_1 B(\pi,\pi)}{n\pi(1+a_1^2)} + O\left(\frac{1}{n^2}\right)$$

This completes the proof.

Theorem 3. *Let consider the two problems $E(p,q,a_0,a_1)$, $E(p,\widetilde{q},a_0,a_1)$ and their spectrums $\sigma(p,q,a_0,a_1)$, $\sigma\left(p,\widetilde{q},a_0,a_1\right)$, respectively. Assume $\sigma(p,q,a_0,a_1) = \sigma\left(p,\widetilde{q},a_0,a_1\right)$, then $\int_0^1 [q-\widetilde{q}]dx = 0$.*

Proof. By the hypotesis $\sigma(p,q,a_0,a_1) = \sigma\left(p,\widetilde{q},a_0,a_1\right)$, then it follows $\lambda_n \in \sigma(p,q,a_0,a_1) = \sigma\left(p,\widetilde{q},a_0,a_1\right)$. Let's consider the problems $E(p,q,a_0,a_1)$ and $E(p,\widetilde{q},a_0,a_1)$, multiply the first equation by $\widetilde{y}$, second by y subracting them after integration on $[0,\pi]$;

$$\int_0^\pi [q-\widetilde{q}]y\widetilde{y}dx = \left(\widetilde{y}'y - y'\widetilde{y}\right)\Big|_0^\pi,$$

using the conditions (1) and (4) as $y'(\pi) = -(a_0 + a_1\lambda)y(\pi)$, $\widetilde{y}'(\pi) = -(a_0 + a_1\lambda)\widetilde{y}(\pi)$ and inserting in above, we obtain that $\int_0^\pi [q-\widetilde{q}]y\widetilde{y}dx = 0$.

Let multiply y and $\widetilde{y}$ in (5) and using some trigonometric identities, we can get easily see that

$$\frac{1}{2}\int_0^\pi (q-\widetilde{q})\,dx + \int_0^\pi (q-\widetilde{q})\cos(\lambda x - \alpha(x))\,dx +$$

$$\int_0^\pi (q-\widetilde{q})\int_0^s H(s,\tau)\cos 2\left[\lambda\tau - \alpha(\tau)\right]d\tau dx = 0,$$

where $H(s,t)$ depends on $A(x,t)$, $B(x,t)$, $\frac{\partial}{\partial x,t}A(x,t)$ and $\frac{\partial}{\partial x,t}B(x,t)$,then the first and second terms goes to zero as $\lambda \to \infty$ because of the Riemann-Lebesque lemma. This completes the proof.

Theorem 4. *Assume that $p,q \in C[0,\pi]$, and $\sigma(p,q,a_0,a_1)$, $\sigma(0,p,a_0,a_1)$. Then $q(x) = 0$ on $[0,\pi]$.*

Proof. By assumption Theorem 2, we obtain that

$$\int_0^{\pi} q(x)dx = 0.$$

The rest of proof is the same as in [18]. Then, this completes the proof.

3 Conclusion

In this short note, we solve inverse problem for Sturm-Liouville problem energy dependent potential containing the spectral parameter in boundary condition. We note that, results are more general then the results obtained in [18].

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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