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Non-polynomial spline method for solving the generalized regularized long wave equation

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Abstract: In this paper, non-polynomial spline method for solving the generalized regularized long wave (GRLW) equation are presented. In this paper, we take deferent spline functions. The stability analysis using Von-Neumann technique shows the scheme is marginally stable. To test accuracy the error normsL_2, L_∞ are computed. Also, the change in conservation quantities are evaluated which are found to be very small. To illustrate the applicability and efficiency of the basis, we compare obtained numerical results with other existing recent methods. Moreover, interaction two and three solitary waves are shown. The development of the Maxwellian initial condition into solitary waves is also shown and we show that the number of solitons which are generated from the Maxwellian initial condition can be determined.

Keywords: Non-polynomial spline, generalized regularized long wave equation, solitary waves, solitons.

1 Introduction

Nowadays, the solitary wave and in particular soliton has been becoming a hot and attractive topic and is studied both theoretically and experimentally [1-36]. A soliton is a very special type of a solitary wave, which is of permanent form, is localized within a region, and can interact with other solitons and emerge from the collision unchanged, except for a phase shift.

GRLW equation is a nonlinear evolution equation of the form

$$u_t + u_x + p(p+1)u^p u_x - \mu u_{xxt} = 0, \tag{1}$$

which is useful in describing various phenomena in science and engineering. It is also one of the weakly nonlinear dispersive partial differential equations which have many applications in several areas, e.g., ion-acoustic waves in plasma, magneto hydro dynamics waves in plasma, longitudinal dispersive waves in elastic rods, pressure waves in liquid-gas bubble mixtures, and rotating flow down a tube. The GRLW equation is studied by few authors, Mokhtari used Sinc-collocation [17], Kaya used a numerical simulation of solitary wave solutions [18], El-Danaf et al, used Adomian decomposition method (ADM) [19] and Thoudam Roshan used a petrov-Galerkin method [20], Mohammadi used the basis of a reproducing kernel space [21] and Zhang used finite difference method for a Cauchy problem [22]. Applying non- polynomial spline functions to solve some partial differential equations does not regard as a new subject, as someone can pursue this subject in the literature, e.g., using NPS in solving Burgers' equation, cubic nonlinear

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Schrödinger equation, nonlinear Klein–Gordon equation, variable coefficient fourth-order wave equations, and Bratu's problem [23-26]. However, according to our knowledge, there are not yet any publications relevant to applying NPS for solving GRLW equations. Perhaps, the existence of u_t term together with u_{xxt}term in the GRLW equation incommodes numerical analysts to design such methods. In the next section, with the aid of the well-known Cranck–Nicolson scheme, we can apply non-polynomial spline functions to develop a numerical method for solving the nonlinear GRLW equation. In this paper, we take deferent spline functions as forms.

$$T_3 = \text{span}\{1,\ \text{x},\ \sin(\omega\text{x}),\ \cos(\omega\text{x})\},\quad T_3 = \text{span}\{1,\ \text{x},\ \cosh(\omega\text{x}),\ \sin(\omega\text{x})\},$$

$$T_3 = \text{span}\{1,\ \text{x},\ \cosh(\omega\text{x}),\ \sinh(\omega\text{x})\},\quad T_3 = \text{span}\{1,\ \text{x},\ \tanh(\omega\text{x}),\ \text{sech}(\omega\text{x})\},$$

and $T_3 = \text{span}\{1,\ \text{x},\ e^{\theta},\ e^{-\theta}\}$, where ω is the frequency of the trigonometric part of the spline functions which will be used to raise the accuracy of the method. The purpose of this paper is to present fourier stability analysis of the linearized scheme shows that it is unconditionally stable. Also, the local truncation error of the method is investigated. The interaction of solitary waves and other properties of the GRLW equation are studied. The development of the Maxwellian initial condition into solitary waves is also shown and we show that the number of solitons which are generated from the Maxwellian initial condition can be determined.

2 The problem and analytical solution

The GRLW Eq. 1 can be written it in this form [20]

$$u_t + u_x + \varepsilon u^p u_x - \mu u_{xxt} = 0, \tag{2}$$

where $\varepsilon = p(p+1)$ and subscripts x and t denote differentiation, is considered with the boundary and initial conditions $u \to 0$ as $x \to \pm\infty$. In this work, the initial and boundary conditions on the region $a \le x \le b$ are assumed in the form:

$$u(a,t) = u(b,t) = 0, \quad t \ge 0, \tag{3}$$

$$u(x,0) = f(x), \quad a \le x \le b, \tag{4}$$

and then the analytical solution of Eq. 2 take the form. [20]

$$u(x,t) = \sqrt[p]{\frac{(p+2)c}{2p}\,\text{sec}h\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x-(c+1)t-x_0)\right)}, \tag{5}$$

where x_0 is an arbitrary constant. It is not always available to get an analytic solution for nonlinear partial differential equations, so we try to provide numerical methods to solve such problems.

2.1 Derivation of the numerical method

In this section we given theoretically discussed for the numerical method using deferent spline functions.

Case 1. We take spline function in this form $T_3 = \text{span}\{1,\ x,\ \sin(\omega x),\ \cos(\omega x)\}$. To set up the non-polynomial spline method, we select an integer $N > 0$ and a time step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_j, t_j) are $x_j = a + j\,h$ and $t_n = n\,k$, where $n = 0, 1, ...,$ and $j = 0, 1, ..., N+1$. Let U_j^n be an approximation to $u(x_j, t_n)$, obtained by the segment $p_j(x, t_n)$ of the mixed spline function passing through the points (x_j, U_j^n) and (x_{j+1}, U_{j+1}^n). Each segment has the form

$$p_j(x,t_n) = a_j(t_n)\cos\omega(x-x_j) + b_j(t_n)\sin\omega(x-x_j) + c_j(t_n)(x-x_j) + d_j(t_n), \tag{6}$$

for each $j = 0, 1, ..., N$. To obtain expressions for the coefficients of Eq. 6 in terms of U_j^n, U_{j+1}^n, S_j^n and S_{j+1}^n which are as follows:

$$U_j^n = p_j(x_j,t_n), U_{j+1}^n = p_j(x_{j+1},t_n), \quad S_j^n = p_j''(x_j,t_n) \quad \text{and} \quad S_{j+1}^n = p_j''(x_{j+1},t_n). \tag{7}$$

Using Eqs. 6 and 7, we get

$$\begin{aligned} &a_j + d_j = U_j^n, \\ &a_j\cos\theta + b_j\sin\theta + c_j h + d_j = U_{j+1}^n, \\ &-a_j\omega^2 + d_j = S_j^n, \\ &-a_j\omega^2\cos\theta - b_j\omega^2\sin\theta = S_{j+1}^n, \end{aligned} \tag{8}$$

where $a_j \equiv a_j(t_n), \quad b_j \equiv b_j(t_n), \quad c_j \equiv c_j(t_n), \quad d_j \equiv d_j(t_n) \quad$ and $\quad \theta = wh$. by solving the last four equations in 8. We obtain expressions for the coefficients as:

$$\begin{aligned} &a_j = \frac{-h^2}{\theta^2}S_j^n, \quad b_j = \frac{h^2(S_j^n\cos\theta - S_{j+1}^n)}{\theta^2\sin\theta}, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{h(S_{j+1}^n - S_j^n)}{\theta^2}, \quad d_j = \frac{h^2}{\theta^2}S_j^n + U_j^n, \\ &a_j = \frac{-h^2}{\theta^2}S_j^n, \quad b_j = \frac{h^2(S_j^n\cos\theta - S_{j+1}^n)}{\theta^2\sin\theta}, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{h(S_{j+1}^n - S_j^n)}{\theta^2}, \quad d_j = \frac{h^2}{\theta^2}S_j^n + U_j^n, \end{aligned} \tag{9}$$

Using the continuity condition of the first derivative at,$x = x_j$ that is $p_j'(x_j, t_n) = p_{j-1}'(x_j, t_n)$, we get the flowing equations:

$$b_j\omega + c_j = -a_{j-1}\omega\sin\theta + b_{j-1}\omega\cos\theta + c_{j-1}, \quad j = 1, ..., N. \tag{10}$$

Using Eq. 9, after slight rearrangements, then Eq. 10 becomes

$$U_{j+1}^n - 2U_j^n + U_{j-1}^n = \alpha S_{j+1}^n + \beta S_j^n + \alpha S_{j-1}^n, \quad j = 1, ..., N, \tag{11}$$

where

$$\alpha = \frac{h^2}{\theta\sin\theta} - \frac{h^2}{\theta^2}, \quad \beta = -\frac{2h^2\cos\theta}{\theta\sin\theta} + \frac{2h^2}{\theta^2} \quad \text{and} \quad \theta = \omega h.$$

Remark. As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha, \beta) \to (\frac{h^2}{6}, \frac{4h^2}{6})$.

Case 2. We take spline function in this form $T_3 = \text{span}\{1,\ x,\ \cosh(\omega x),\ \sin(\omega x)\}$.

To set up the non-polynomial spline method, we select an integer $N > 0$ and a time step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_j, t_n) are $x_i = a + i\,h \quad$ and $\quad t_n = n\,k$, where $n = 0, 1, ...,$ and $j = 0, 1, ..., N+1$. Let U_j^n be an approximation to $u(x_j, t_n)$, obtained by the segment $p_j(x, t_n)$ of the mixed spline function passing through the points (x_j, U_j^n) and (x_{j+1}, U_{j+1}^n). Each segment has the form

$$p_j(x,t_n) = a_j(t_n)\cosh\omega(x-x_j) + b_j(t_n)\sin\omega(x-x_j) + c_j(t_n)(x-x_j) + d_j(t_n), \tag{12}$$

for each $j = 0,1,\ldots,N$. To obtain expressions for the coefficients of Eq. 12 in terms of U_j^n,U_{j+1}^n,S_j^n and S_{j+1}^n which are as follows:

$$U_j^n = p_j(x_j,t_n), U_{j+1}^n = p_j(x_{j+1},t_n), S_j^n = p_j''(x_j,t_n) \quad \text{and} \quad S_{j+1}^n = p_j''(x_{j+1},t_n). \tag{13}$$

Using Eqs. 12 and 13, we get

$$\begin{aligned} &a_j + d_j = U_j^n, \\ &a_j \cosh\theta + b_j \sin\theta + c_j h + d_j = U_{j+1}^n, \\ &a_j \omega^2 = S_j^n, \\ &-a_j\,\omega^2 \cosh\theta - b_j\,\omega^2 \sin\theta = S_{j+1}^n, \end{aligned} \tag{14}$$

where $a_j \equiv a_j(t_n), b_j \equiv b_j(t_n), c_j \equiv c_j(t_n), \quad d_j \equiv d_j(t_n)$ and $\theta = wh$. by solving the last four equations in 14. We obtain expressions for the coefficients as:

$$\begin{aligned} &a_j = \frac{S_j^n}{\omega^2}, \quad b_i = \frac{S_j^n \cosh\theta - S_{j+1}^n}{\omega^2 \sin\theta}, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{S_{j+1}^n + S_j^n - 2S_j^n \cosh\theta}{\omega^2 h}, \quad d_j = \frac{-S_j^n}{\omega^2} + U_j^n, \\ &a_j = \frac{S_j^n}{\omega^2}, \quad b_i = \frac{S_j^n \cosh\theta - S_{j+1}^n}{\omega^2 \sin\theta}, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{S_{j+1}^n + S_j^n - 2S_j^n \cosh\theta}{\omega^2 h}, \quad d_j = \frac{-S_j^n}{\omega^2} + U_j^n, \end{aligned} \tag{15}$$

using the continuity condition of the first derivative at $x = x_j$ that is $p_j'(x_j,t_n) = p_{j-1}'(x_j,t_n)$, we get the flowing equations:

$$b_j\omega + c_j = -a_{j-1}\omega \sinh\theta + b_{j-1}\omega \cos\theta + c_{j-1}, \; j = 1,\ldots,N. \tag{16}$$

Using Eq. 15, after slight rearrangements, then Eq. 16 becomes

$$U_{j+1}^n - 2U_j^n + U_{j-1}^n = \alpha S_{j+1}^n + \beta S_i^n + \gamma S_{j-1}^n, \; j = 1,\ldots,N, \tag{17}$$

where

$$\begin{aligned} &\alpha = \frac{h^2}{\theta \sin\theta} - \frac{h^2}{\theta^2}, \quad \beta = \frac{-h^2 \cos\theta}{\theta \sin\theta} + \frac{2h^2 \cosh\theta}{\theta^2} - \frac{h^2 \cos\theta}{\theta \sin\theta}, \\ &\gamma = \frac{h^2 \sin\theta}{\theta} + \frac{h^2 \cos\theta \cosh\theta}{\theta \sin\theta} + \frac{h^2}{\theta^2} - \frac{2h^2 \cosh\theta}{\theta^2}, \; \mathit{and}\ \theta = \omega h. \end{aligned}$$

Remark. As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha,\beta,\gamma) \to (\frac{h^2}{6}, \frac{4h^2}{6}, \frac{h^2}{6})$.

Case 3. We take spline function in this form $T_3 = \mathrm{span}\{1,\ x,\ \cosh(\omega x),\ \sinh(\omega x)\ \}$.

To set up the non-polynomial spline method, we select an integer $N > 0$ and a time step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_j,t_n) are $x_j = a + j\,h$ and $t_n = n\,k$, where $n = 0,1,\ldots,$ and $j = 0,1,\ldots,N+1$. Let U_j^n be an approximation to $u(x_j,t_n)$, obtained by the segment $p_j(x,t_n)$ of the mixed spline function passing through the points (x_j,U_j^n) and (x_{j+1},U_{j+1}^n). Each segment has the form

$$p_j(x,t_n) = a_j(t_n) \cosh\omega(x-x_j) + b_j(t_n) \sinh\omega(x-x_j) + c_j(t_n)(x-x_j) + d_j(t_n), \tag{18}$$

for each $j = 0,1,...,N$. To obtain expressions for the coefficients of Eq. 18 in terms of U_j^n,U_{j+1}^n,S_j^n and S_{j+1}^n which are as follows:

$$U_j^n = p_j(x_j,t_n), U_{j+1}^n = p_j(x_{j+1},t_n), \quad S_j^n = p_j''(x_j,t_n) \;\; and \;\; S_{j+1}^n = p_j''\,(x_{j+1},t_n). \tag{19}$$

Using Eqs. 18 and 19, we get

$$\begin{aligned} &a_j + d_j = U_j^n, \\ &a_j \cosh\theta + b_j \sinh\theta + c_j h + d_j = U_{j+1}^n, \\ &a_j \omega^2 = S_j^n, \\ &a_j\,\omega^2 \cosh\theta + b_j\,\omega^2 \sinh\theta = S_{j+1}^n, \end{aligned} \tag{20}$$

where $a_j \equiv a_j(t_n), \quad b_j \equiv b_j(t_n), \quad c_j \equiv c_j(t_n), \quad d_j \equiv d_j(t_n)$ and $\theta = wh$. by solving the last four equations in 20 We obtain expressions for the coefficients as:

$$\begin{aligned} &a_j = \tfrac{h^2}{\theta^2} S_j^n, \quad b_j = \tfrac{h^2(-S_j^n \cosh\theta + S_{j+1}^n)}{\theta^2 \sin\theta}, \\ &c_j = \tfrac{U_{j+1}^n - U_j^n}{h} + \tfrac{h(-S_{j+1}^n + S_j^n)}{\theta^2}, \quad d_j = -\tfrac{h^2}{\theta^2} S_j^n + U_j^n, \\ &a_j = \tfrac{h^2}{\theta^2} S_j^n, \quad b_j = \tfrac{h^2(-S_j^n \cosh\theta + S_{j+1}^n)}{\theta^2 \sin\theta}, \\ &c_j = \tfrac{U_{j+1}^n - U_j^n}{h} + \tfrac{h(-S_{j+1}^n + S_j^n)}{\theta^2}, \quad d_j = -\tfrac{h^2}{\theta^2} S_j^n + U_j^n, \end{aligned} \tag{21}$$

using the continuity condition of the first derivative at $x = x_j$, that is $p_j'(x_j,t_n) = p_{j-1}'(x_j,t_n)$, we get the flowing equations:

$$b_j\omega + c_j = -a_{j-1}\omega \sinh\theta + b_{j-1}\omega \cosh\theta + c_{j-1}, \; j = 1,...,N. \tag{22}$$

Using Eq. 21, after slight rearrangements, then Eq. 22 becomes

$$U_{j+1}^n - 2U_j^n + U_{j-1}^n = \alpha S_{j+1}^n + \beta S_j^n + \alpha S_{j-1}^n, \quad j = 1,...,N, \tag{23}$$

where

$$\alpha = -\frac{h^2}{\theta \sinh\theta} + \frac{h^2}{\theta^2}, \; \beta = \frac{2h^2 \cosh\theta}{\theta \sinh\theta} - \frac{2h^2}{\theta^2}, and\, \theta = \omega h.$$

Remark. As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha,\beta) \to (\frac{h^2}{6}, \frac{4h^2}{6})$.

Case 4. We take spline function in this form $T_3 = \mathrm{span}\{1,\ \mathrm{x},\ \tanh(\omega \mathrm{x}),\ \mathrm{sech}(\omega \mathrm{x})\}$. To set up the non-polynomial spline method, we select an integer $N > 0$ and a time step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_j,t_n) are $x_j = a + j\,h$ and $t_n = n\,k$, where $n = 0,1,...,$ and $j = 0,1,...,N+1$. Let U_j^n be an approximation to $u(x_j,t_n)$, obtained by the segment $p_j(x,t_n)$ of the mixed spline function passing through the points (x_j,U_j^n) and (x_{j+1},U_{j+1}^n). Each segment has the form

$$p_j(x,t_n) = a_j(t_n) \tanh\omega(x - x_j) + b_j(t_n)\,\mathrm{sec}\,h\omega(x - x_j) + c_j(t_n)(x - x_j) + d_j(t_n), \tag{24}$$

for each $j = 0,1,...,N$. To obtain expressions for the coefficients of Eq. 24 in terms of U_j^n,U_{j+1}^n, S_j^n and S_{j+1}^n which are as follows:

$$U_j^n = p_j(x_j,t_n), U_{j+1}^n = p_j(x_{j+1},t_n), S_j^n = p_j''(x_j,t_n) \;\; and \;\; S_{j+1}^n = p_j''\,(x_{j+1},t_n). \tag{25}$$

Using Eqs. 24 and 25, we get

$$\begin{aligned} &a_j + d_j = U_j^n, \\ &a_j \tanh\theta + b_j \sec h\theta + c_j h + d_j = U_{j+1}^n, \\ &-b_j\omega^2 = S_j^n, \\ &-2a_j\omega^2 \sec h^2\theta \tanh\theta - b_j\omega^2 \sec h^3\theta + b_j\omega^2 \sec h\theta \tanh^2\theta = S_{j+1}^n, \end{aligned} \tag{26}$$

where $a_j \equiv a_j(t_n), b_j \equiv b_j(t_n), c_j \equiv c_j(t_n)$, $d_j \equiv d_j(t_n)$ *and* $\theta = wh$. by solving the last four equations in 26.

$$\begin{aligned} &b_j = \frac{-h^2}{\theta^2}S_j^n,\ a_i = \frac{h^2(S_j^n \sec h^3\theta - S_j^n \sec h\theta \tanh^2\theta - S_{j+1}^n)}{2\theta^2 \sec h^2\theta \tanh\theta},\ d_j = \frac{h^2}{\theta^2}S_j^n + U_j^n, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{h(S_j^n \sec h\theta - S_j^n)}{\theta^2} + \frac{h(S_j^n \sec h^3\theta + S_j^n \sec h\theta \tanh^2\theta + S_{j+1}^n)}{2\theta^2 \sec h^2\theta}, \\ &b_j = \frac{-h^2}{\theta^2}S_j^n,\ a_i = \frac{h^2(S_j^n \sec h^3\theta - S_j^n \sec h\theta \tanh^2\theta - S_{j+1}^n)}{2\theta^2 \sec h^2\theta \tanh\theta},\ d_j = \frac{h^2}{\theta^2}S_j^n + U_j^n, \\ &c_j = \frac{U_{j+1}^n - U_j^n}{h} + \frac{h(S_j^n \sec h\theta - S_j^n)}{\theta^2} + \frac{h(S_j^n \sec h^3\theta + S_j^n \sec h\theta \tanh^2\theta + S_{j+1}^n)}{2\theta^2 \sec h^2\theta}, \end{aligned} \tag{27}$$

We obtain expressions for the coefficients as: using the continuity condition of the first derivative at $x = x_j$, that is $p'_j(x_j,t_n) = p'_{j-1}(x_j,t_n)$, we get the flowing equations:

$$a_j\omega + c_j = a_{j-1}\omega \sec h^2\theta - b_{j-1}\omega \sec h\theta \tanh\theta + c_{j-1}, \qquad j = 1,...,N. \tag{28}$$

Using Eq. 27, after slight rearrangements, then Eq. 28 becomes

$$U_{j+1}^n - 2U_j^n + U_{j-1}^n = \alpha S_{j+1}^n + \beta S_j^n + \gamma S_{j-1}^n, \quad j = 1,...,N, \tag{29}$$

where

$$\begin{aligned} &\alpha = \frac{h^3\omega}{2\theta^2 \sec h^2\theta \tanh\theta} - \frac{h^2}{2\theta^2 \sec h^2\theta}, \\ &\beta = \frac{-h^3\omega \sec h^3\theta}{2\theta^2 \sec h^2\theta \tanh\theta} + \omega \sec h\theta \tanh^2\theta 2\theta^2 \sec h^2\theta \tanh\theta + \frac{h^2}{\theta^2} \\ &- \frac{h^2 \sec h\theta}{2\theta^2} + \frac{h^2 \sec h^3\theta}{2\theta^2} - \frac{h^2 \sec h\theta \tanh^2\theta}{2\theta^2 \sec h^2\theta} - \frac{h^2\omega}{2\theta^2 \tanh\theta} + \frac{h^2}{2\theta^2 \sec h^2\theta}, \\ &\gamma = \frac{h^3\omega(\sec h^3\theta - \sec h\theta \tanh^2\theta)}{2\theta^2 \tanh\theta} + \frac{h^3\omega \sec h\theta \tanh\theta}{\theta^2} + \frac{h^2(\sec h\theta - 1)}{\theta^2} + \frac{h^2(-\sec h^3\theta + \sec h\theta \tanh^2\theta)}{2\theta^2 \sec h^2\theta}, \\ &\textit{and} \\ &\theta = \omega h. \end{aligned}$$

Remark. As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha,\beta,\gamma) \to (\frac{h^2}{6}, \frac{4h^2}{6}, \frac{h^2}{6})$.

Case 5. We take spline function in this form $T_3 = \mathrm{span}\{1,\ x,\ e^{\theta},\ e^{-\theta}\}$.

To set up the non-polynomial spline method, we select an integer $N > 0$ and a time step size $k > 0$. With $h = \frac{b-a}{N+1}$, the mesh points (x_j,t_n) are $x_j = a + j\,h$ and $t_n = n\,k$, where $n = 0,1,...,$ and $j = 0,1,...,N+1$. Let U_j^n be an approximation to$u(x_j,t_n)$, obtained by the segment $p_j(x,t_n)$ of the mixed spline function passing through the points (x_j,U_j^n) and

(x_{j+1}, U^n_{j+1}). Each segment has the form

$$p_j(x,t_n) = a_j(t_n)e^{\omega(x-x_j)} + b_j(t_n)e^{-\omega(x-x_j)} + c_j(t_n)(x-x_j) + d_j(t_n), \tag{30}$$

for each $j = 0,1,...,N$. To obtain expressions for the coefficients of Eq. 30 in terms ofU^n_j,U^n_{j+1},S^n_j and S^n_{j+1} which are as follows:

$$U^n_j = p_j(x_j,t_n), U^n_{j+1} = p_j(x_{j+1},t_n), S^n_j = p''_j(x_j,t_n) \ and \ S^n_{j+1} = p''_j(x_{j+1},t_n). \tag{31}$$

Using Eqs. 30 and 31, we get

$$\begin{aligned} &a_j + b_j + d_j = U^n_j, \\ &a_j e^{\theta} + b_j e^{-\theta} + c_j h + d_j = U^n_{j+1}, \\ &a_j\omega^2 + b_j\omega^2 = S^n_j, \\ &a_j\omega^2 e^{\theta} + b_j\omega^2 e^{-\theta} = S^n_{j+1}, \end{aligned} \tag{32}$$

where $a_j \equiv a_j(t_n), b_j \equiv b_j(t_n), c_j \equiv c_j(t_n)$, $d_j \equiv d_j(t_n)$ and $\theta = wh$. by solving the last four equations in 32. We obtain expressions for the coefficients as:

$$\begin{aligned} a_j &= \frac{h^2(S^n_j e^{-\theta} - S^n_{j+1})}{\theta^2(e^{-\theta} - e^{\theta})}, \ b_j = \frac{h^2(-S^n_j e^{\theta} + S^n_{j+1})}{\theta^2(e^{-\theta} - e^{\theta})}, \\ c_j &= \frac{U^n_{j+1} - U^n_j}{h} + \frac{h(-S^n_{j+1} + S^n_j)}{\theta^2}, \ d_j = -\frac{h^2}{\theta^2}S^n_j + U^n_j, \end{aligned} \tag{33}$$

Using the continuity condition of the first derivative at $x = x_j$, that is $p'_j(x_j,t_n) = p'_{j-1}(x_j,t_n)$, we get the flowing equations:

$$a_j\omega - b_j\omega + c_j = a_{j-1}\omega e^{\theta} - b_{j-1}\omega e^{-\theta} + c_{j-1}, \ j = 1,...,N. \tag{34}$$

Using Eq. 33, after slight rearrangements, then Eq. 34 becomes

$$U^n_{j+1} - 2U^n_j + U^n_{j-1} = \alpha S^n_{j+1} + \beta S^n_j + \alpha S^n_{j-1}, \ j = 1,...,N, \tag{35}$$

where

$$\alpha = \frac{2h^3\omega}{\theta^2(e^{-\theta} - e^{\theta})} + \frac{h^2}{\theta^2}, \ \beta = \frac{2h^3\omega(-e^{-\theta} - e^{\theta})}{\theta^2(e^{-\theta} - e^{\theta})} - \frac{2h^2}{\theta^2}, and \theta = \omega h.$$

Remark. As $\omega \to 0$, that is $\theta \to 0$, then $(\alpha,\beta) \to (\frac{h^2}{6}, \frac{4h^2}{6})$.

After we studied the five cases we show that all five cases imply that the same equation and the same values of (α,β).

i.e. The equations 11, 17, 23, 29 and 35 are same equation in this form

$$U^n_{j+1} - 2U^n_j + U^n_{j-1} = \alpha S^n_{j+1} + \beta S^n_j + \alpha S^n_{j-1}, \quad j = 1,...,N, \tag{36}$$

where $(\alpha,\beta) \to (\frac{h^2}{6}, \frac{4h^2}{6})$.

Then we consider Eq. 36 at two-time level n, $n+1$ and subtract them to obtain the following relation

$$(U_{j+1}^{n+1}-U_{j+1}^{n})-2(U_{j}^{n+1}-U_{i}^{n})+(U_{j-1}^{n+1}-U_{j-1}^{n})=\alpha(S_{j+1}^{n+1}-S_{j+1}^{n})+\beta(S_{j}^{n+1}-S_{j}^{n})+\alpha(S_{j-1}^{n+1}-S_{j-1}^{n}),\ j=1,...,N. \tag{37}$$

On the other hand, we rewrite Eq. 2 as

$$\frac{\partial}{\partial t}\left(\frac{\partial u(x,t)}{\partial x^2}-\frac{1}{\mu}u(x,t)\right)=\frac{1}{\mu}\left(\frac{\partial u(x,t)}{\partial x}+\varepsilon(u(x,t))^p\frac{\partial u(x,t)}{\partial x}\right),$$

and follow the famous Cranck–Nicolson scheme to derive

$$\left(\frac{\partial u(x_j,t_{n+1})}{\partial x^2}-\frac{1}{\mu}u(x_j,t_{n+1})\right)-\left(\frac{\partial u(x_j,t_n)}{\partial x^2}-\frac{1}{\mu}u(x_j,t_n)\right)\cong\frac{k}{4\mu h}\left(1+\varepsilon(u(x_j,t_{n+1}))^p\right)\left(u(x_{j+1},t_{n+1})-u(x_{j-1},t_{n+1})\right)$$
$$+\frac{k}{4\mu h}\left(1+\varepsilon(u(x_j,t_n))^p\right)\left(u(x_{j+1},t_n)-u(x_{j-1},t_n)\right).$$

From this equation, the following difference equation can be extracted:

$$S_j^{n+1}-S_j^n=\frac{1}{\mu}(U_j^{n+1}-U_j^n)-r(1+\varepsilon(U_j^{n+1})^p)(U_{j+1}^{n+1}-U_{j-1}^{n+1})+r(1+\varepsilon(U_j^n)^p)(U_{j+1}^n-U_{j-1}^n), \tag{38}$$

where $r=\frac{k}{4\mu h}$ and $j=1,...,N$. Substituting Eq. 38 in Eq. 37 and doing some calculations, we get

$$A_jU_{j-1}^{n+1}+B_jU_j^{n+1}+C_jU_{j+1}^{n+1}=D_jU_{j-1}^n+E_jU_j^n+F_jU_{j+1}^n, \tag{39}$$

where

$$\begin{aligned}
A_j&=1-\tfrac{\alpha}{\mu}+2\alpha r(1+\varepsilon(U_{j-1}^{n+1})^p)+\beta r(1+\varepsilon(U_j^{n+1})^p),\\
B_j&=-2-\tfrac{\beta}{\mu}+2\alpha\varepsilon r((U_{j+1}^{n+1})^p-(U_{j-1}^{n+1})^p),\\
C_j&=1-\tfrac{\alpha}{\mu}-2\alpha r(1+\varepsilon(U_{j+1}^{n+1})^p)-\beta r(1+\varepsilon(U_j^{n+1})^p),\\
D_j&=1-\tfrac{\alpha}{\mu}-2\alpha r(1+\varepsilon(U_{j-1}^{n})^p)-\beta r(1+\varepsilon(U_j^{n})^p),\\
E_j&=-2-\tfrac{\beta}{\mu}-2\alpha\varepsilon r((U_{j+1}^{n})^p-(U_{j-1}^{n})^p),\\
F_j&=1-\tfrac{\alpha}{\mu}+2\alpha r(1+\varepsilon(U_{j+1}^{n})^p)+\beta r(1+\varepsilon(U_j^{n})^p),
\end{aligned}$$

now to solving this system we using initial conditions Eq. 4 to find the value of U_j^0, where $U_j^0=f(x_j)$, for each $j=0,1,...,N+1$. If the procedure is reapplied all the approximation U_j^1 are known, the values of U_j^2, U_j^3, U_j^4,... can be obtained in a similar manner .

2.2 Truncation error

Theorem 1. *The difference 39 has the local truncation error (LTE)* $kh^2(1+k)$.

Proof. Using the difference scheme 39 at $p=3$ we obtain the truncation error

$$T_j^n=A_ju_{j-1}^{n+1}+B_ju_j^{n+1}+C_ju_{j+1}^{n+1}-D_ju_{j-1}^n-E_ju_j^n-F_ju_{j+1}^n, \tag{40}$$

where

$$A_j = 1-\frac{\alpha}{\mu}+2\alpha r(1+\varepsilon(u_{j-1}^{n+1})^3)+\beta r(1+\varepsilon(u_j^{n+1})^3),$$
$$B_j = -2-\frac{\beta}{\mu}+2\alpha\varepsilon r((u_{j+1}^{n+1})^3-(u_{j-1}^{n+1})^3),$$
$$C_j = 1-\frac{\alpha}{\mu}-2\alpha r(1+\varepsilon(u_{j+1}^{n+1})^3)-\beta r(1+\varepsilon(u_j^{n+1})^3),$$
$$D_j = 1-\frac{\alpha}{\mu}-2\alpha r(1+\varepsilon(u_{j-1}^{n})^3)-\beta r(1+\varepsilon(u_j^{n})^3),$$
$$E_j = -2-\frac{\beta}{\mu}-2\alpha\varepsilon r((u_{j+1}^{n})^3-(u_{j-1}^{n})^3),$$
$$F_j = 1-\frac{\alpha}{\mu}+2\alpha r(1+\varepsilon(u_{j+1}^{n})^3)+\beta r(1+\varepsilon(u_j^{n})^3),$$

Expanding this equation in Taylor series in terms $u(x_j,t_n)$ and its derivatives after simplifying, we obtain the following expression

$$T_j^n = (h^2-2\alpha-h^2\alpha-\beta)kD_{xxt}u_j^n+(\frac{h^2}{2}-\frac{h^2\alpha}{2})k^2D_{xxt}u_j^n+frach^2 6-\frac{h^2\alpha}{6})k^3D_{xxttt}u_j^n+\ldots, \tag{41}$$

then the local truncation error of the scheme is O $(kh^2(1+k))$.

2.3 Stability analysis of the method

In this section, the standard Von-Neumann concept is applied to investigate the stability analysis of the scheme. At first, we must linearize the nonlinear term of the GRLW equation by making $\varepsilon(U_j^n)^p$ as a local constant λ. According to the Von- Neumann method, we get

$$U_j^n = \zeta^n \exp(ij\phi), \tag{42}$$

$$g = \frac{\zeta^{n+1}}{\zeta^n},$$

where $\phi = kh$, k is the mode number, $i=\sqrt{-1}$ and g is the amplification factor of the scheme. Substituting Eq. 42 into the difference Eq. 39, we get

$$\zeta^{n+1}(A\exp(i(j-1)\phi)+B\exp(ij\phi)+C\exp(i(j+1)\phi) = \zeta^n(D\exp(i(j-1)\phi)+E\exp(ij\phi)+F\exp(i(j+1)\phi), \tag{43}$$

where

$$A = 1-\frac{\alpha}{\mu}+2\alpha r(1+\lambda)+\beta r(1+\lambda),$$
$$B = -2-\frac{\beta}{\mu},\quad C = 1-\frac{\alpha}{\mu}-2\alpha r(1+\lambda)-\beta r(1+\lambda),$$
$$D = 1-\frac{\alpha}{\mu}-2\alpha r(1+\lambda)-\beta r(1+\lambda),\quad E = -2-\frac{\beta}{\mu},$$
$$F = 1-\frac{\alpha}{\mu}+2\alpha r(1+\lambda)+\beta r(1+\lambda),$$

after simple calculations, Eq. 43 leads to

$$g = \frac{D\exp(-i\phi)+E+F\exp(i\phi)}{A\exp(-i\phi)+B+C\exp(i\phi)},$$

using the well-known Euler's formula, we have

$$g = \frac{X+iY}{X-iY},$$

where $X = (2-\frac{2\alpha}{\mu})\cos\phi-2-\frac{\beta}{\mu}$,and$Y = r(1+\lambda)(2\alpha+\beta)\sin\phi$. Obviously,

$$|g| = 1, \tag{44}$$

then, the linearized scheme is marginally stable.

2.4 Numerical tests and results of GRLW equation

In this section, we present some numerical examples to test validity of our scheme for solving GRLW Eq. 2. For this purpose, we aim to simulate motion of single solitary wave, interaction of two and three solitary waves, wave undulation and propagation of wave with the Maxwellian initial condition.

2.4.1 The motion of single solitary waves

In previous section, we have provided the non-polynomial scheme for the GRLW equation, and we can take the following as an initial condition.

$$u(x,0)=\sqrt[p]{\frac{(p+2)c}{2p}\operatorname{sec}h^2\left(\frac{p}{2}\sqrt{\frac{c}{\mu(c+1)}}(x-x_0)\right)}, \tag{45}$$

The norms L_2-norm and L_∞-norm are used to compare the numerical solution with the analytical solution

$$\begin{aligned} L_2 &= \left\|u^E-u^N\right\| = \sqrt{h\sum_{i=0}^{N}(u_j^E-u_j^N)^2}, \\ L_\infty &= \max_j \left|u_j^E-u_j^N\right|, \ j=0,\ 1,\cdots,N. \end{aligned} \tag{46}$$

Where u^E is the exact solution u and u^N is the approximation solution U_N. and the quantities I_1, I_2 and $I_3$are shown to measure conservation for the schemes.

$$\left.\begin{aligned} I_1 &= \int_{-\infty}^{\infty} u(x,t)dx \cong h\sum_{j=0}^{N} U_j^n, \\ I_2 &= \int_{-\infty}^{\infty}(u^2+\mu u_x^2)dx \cong h\sum_{j=0}^{N}((U^2)_j^n+\mu(U_x^2)_j^n), \\ I_3 &= \int_{-\infty}^{\infty}(u^{p+2}-\tfrac{(p+1)(p+2)\mu}{2\,\varepsilon}u_x^2)dx \cong h\sum_{j=0}^{N}((U^{p+2})_j^n-\tfrac{(p+1)(p+2)\mu}{2\,\varepsilon}(U_x^2)_j^n), \end{aligned}\right\}. \tag{47}$$

Now we consider two test problems.

Problem 1. Now, we consider a test problem where $p=3,\ c=0.1,\ \mu=1, h=0.1,\ x_0=40,\ \Delta t=k=0.1$ with range [0, 80]. The simulations are done up to $t=5$. The changes of the invariants $I_1,\ I_2$ and I_3 are approach to zero in the computer program for the scheme. Errors, also, at $t=5$ are satisfactorily small L_2-error =3.63674 $\times 10^{-4}$ and L_∞-error = 2.06037×10^{-4} for the scheme. Our results are recorded in Table 1 and the motion of solitary wave is plotted at different time levels in Fig. 1.

Problem 2. Now, for comparison, we consider a test problem where$p=3,\ c=1.2,\ \mu=1,\ h=0.1,\ x_0=40, \Delta t=k=0.025$ with range [0,100]. The simulations are done up to $t=2.5$. The changes of the invariants I_1 and I_2 approach to zero and the invariant I_3 changed by less 3.6×10^{-5} in the computer program for the scheme. Errors, also, at $t=2.5$ are satisfactorily small L_2-error =2.38109×10^{-3} and L_∞-error = 1.60851×10^{-3} for the scheme. Our results are recorded in Table 2. The motion of solitary wave is plotted at different time levels in Fig. 2.

In the next table we make comparison between the results of third scheme and the results have been published in Search [20]. We find that our results are related to the results of research [20].

2.4.2 Interaction of two solitary waves

The interaction of two GRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider GRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x,0)=\sum_{j=1}^{2}\sqrt[p]{\frac{(p+2)c_j}{2p}\operatorname{sec}h^2\left(\frac{p}{2}\sqrt{\frac{c_j}{\mu(c_j+1)}}(x-x_j)\right)}, \tag{48}$$

where, $j=1,2,\;\; x_j$ and $c_j$are arbitrary constants. In our computational work.

Now, we choose $c_1=1,\; c_2=0.5,\; x_1=15,\; x_2=35\;,\mu=1,\;\; h=0.1, k=0.1$ with interval [0, 80]. In Fig. 3, the interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 4. The invariants I_1, I_2 and I_3 are changed by less than 1.06×10^{-3}, 3.5×10^{-4} and 3.13×10^{-4}, respectively for the scheme.

2.4.3 Interaction of three solitary waves

The interaction of three GRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the GRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes:

$$u(x,0)=\sum_{j=1}^{3}\sqrt[p]{\frac{(p+2)c_j}{2p}\operatorname{sec}h^2\left(\frac{p}{2}\sqrt{\frac{c_j}{\mu(c_j+1)}}(x-x_j)\right)}, \tag{49}$$

where, $j=1,2,\;3,\; x_j$ and $c_j$are arbitrary constants. In our computational work. Now, we choose $c_1=1, c_2=0.75, c_3=0.5,\; x_1=15, x_2=35,\;\; x_3=45$ with interval [0, 80]. In Fig. 4. The interactions of these solitary waves are plotted at different time levels. We also, observe an appearance of a tail of small amplitude after interaction and the three invariants for this case are shown in Table 5. The invariants $I_1,\; I_2$ and I_3 are changed by less than 1.2×10^{-3}, 1.72×10^{-3} and 1.08×10^{-3}, respectively for the scheme.

2.4.4 The Maxwellian initial condition

In final series of numerical experiments, the development of the Maxwellian initial condition

$$u(x,0)=\exp(-(x-40)^2)\quad , \tag{50}$$

into a train of solitary waves is examined. We apply it to the problem for different cases: (I) $\mu=0.1$, (II)$\mu=0.04$, (III) $\mu=0.01$ and (IV)$\mu=0.005$. When μ is large such as case (I), only single soliton is generated as shown in Fig. 5a. However, when μ is reduced, increasingly solitary waves are formed, since for case (II), two solitary waves are generated as shown in Fig. 5b, and for case (III) the Maxwellian pulse breaks up into a train of at least three solitary waves as shown in Fig. 6a. Finally, for (IV) case, the Maxwellian initial condition has decayed into four stable solitary waves as shown in Fig. 6b. The peaks of the well-developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and we observe a small oscillating tail appearing behind the last wave as shown in the figures 5 and 6, and all states at $t=5$. Moreover, the total number of solitary waves which are generated from the Maxwellian initial condition according to the results obtained from the numerical scheme in test problem as shown in Table 7, can be shown to follow approximately the relation

$$N \cong \left[\frac{1}{\sqrt[4]{\mu}}\right]. \tag{51}$$

3 Conclusions

In this paper, we applied non-polynomial spline functions to develop a numerical method for solving GRLW equation and shown that the schemes are marginally stable. We tested our schemes through a single solitary wave in which the analytic solution is known, then extend it to study the interaction of solitons where no analytic solution is known during the interaction and its accuracy was shown by calculating error norms L_2 and L_∞. The Maxwellian initial condition has been used and a relation between μ and the number of waves was explored.

4 Tables and figures

Table 1: Invariants and errors for single solitary wave $p = 3, c = 0.1, h = 0.1, k = 0.1,$ and $x_0 = 40,\ 0 \le x \le 80.$

t	I_1	I_2	I_3	L_2-norm	L_∞-norm
0	4.06257	1.13382	0.0165781	0.0	0.0
1	4.06257	1.13382	0.0165782	8.15023E-5	5.10793E-5
2	4.06256	1.13382	0.0165782	1.59786E-4	1.01785 E-4
3	4.06255	1.13382	0.0165782	2.32965E-4	1.44328 E-4
4	4.06254	1.13382	0.0165782	3.00723E-4	1.78179E-4
5	4.06252	1.13382	0.0165782	3.63674E-4	2.06037E-4

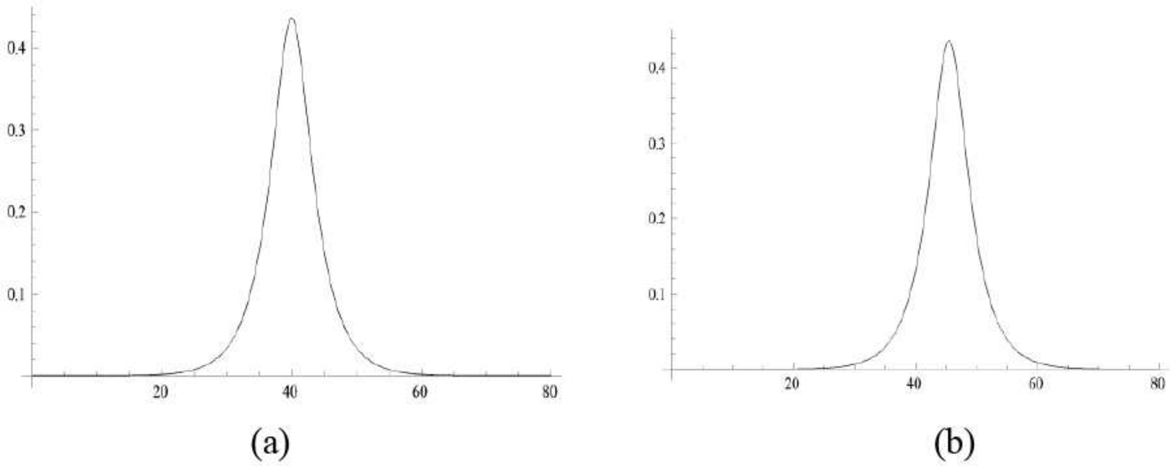


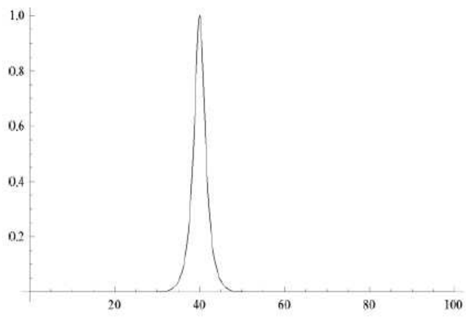
Fig. 1: Single solitary wave with $c = 0.1, h = 0.1,\ k = 0.1andx_0 = 40,\ 0 \le x \le 80, t = 0,\ 5$ respectively

Competing interests

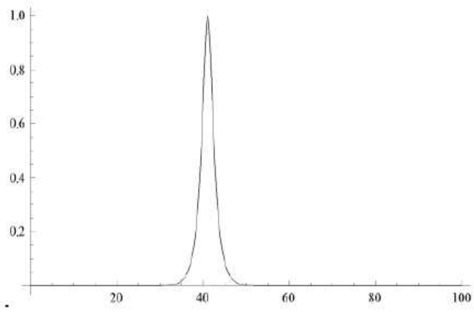
The authors declare that they have no competing interests.

Table 2: Invariants and errors for single solitary wave $p = 3,\ c = 1.2,\ \ h = 0.1,\ k = 0.025$ and $x_0 = 40,\ 0 \le x \le 100$.

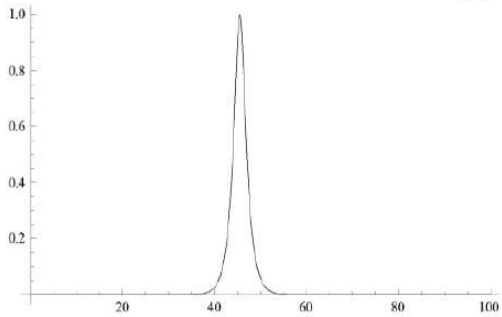
t	I_1	I_2	I_3	L_2-norm	L_∞-norm
0	3.79713	2.87871	0.881628	0.0	0.0
0.5	3.79713	2.87871	0.881615	8.13695E-4	8.09267E-4
1.0	3.79713	2.87871	0.881615	1.39242E-3	1.05441E-3
1.5	3.79713	2.87871	0.881594	1.78371E-3	1.30178E-3
2	3.79713	2.87871	0.881592	2.09581E-3	1.45102E-3
2.5	3.79713	2.87871	0.881592	2.38109E-3	1.60851E-3



(a)



(b)



(c)

Fig. 2: Single solitary wave with $c = 1.2, h = 0.1, k = 0.025$ and $x_0 = 40,\ 0 \le x \le 100, t = 0,\ 1,\ 2$ respectively.

Table 3: Invariants and errors for single solitary wave $c = 1.2, h = 0.1, k = 0.025$ and $x_0 = 40,\ 0 \le x \le 100, t = 2$.

Method	I_1	I_2	I_3	L_2-norm	L_∞-norm
Analytical	3.79713	2.87871	0.881628	0.0	0.0
Our scheme	3.79713	2.87871	0.881592	2.09581E-3	1.45102E-3
[20]	3.79713	2.88122	0.972388	2.13434E-3	1.47042E-3

Table 4: Invariants of interaction two solitary waves of GRLW equation, $c_1 = 1, c_2 = 0.5, x_1 = 15, x_2 = 35, 0 \le x \le 80$.

t	I_1	I_2	I_3
0	7.35999	4.51990	0.847824
2	7.35999	4.51004	0.847634
4	7.35977	4.51014	0.847571
6	7.35958	4.51014	0.847553
8	7.35933	4.51017	0.847563
10	7.35893	4.51025	0.847501

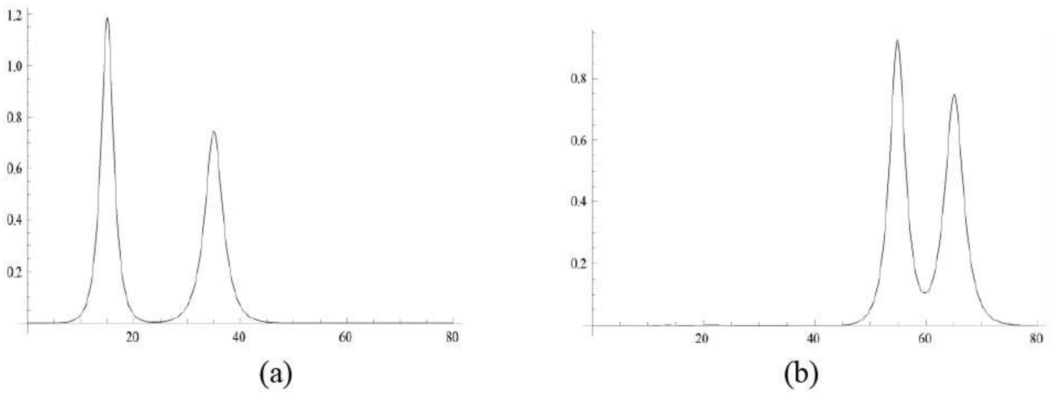


Fig. 3: interaction two solitary waves with
$c_1 = 1,\ c_2 = 0.5,\ x_1 = 15,\ x_2 = 35,\ \ 0 \leq x \leq 80$ **at time** $t = 0,\ \ 20$ **respectively.**

Table 5: Invariants of interaction three solitary waves of GRLW equation. $c_1 = 1,\ c_2 = 0.75,\ c_3 = 0.5,\ x_1 = 15,\ x_2 = 35,\ x_3 = 45,\ 0 \leq x \leq 80$.

t	I_1	I_2	I_3
0	11.0225	6.85159	1.28631
2	11.0223	6.85208	1.28589
4	11.0221	6.85251	1.28563
6	11.0219	6.85276	1.28555
8	11.0216	6.85302	1.28542
10	11.0213	6.85331	1.28523

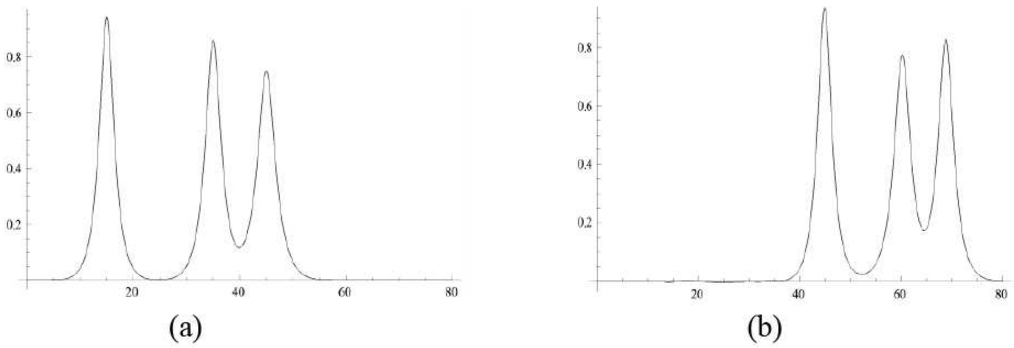


Fig. 4: interaction three solitary waves with
$c_1 = 1, c_2 = 0.75,\ c_3 = 0.5, x_1 = 15,\ \ x_2 = 35, x_3 = 45, 0 \leq x \leq 80$ **at times** $t = 0,\ 5,\ 10,\ 15$ **respectively.**

Authors’ contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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Table 6: The values of the quantities $I_1,\ I_2$ and $I_3$for the cases: $\mu = 0.1$,$\mu = 0.04$,$\mu = 0.01$ and $\mu = 0.005$.

μ	t	I_1	I_2	I_3
0.1	3	1.78293	1.29611	0.712689
	4	1.78287	1.29611	0.712671
	5	1.78283	1.29611	0.712663
0.04	3	1.79657	1.16061	0.720647
	4	1.79671	1.16087	0.720781
	5	1.79682	1.16114	0.720913
0.01	3	1.82568	1.08068	0.655401
	4	1.82537	1.08194	0.656731
	5	1.82437	1.08354	0.657868
0.005	3	1.85124	1.06321	0.630933
	4	1.85535	1.06338	0.632811
	5	1.85773	1.06356	0.633704

Table 7: Solitary Waves Generated from a Maxwellian initial condition.

μ	Number of solitary waves
0.1	1
0.04	2
0.01	3
0.005	4

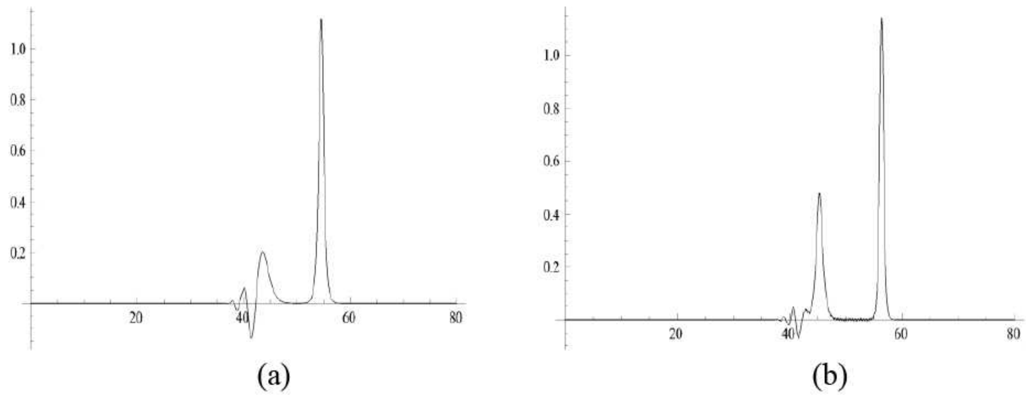

Fig. 5: The Maxwellian initial condition at (a) $\mu = 0.1$, (b) $\mu = 0.04$ and $t = 5$.

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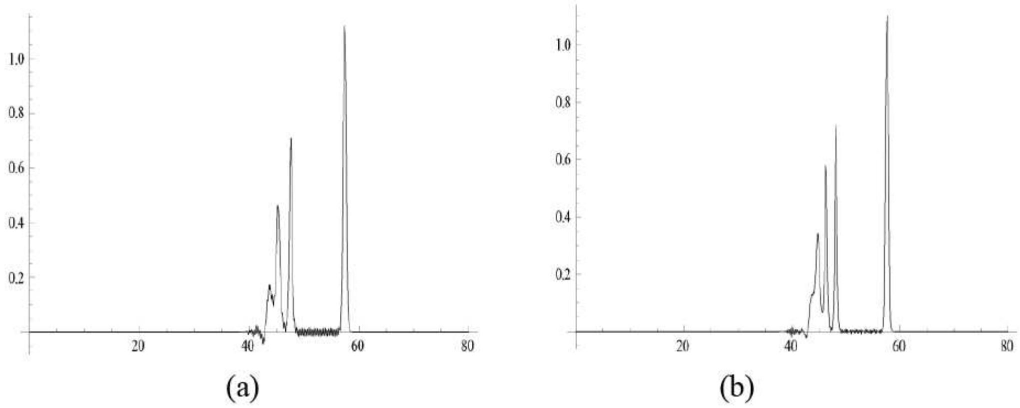

Fig. 6: The Maxwellian initial condition at (a) $\mu = 0.01$, (b) $\mu = 0.005$ and $t = 5$.

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