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Fixed point approach to Bagley Torvik problem

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Abstract: In the present paper, a sufficient condition for existence and uniqueness of Bagley Torvik problem is obtained. The theorem on existence and uniqueness is established. This approach permits us to use fixed point iteration method to solve problem for differential equation involving derivatives of nonlinear order.

Keywords: Fixed Point, initial value problem, Bagley-Torvik equation, Riemann-Liouville derivative.

1 Introduction

It is well known that differential equations involving derivatives of non-integer order are used in modelling of various physical phenomena in areas like diffusion processes, damping laws, etc. (see *[1]*-*[7]*). Methods of solutions of problems for fractional differential equations have been studied extensively by many researchers (see *[1]*-*[12]* and references therein).

Let us give definitions of fractional derivatives and fractional powers of positive operators that will be needed below *[11]*.

Definition 1. *If $x(t) \in C([a,b])$ and $a < t < b$, then*

$$I_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{x(s)}{(t-s)^{1-\alpha}} ds,$$

where $\alpha \in (-\infty, \infty)$ is called the Riemann-Liouville fractional integral of order α. In the same manner for $\alpha \in (0,1)$

$$D_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{x(s)}{(t-s)^{\alpha}} ds,$$

is called the Riemann-Liouville fractional derivative of order α.

Note that if $x(a) = 0$, then we can write

$$D_{a+}^{\alpha} x(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{x'(s)}{(t-s)^{\alpha}} ds,$$

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here,

$$\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s} ds \qquad (\alpha > 0).$$

Consider initial value problem for Bagley-Torvik equation

$$\begin{cases} \frac{d^2x(t)}{dt^2} + D_{a+}^{\alpha} x(t) + a(t)x(t) = f(t),\ 0 < t < T, \\ 0 < \alpha < 1, \\ x(0) = 0, x'(0) = 0. \end{cases}$$

Actually, fractional differential equation corresponds to the Bagley-Torvik problem [3].

We now shortly describe the organization of the paper. In section 2, we give the basic definitions of fixed point, contraction and the basic concepts we need. In section 3, we obtain a sufficient condition for existence and uniqueness of problem and establish the theorem on existence and uniqueness.

2 Fixed point and contraction

Definition 2. *Let $E = (E, d)$ be a metric space. A fixed point of a mapping $A : E \longrightarrow E$ of set E into itself is an element*

$x \in E$ which is mapped onto itself, that is, $Ax = x$, the image Ax coincides with x.

Here note that the Banach fixed point theorem to be stated below is an existence and uniqueness theorem for fixed points of certain mappings, and it also gives a constructive procedure for obtaining better and better approximations to the solution of the equation

$$x = Ax.$$

Actually, we choose an arbitrary $x_0 \in E$ and determine successively a sequence $\{x_n\}_{n=0}^{\infty}$ defined by the relation

$$x_n = Ax_{n-1}\ ,\ n \in \mathbb{N}. \tag{1}$$

This procedure is called an iteration. Iteration procedure are used in many fields of applied mathematics. Banach's fixed-point theorem gives sufficient conditions for the existence and uniqueness of a fixed point of a class of mappings, called contractions.

Definition 3. *A maping $A : E \longrightarrow E$ is called a contraction on E, if there is a positive real number $c < 1$ such that for all $x, y \in E$*

$$d(Ax, Ay) \le cd(x, y).$$

Theorem 1. [13] *Assume that $E \neq \varnothing$ is complete and let A be a contraction mapping on E. Then A precisely one fixed point.*

For proof of the following result, *[14]* can be looked at.

Corollary 1. *Under the assumptions of Theorem 4, the squence $\{x_n\}_{n=0}^{\infty}$ defined by recursive formula (1) with arbitrary $x_0 \in E$ converges to the unique fixed point x of the mapping A. Error estimates are a pirior estimate*

$$d(x_n, x) \le \frac{c^n}{1-c} d(x_0, x_1),\ n \in \mathbb{N},$$

and a posteriori estimate

$$d(x_n,x) \leq \frac{c}{1-c} d(x_{n-1},x_n),\ n \in \mathbb{N}.$$

Now, we state the existence uniqueness theorem the most important application of the Fixed Point Theorem to ordinary differential equations. We will consider the initial value problem of the form

$$\frac{dx(t)}{dt} = f(t,x(t)),\ |t-t_0| \leq a,\ x(t_0) = x_0.$$

The problem for ordinary differential equations will be converted to an integral equation, which defines a mapping A, and the conditions of the theorem will imply that A is a contraction such that its fixed point becomes the solution of problem.

Theorem 2. [13] *Assume that f is continuous on the rectangle*

$$D = \{(t,x) : |t-t_0| \leq a, |x-x_0| \leq b\}$$

and thus bounded on D, i.e,

$$|f(t,x)| \leq k \ \text{for all}\ (t,x) \in D.$$

Suppose that f satisfies a Lipschitz condition on D with respect to its second argument, that is, there is a constant l such that for $(t,x),(t,y) \in D$

$$|f(t,x) - f(t,y)| \leq l\,|x-y|.$$

Then, initial value problem has a unique solution x defined on $|t-t_0| \leq \beta$, where

$$\beta < \min\left\{a, \frac{b}{k}, \frac{1}{l}\right\}.$$

This function x is the limit of iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$x_n(t) = x_0 + \int_{t_0}^{t} f(s,x_{n-1}(s))\ ds,\ n \in \mathbb{N},$$

where $x_0(t)$ is an arbitrary continuous function. Error bounds are

$$d(x_n,x) \leq \frac{c^n}{1-c} e^{2cL} d(x_0,x_1),$$
$$d(x_n,x) \leq \frac{c}{1-c} e^{2cL} d(x_{n-1},x_n), \quad n \in \mathbb{N},$$

where $c = l\beta$.

3 Main results

Definition 4. *Let $C^1_{[0,T]}$ be the complete space of all continuously differentiable functions defined on the interval $[0,T]$ with the metric d defined by*

$$d(x,y) = \max_{t\in[0,T]} |x(t)-y(t)| + \max_{t\in[0,T]} \left|\frac{dx(t)}{dt} - \frac{dy(t)}{dt}\right|. \tag{2}$$

Now, we consider the following initial value problem for Bagey-Torvik equation.

$$\begin{cases} \frac{d^2x(t)}{dt^2} + D^{\alpha}_{a+}x(t) + a(t)x(t) = f(t) \\ 0 < t < T, \quad 0 < \alpha < 1 \\ x(0) = 0, \quad x'(0) = 0 \end{cases} \tag{3}$$

Theorem 3. *Let f and a functions are continuous on*

$$P = \{(t,x,y) : t \in [0,T], \ |x - x_0| < \infty, \ |y - y_0| < \infty\} \subseteq \mathbb{R}^3,$$

and $|f(t)| \leq k$, $|a(t)| \leq m$ such that $k > 0$ and $m > 0$ for all $t \in [0,T]$. Moreover, f satisfies a Lipschitz condition on P with respect to its second and third arguments, i.e, there is a pozitive constant l such that for arbitrary $(t,x,u), (t,y,w) \in P$

$$|f(t,x,u) - f(t,y,w)| \leq l(|x-y| + |u-w|) \tag{4}$$

is valid. Moreover, let

$$g(\alpha,T,l,m) = \frac{T^{-\alpha+3}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)(-\alpha+3)} + \frac{mT^2}{2} + Tl$$

and suppose that

$$g(\alpha,T,l,m) < 1. \tag{5}$$

Then, Bagley-Torvik problem (3) has a unique solution $x \in C^1_{[0,T]}$.

Proof. By integrating both sides of Bagley-Torvik equation (3), we obtain integral equation

$$\int_0^v \frac{d^2x(p)}{dt^2}dp + \int_0^v D^{\alpha}_0 x(p)\,dp + \int_0^v a(p)x(p)\,dp = \int_0^v f(p,x(p),u(p))\,dp$$

$$\left.\frac{dx(p)}{dt}\right|_0^v + \int_0^v D^{\alpha}_0 x(p)\,dp + \int_0^v a(p)x(p)\,dp = \int_0^v f(p,x(p),u(p))\,dp$$

$$\frac{dx(v)}{dt} + \int_0^v D^{\alpha}_0 x(p)\,dp + \int_0^v a(p)x(p)\,dp = \int_0^v f(p,x(p),u(p))\,dp$$

$$\int_0^t \frac{dx(v)}{dt}dv + \int_0^t\int_0^v D^{\alpha}_0 x(p)\,dp\,dv + \int_0^t\int_0^v a(p)x(p)\,dp\,dv = \int_0^t\int_0^v f(p,x(p),u(p))\,dp\,dv$$

$$x(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^v\int_0^p (p-s)^{-\alpha}x'(s)\,ds\,dp\,dv - \int_0^t\int_0^v a(p)x(p)\,dp\,dv + \int_0^t\int_0^v f(p,x(p),u(p))\,dp\,dv. \tag{6}$$

This function x is the limit of the iterative sequence $\{x_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$x_n(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^v\int_0^p (p-s)^{-\alpha}x'_{n-1}(s)\,ds\,dp\,dv - \int_0^t\int_0^v a(p)x_{n-1}(p)\,dp\,dv + \int_0^t\int_0^v f(p,x_{n-1}(p),u(p))\,dp\,dv, n \in \mathbb{N} \tag{7}$$

where $x_0(t)$ is an arbitrary continuous function. Error bounds are

$$d(x_n,x) \leq \frac{g^n}{1-g} d(x_0,x_1),$$

$$d(x_n,x) \leq \frac{g}{1-g} d(x_{n-1},x_n)\ ,\ n \in \mathbb{N}$$

where $g < 1$. We see that initial value problem (3) can be written in the equivalent integral form (6), which is in the form $x = Ax$, where $A : C^1_{[0,T]} \longrightarrow C^1_{[0,T]}$ is an operator defined by

$$Ax(t) = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp\, dv - \int_0^t \int_0^v a(p)x(p)\, dp\, dv + \int_0^t \int_0^v f(p,x(p),u(p))\, dp\, dv, \tag{8}$$

where f is continuous function on P. Under assumptions of theorem, by using (2), we have

$$\begin{aligned}
|Ax(t)| &= \left| -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp\, dv - \int_0^t \int_0^v a(p)x(p)\, dp\, dv + \int_0^t \int_0^v f(p,x(p),u(p))\, dp\, dv \right| \\
&\leq \frac{1}{|\Gamma(1-\alpha)|} \left| \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp\, dv \right| + \left| \int_0^t \int_0^v a(p)x(p)\, dp\, dv \right| + \left| \int_0^t \int_0^v f(p,x(p),u(p))\, dp\, dv \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \left(\int_0^p |(p-s)|^{-\alpha} \left| x'(s) \right| ds \right) dp + \int_0^t \int_0^v |a(p)x(p)|\, dp\, dv + \int_0^t \int_0^v |f(p,x(p),u(p))|\, dp\, dv \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \left(\int_0^p |p-s|^{-\alpha} \max_{s\in[0,T]} \left| x'(s) \right| ds \right) dp + \int_0^t \int_0^v \max_{p\in[0,T]} |x(p)|\ \ |a(p)|\, dp\, dv + \int_0^t \int_0^v |f(p,x(p),u(p))|\, dp\, dv \\
&\leq \frac{\max_{s\in[0,T]} \left| x'(s) \right|}{\Gamma(1-\alpha)} \int_0^t \int_0^v \left(\left. \frac{-(p-s)^{-\alpha+1}}{-\alpha+1} \right|_0^p \right) dp + \max_{p\in[0,T]} |x(p)| \int_0^t \int_0^v m\, dp\, dv + \int_0^t \int_0^v c\, dp\, dv \\
&= \frac{\max_{s\in[0,T]} \left| x'(s) \right|}{\Gamma(1-\alpha)} \frac{t^{-\alpha+3}}{(-\alpha+1)(-\alpha+2)(-\alpha+3)} - \max_{p\in[0,T]} |x(p)|\, m \frac{t^2}{2} + c \frac{t^2}{2}.
\end{aligned}$$

Thus, we hawe show that $Ax \in C^1_{[0,T]}$ if $x \in C^1_{[0,T]}$; *i.e.*,A maps the set $C^1_{[0,T]}$ itself. Now, we show that, A is a contraction map on $C^1_{[0,T]}$. Applying the Lipschitz condition (4), we get

$$\begin{aligned}
|Ax(t) - Ay(t)| = &\left| -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s)\, ds\, dp\, dv - \int_0^t \int_0^v a(p)x(p)\, dp\, dv + \int_0^t \int_0^v f(p,x(p),u(p))\, dp\, dv \right. \\
&\left. + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} y'(s)\, ds\, dp\, dv. - \int_0^t \int_0^v a(p)y(p)\, dp\, dv + \int_0^t \int_0^v f(p,y(p),w(p))\, dp\, dv \right| \\
&\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p |(p-s)|^{-\alpha} \left| x'(s) - y'(s) \right|\ ds\, dp\, dv + \int_0^t \int_0^v |x(p) - y(p)|\, |a(p)|\, dp\, dv \\
&+ \int_0^t |f(p,x(p),u(p)) - f(p,y(p),w(p))|\, dp
\end{aligned}$$

$$\le \frac{1}{\Gamma(1-\alpha)} \max_{s\in[0,T]} \left| x'(s) - y'(s) \right| \int_0^t \int_0^v \int_0^p |(p-s)|^{-\alpha} \, ds\,dp\,dv$$
$$+ \max_{p\in[0,T]} |x(p) - y(p)| \int_0^t \int_0^v |a(p)| \, dp\,dv + \int_0^t |f(p, x(p), u(p)) - f(p, y(p), w(p))| \, dp$$
$$= \frac{1}{\Gamma(1-\alpha)} \max_{s\in[0,T]} \left| x'(s) - y'(s) \right| \frac{t^{-\alpha+3}}{(-\alpha+1)(-\alpha+2)(-\alpha+3)}$$
$$+ \max_{p\in[0,T]} |x(p) - y(p)| \, m\frac{t^2}{2} + lt\left(|x(p) - y(p)| + |u(p) - w(p)|\right)$$
$$\le \frac{t^{-\alpha+3}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)(-\alpha+3)} \, d(x,y) + m\frac{t^2}{2} \, d(x,y) + tl \, d(x,y)$$
$$= \left(\frac{t^{-\alpha+3}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)(-\alpha+3)} + m\frac{t^2}{2} + tl \right) d(x,y).$$

Taking maximum from both sides we have

$$\begin{aligned} d(Ax, Ay) &\le \max_{t\in[0,T]} \left(\frac{t^{-\alpha+3}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)(-\alpha+3)} + m\frac{t^2}{2} + tl \right) d(x,y) \\ &= \left(\frac{T^{-\alpha+3}}{\Gamma(1-\alpha)(-\alpha+1)(-\alpha+2)(-\alpha+3)} + m\frac{T^2}{2} + Tl \right) d(x,y) \\ &\le g(\alpha, T, l, m) \, d(x,y) \end{aligned}$$

From (5) we see that $g(\alpha, T, l, m) < 1$, so

$$d(Ax, Ay) \le g d(x,y).$$

Thus, A is a contraction on $C^1_{[0,T]}$. Therefore, A has a unique fixed point $x \in C^1_{[0,T]}$, that is, a continuous function on $[0,T]$ satisfying $x = Ax$. By (7), we have

$$x(t) = -\frac{1}{\Gamma(1-\alpha)} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s) \, ds\,dp\,dv - \int_0^t \int_0^v a(p) x(p) \, dp\,dv + \int_0^t \int_0^v f(p, x(p), u(p)) \, dp\,dv. \tag{9}$$

Example 1. Solve initial value problem for Bagley-Torvik equation

$$\begin{cases} \frac{d^2x(t)}{dt^2} + D_{0+}^{\frac{1}{2}} x(t) - \frac{8}{3\sqrt{\pi}} t^{-\frac{1}{2}} x(t) = 2, \\ 0 < t < T, \\ x(0) = 0, \quad x'(0) = 0 \end{cases} \tag{10}$$

by the iteration method.

Solution 1. By integrating both sides of Bagley-Torvik equation (10),

$$x(t) = -\frac{1}{\sqrt{\pi}} \int_0^t \int_0^v \int_0^p (p-s)^{-\alpha} x'(s) \, ds\,dp\,dv + \frac{8}{3\sqrt{\pi}} \int_0^t \int_0^v p^{-\frac{1}{2}} x(p) \, dp\,dv + \int_0^t \int_0^v 2 \, dp\,dv$$

Since $a(t) = -\frac{8}{3\sqrt{\pi}}t^{-\frac{1}{2}}$, $f(t) = 2$, $0 < t < T$, $\alpha = \frac{1}{2}$, we have that $l = 1$. Let $x_0(t) = 0$. Now, $x_n(t)$ is defined by formula

$$x_n(t) = -\frac{1}{\Gamma(1-\alpha)}\int_0^t\int_0^v\int_0^p (p-s)^{-\alpha}x'_{n-1}(s)\,ds\,dp\,dv - \int_0^t\int_0^v a(p)\,x_{n-1}(p)\,dp\,dv + \int_0^t\int_0^v f(p, x_{n-1}(p), u(p))\,dp\,dv,$$

for $n \in \mathbb{N}$. Then,

$$\begin{aligned} x_1(t) &= -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\int_0^v\int_0^p (p-s)^{-\frac{1}{2}}x'_0(s)\,ds\,dp\,dv + \frac{8}{3\sqrt{\pi}}\int_0^t\int_0^v p^{-\frac{1}{2}}x_0(p)\,dp\,dv + \int_0^t\int_0^v f(p, x_0(p), u(p))\,dp\,dv \\ &= -\frac{1}{\sqrt{\pi}}\int_0^t\int_0^v\int_0^p (p-s)^{-\frac{1}{2}}\,0\,ds\,dp\,dv + \frac{8}{3\sqrt{\pi}}\int_0^t\int_0^v p^{-\frac{1}{2}}\,0\,dp\,dv + \int_0^t\int_0^v 2\,dp\,dv = t^2, \end{aligned}$$

$$\begin{aligned} x_2(t) &= -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\int_0^v\int_0^p (p-s)^{-\frac{1}{2}}x'_1(s)\,ds\,dp\,dv + \frac{8}{3\sqrt{\pi}}\int_0^t\int_0^v p^{-\frac{1}{2}}x_1(p)\,dp\,dv + \int_0^t\int_0^v f(p, x_1(p), u(p))\,dp\,dv \\ &= -\frac{1}{\sqrt{\pi}}\int_0^t\int_0^v\int_0^p (p-s)^{-\frac{1}{2}}\,2s\,ds\,dp\,dv + \frac{8}{3\sqrt{\pi}}\int_0^t\int_0^v p^{-\frac{1}{2}}p^2\,dp\,dv + \int_0^t\int_0^v 2\,dp\,dv \\ &= -\frac{2^3}{3\sqrt{\pi}}\int_0^t\int_0^v p^{\frac{3}{2}}dp\,dv + \frac{2^3}{3\sqrt{\pi}}\int_0^t\int_0^v p^{\frac{3}{2}}\,dp\,dv + \int_0^t\int_0^v 2\,dp\,dv = t^2, \end{aligned}$$

$$x_3(t) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t\int_0^v\int_0^p (p-s)^{-\frac{1}{2}}x'_2(s)\,ds\,dp\,dv + \frac{8}{3\sqrt{\pi}}\int_0^t\int_0^v p^{-\frac{1}{2}}x_2(p)\,dp\,dv + \int_0^t\int_0^v f(p, x_2(p), u(p))\,dp\,dv = t^2,$$

In similar manner, it can be showed

$$x_n(t) = t^2, \quad n \in \mathbb{N}.$$

Hence,

$$x(t) = \lim_{n\to\infty} x_n(t) = \lim_{n\to\infty} t^2 = t^2.$$

4 Conclusion

In this work, we consider initial value problem for Bagley-Torvik equation. We obtain a sufficient condition for existence and uniqueness of this problem and establish the theorem on existence and uniqueness. This approach permit us to use fixed point iteration method to solve problem for differential equation involving derivatives of nonlinear order.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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