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Bounds of Coefficient Functional of Certain Subclasses of Analytic Functions Associated with Cardioid Domain

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Abstract. In the present paper with the aid of subordination, the authors introduce a new subclass of univalent functions namely; starlike functions with respect to symmetric points linked with cardioid domain defined by $S^{**}_{s,e} := \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z)-f(-z)} \prec 1 + ze^z =: p(z) \right\}$, where the function $p(z)$ maps unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ onto a cardioid domain in the right half plane. We investigate the sharp upper bounds of some of the initial coefficients, Fekete-Szegö functional and Hankel determinant involving initial coefficients of function f for the class $S^{**}_{s,e}$. Further, we determine some of the sharp bounds of logarithmic inverse coefficients, Hankel, Toeplitz, Hermitian-Toeplitz determinant, Zalcman functional, Kruskal inequality as well as the lower and upper bounds for modulo difference of second and the first logarithmic inverse coefficient for such family. Also we obtained some of our results are sharp and respective extremal functions are mentioned.

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1. Introduction and Motivation

Let $\mathcal{A}$ denote the class of functions f which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ having normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Then, the function f can admits Taylor-Maclaurin's series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

The subclass of $\mathcal{A}$ that consists of analytic and univalent functions in the open unit disk $\mathbb{D}$ is denoted by $\mathcal{S}$. In 1959, Sakaguchi [29] introduced the class of starlike functions with respect to symmetric point as:

$$S^*_s := \left\{ f \in \mathcal{S} : \Re\left(\frac{zf'(z)}{f(z) - f(-z)} \right) > 0; z \in \mathbb{D} \right\}.$$

These functions are also known as Sakaguchi functions which are close-to-convex and hence univalent. In 2004, making use of subordination between two analytic functions Ravichandran [25] introduced unified class $S^*_s(\phi)$ as:

$$S^*_s(\phi) = \left\{ f \in \mathcal{S} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z) \quad (z \in \mathbb{D}) \right\},$$

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where ϕ is univalent starlike function with respect to 1 which maps $\mathbb{D}$ onto a symmetric region with respect to real axis in the right half plane. For details, see [11] and reference within.
Motivated by aforementioned works, we introduce the following subclass of the class S_s^*.

Definition 1.1. A function $f \in \mathcal{S}$ given by (1.1) is said to be the member of the class $S_{s,e}^{**}$ if the following subordination condition holds:

$$\frac{2zf'(z)}{f(z)-f(-z)} \prec 1 + ze^z = p(z) \quad (z \in \mathbb{D}),$$

where the function p maps the unit disk $\mathbb{D}$ onto a cardioid domain in the right half plane.

Remark 1.2. The left hand side of the subordination appeared in the definition of starlike functions with respect to symmetrical points. Hence, because the right hand side of this subordination is a function with real positive part, it follows that these classes are subclasses of the starlike functions with respect two symmetrical points.

Sivaprasad Kumar and Gangania [16] studied radius of convexity and inclusion relation for the class of starlike functions related to cardioid domain. Also, Shi et. al. [31] determined the sharp bounds of coefficient functionals related to the Carathéodory functions and investigated the initial coefficient bounds and Fekete-Szegö functional on a subclass of bounded turning functions associated with cardioid domain.

We would like to emphasize here that the class $S_{s,e}^{**}$ is not empty. First, we have to show that the function $p(z) := 1 + ze^z$ is also correctly choose because $p'(z) = (1+z)e^z$, hence $p'(0) = 1 \neq 0$. Also, it's easy to see that

$$\operatorname{Re}\frac{zp'(z)}{p(z)-p(0)} = \operatorname{Re}\frac{zp'(z)}{p(z)-1} = \operatorname{Re}(1+z) > 0,\ z \in \mathbb{D}.$$

Using this fact together with $p'(0) = 1 \neq 0$ it follows that $p(z) = 1 + ze^z$ is also a starlike (univalent) function in $\mathbb{D}$ (see Fig. 1) and because $p(\overline{z}) = \overline{p(z)}$, $z \in \mathbb{D}$, the domain $p(\mathbb{D})$ is symmetric with respect to the real axis. Further, from Figure 2 we observe that $p(z) \neq 0$ for all $z \in \mathbb{D}$.

To show that $S_{s,e}^{**}$ is non empty for some appropriate choices of $\mathfrak{q}$. Let us consider the functions $\mathfrak{q}(z) := z+0.18z^2 \in \mathcal{A}$.

From the Figure 3, $\mathfrak{q}(\mathbb{D}) \subset p(\mathbb{D})$ with the univalency of $p(z) = 1 + ze^z$ seen previously for $S_{s,e}^{**}$ class, leads to $\mathfrak{q}(z) := z + 0.18z^2 \prec p(z)$.

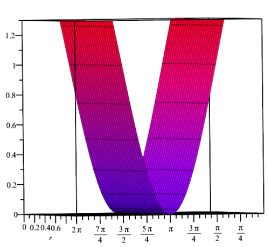

FIGURE 1. Starlike function of $p(z)$

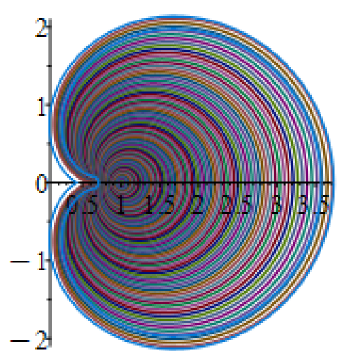

FIGURE 2. The images of $p(\mathbb{D})$ (multi color)

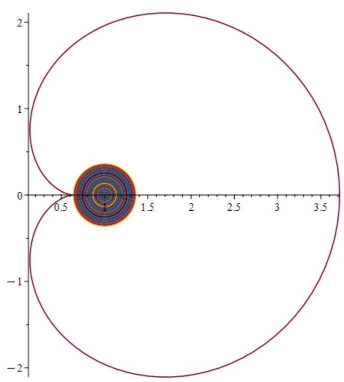

FIGURE 3. The images of $\mathfrak{p}\left(e^{it}\right)$ (red color) and $\mathfrak{q}\left(e^{it}\right)$ (multi color), $t \in [0, 2\pi)$

It may be noted that for the Schwarz function $w(z) = z$, the corresponding extremal function

$$f_1(z) = z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \frac{1}{4}z^4 + \frac{1}{6}z^5 \cdots, \tag{1.2}$$

and for the Schwarz function $w(z) = z^2$, the corresponding extremal function

$$f_2(z) = z + \frac{1}{2}z^3 + \frac{3}{8}z^5 + \cdots, \tag{1.3}$$

are belong to the class $S_{s,e}^{**}$.

For each functions $f \in \mathcal{S}$ defined on $\mathbb{D}$, the famous one-quarter theorem of Koebe (see [9]) asserts that its inverse f^{-1} exists at least on a disk of radius $\frac{1}{4}$. If $f \in \mathcal{S}$, the function f^{-1} which is the inverse of f has expression given by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \quad (|w| < \frac{1}{4}).$$

Some of the initial coefficients of f^{-1} are given by

$$\begin{aligned} &A_2 = -a_2, \qquad A_3 = 2a_2^2 - a_3, \qquad A_4 = -a_4 + 5a_2a_3 - 5a_2^3 \\ &A_5 = -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.4}$$

The inverse functions are studied by several authors in various subclasses of analytic functions. (see, for details [1] and reference therein).

The logarithmic coefficients $\gamma_n := \gamma_n(f)$ $(n \in \mathbb{N})$ of the function $f \in \mathcal{S}$ are defined as

$$F_f(z) = log\frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n, \quad (z \in \mathbb{D}), \qquad where \quad log1 = 0. \tag{1.5}$$

If f is given by (1.1), then comparing the coefficients of z^n in (1.5) for $n = 1, 2, 3, 4$ it give

$$\begin{aligned} &\gamma_1 = \frac{a_2}{2}, \qquad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{a_2^2}{2}\right), \qquad \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \\ &\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right). \end{aligned}$$

Very recently, the upper bounds of logarithmic coefficients of functions f in some subclasses of the class $\mathcal{S}$ have been obtained by various authors [1, 22, 37].

Further, the concept of inverse logarithmic coefficients i.e. logarithmic coefficients of inverse of f has been introduced by Ponnusamy et al. [24]. The inverse logarithmic coefficients Γ_n $(n \in \mathbb{N})$ of f are given by the relation:

$$F_{f^{-1}}(w) = log\frac{f^{-1}(w)}{w} = 2\sum_{n=1}^{\infty} \Gamma_n w^n \quad (|w| < \frac{1}{4}). \tag{1.6}$$

By differentiating (1.6) together with (1.4) one may get

$$\begin{aligned} &\Gamma_1 = -\frac{1}{2}a_2, \quad \Gamma_2 = -\frac{1}{2}\left(a_3 - \frac{3}{2}a_2^2\right), \quad \Gamma_3 = -\frac{1}{2}\left(a_4 - 4a_2a_3 + \frac{10}{3}a_2^3\right), \\ &\Gamma_4 = -\frac{1}{2}\left(a_5 - 5a_4a_2 + 15a_3a_2^2 - \frac{5}{2}a_3^2 - \frac{35}{4}a_2^4\right), \\ &\Gamma_5 = -\frac{1}{2}\left(-6a_2a_5 - 56a_2^3a_3 + 21a_2^2 + 21a_2a_3^2 + a_6 + \frac{126}{5}a_2^5\right). \end{aligned} \tag{1.7}$$

Hankel matrices and determinants play an important role in several branches of mathematics and have many applications [36]. The Toeplitz determinants are closely related to Hankel determinants. Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. For a good summary of the Toeplitz determinant and its applications to wide range of areas of pure and applied Mathematics, we refer to [36]. Recently, Thomas and Halim [28] have introduced the concept of the symmetric Toeplitz determinant for analytic functions f of the form (1.1). Finding the upper bounds for the modules of Hankel determinants for various subclasses of analytic univalent functions is an active area of research in Geometric Function Theory. In 1976, Noonan and Thomas [20] stated the q-th Hankel determinant for $q \geq 1$ and $n \geq 1$ of functions $f \in \mathcal{A}$ represented by (1.1) denoted by $H_q(n)(f)$ is defined as:

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} (q,\ n \in \mathbb{N} = \{1, 2, 3, ...\}).$$

It may be noted that for $q = 2,\ n = 1$ and $q = 2,\ n = 2$ we have

$$H_2(1)(f) = a_3 - a_2^2 \text{ and } H_2(2)(f) = a_2a_4 - a_3^2 \tag{1.8}$$

are popularly known as Fekete-Szegö functional and second Hankel determinant, respectively. Fekete-Szegö inequality for the coefficient of univalent analytic functions found by Fekete and Szegö in 1933, which is related to Biberbach conjecture. Fekete and Szegö [10] have proved that

$$\max_{f\in\mathcal{S}} |a_3 - \mu a_2^2| = \begin{cases} 1 + 2\exp\left(-\frac{2\mu}{1-\mu}\right), & 0 \le \mu < 1, \\ 1, & \mu = 1, \end{cases}$$

hold for any function $f \in \mathcal{A}$ of the form (1.1). Recently Amourah et. al. [3] introduced a subfamily of bi-univalent functions connected to the Jacobi polynomial through the imaginary error function and derived the initial coefficients and the Fekete-Szegö inequality. Buyankara and Çağlar [6] determined the sharp bounds for the second Hankel determinant and some Toeplitz determinant for of a subclass of analytic functions(also see [5, 30]). Recently, Kowalczyk and Lecko [12] proposed a Hankel determinant whose elements are the logarithmic coefficients of $f \in \mathcal{S}$. The concept of the Hankel determinant $H_q(n)(F_{f^{-1}}/2)$ [34], Toeplitz determinant $T_q(n)(F_{f^{-1}}/2)$ [15, 35] and Hermitian-Toeplitz determinant $T_{q,n}(F_{f^{-1}}/2)$ [19] where the elements of the determinants are logarithmic coefficients of the inverse functions $f \in \mathcal{S}$ are expressed as:

$$H_q(n)(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_n & \Gamma_{n+1} & \cdots & \Gamma_{n+q-1} \\ \Gamma_{n+1} & \Gamma_{n+2} & \cdots & \Gamma_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n+q-1} & \Gamma_{n+q} & \cdots & \Gamma_{n+2q-2} \end{vmatrix}, \tag{1.9}$$

$$T_q(n)(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_n & \Gamma_{n+1} & \cdots & \Gamma_{n+q-1} \\ \Gamma_{n+1} & \Gamma_{n} & \cdots & \Gamma_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{n+q-1} & \Gamma_{n+q-2} & \cdots & \Gamma_{n} \end{vmatrix}, \tag{1.10}$$

and while q^{th} Hermitian-Toeplitz determinant $T_{q,n}(F_{f^{-1}}/2) = [\Gamma_{ij}]$ is given by:

$$T_{q,n}(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_n & \Gamma_{n+1} & \cdots & \Gamma_{n+q-1} \\ \bar{\Gamma}_{n+1} & \Gamma_{n} & \cdots & \Gamma_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\Gamma}_{n+q-1} & \bar{\Gamma}_{n+q-2} & \cdots & \Gamma_{n} \end{vmatrix}, \tag{1.11}$$

where $\Gamma_{ij} = \Gamma_{n+j-i},\ (j \ge i)$ and $\Gamma_{ij} = \bar{\Gamma}_{ij},\ j < i.$

For $q = 2,\ n = 1$, relation (1.9) gives $H_2(1)(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{vmatrix} = \Gamma_1\Gamma_3 - \Gamma_2^2.$

Using (1.7) and after simplification we get

$$H_2(1)(F_{f^{-1}}/2) = \frac{1}{48}(13a_2^4 - 12a_2^2a_3 - 12a_3^2 + 12a_2a_4). \tag{1.12}$$

Similarly, for $q = 2$ and $n = 1$, relation (1.10) yields $T_2(1)(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_1 \end{vmatrix} = \Gamma_1^2 - \Gamma_2^2.$

Making use of the relation (1.4) one may find

$$T_2(1)(F_{f^{-1}}/2) = \frac{1}{16}(-9a_2^4 + 4a_2^2 - 4a_3^2 + 12a_2^2a_3). \tag{1.13}$$

Further, for $q = 2$ and $n = 1$, relation (1.11) yield $T_{2,1}(F_{f^{-1}}/2) = \begin{vmatrix} \Gamma_1 & \Gamma_2 \\ \bar{\Gamma}_2 & \Gamma_1 \end{vmatrix} = \Gamma_1^2 - |\Gamma_2|^2.$

Making use of the relation (1.5) in the above expression, one may get

$$T_{2,1}(F_{f^{-1}}/2) = \frac{1}{16}(-A_2^4 + 4A_2^2 + 4A_2^2\Re(A_3) - 4|A_3|^2). \tag{1.14}$$

Zaprawa [37] obtained the sharp bounds of initial logarithmic coefficients γ_n for functions in the classes $\mathcal{S}_s^*$ and $\mathcal{K}_s$ (also see [12, 33]). Further, results concerning Toeplitz determinants were introduced by Ali et al. [2]. Moreover, Cudna et al. [8] studied the third-order Hermitian-Toeplitz determinant for starlike and convex functions of order α. For more details see [21].

2. Preliminaries Results

Let us define by $\mathcal{P}$ the well-known Carathéodory class i.e the family of holomorphic functions $d \in \mathcal{P}$ that satisfies the condition $\operatorname{Re}\{d(z)\} > 0$, $z \in \mathbb{D}$, and of the form

$$d(z) = 1 + \sum_{n=1}^{\infty} d_n z^n, \quad (z \in \mathbb{D}). \tag{2.1}$$

We need the following lemmas in order to prove our main results.

Lemma 2.1 ([7, 23, 26]). *Let $d \in \mathcal{P}$ be of the form (2.1).*

(i) Then, for $n \geq 1$

$$|d_n| \leq 2. \tag{2.2}$$

The inequality holds for all $n \geq 1$ if and only if $d(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| = 1$.

(ii) Also, if $\mu \geq 0$ then

$$|d_{n+k} - \mu d_n d_k| \leq 2\max\{1; |2\mu - 1|\} = \begin{cases} 2, & \text{if} \quad 0 \leq \mu \leq 1, \\ 2|2\mu - 1|, & \text{otherwise}. \end{cases} \tag{2.3}$$

If $0 < \mu < 1$, the inequality is sharp for the function $d(z) = \dfrac{1+z^{n+k}}{1-z^{n+k}}$. In the other cases, the inequality is sharp for the function $d(z) = \dfrac{1+z}{1-z}$.

Lemma 2.2 ([4], Lemma 2.2.). *If $d \in \mathcal{P}$ has the form (2.1), then*

$$\left|\alpha d_1^3 - \beta d_1 d_2 + \gamma d_3\right| \leq 2\left(|\alpha| + |\beta - 2\alpha| + |\alpha - \beta + \gamma|\right). \tag{2.4}$$

Lemma 2.3 ([32], Proposition 1). *Let $d \in \mathcal{P}$ be given by (2.1). Let B_1, B_2 and B_3 be numbers such that $B_1 \geq 0$, $B_2 \in \mathbb{C}$ and $B_3 \in \mathbb{R}$. Define $\psi_+(d_1, d_2)$ and $\psi_-(d_1, d_2)$ by*

$$\psi_+(d_1, d_2) = |B_2 d_1^2 + B_3 d_2| - |B_1 d_1|, \text{ and } \psi_-(d_1, d_2) = -\psi_+(d_1, d_2).$$

Then,

$$\psi_+(d_1, d_2) \leq \begin{cases} |4B_2 + 2B_3| - 2B_1 & \text{when} \quad |2B_2 + B_3| \geq |B_3| + B_1, \\ 2|B_3| & \text{otherwise}, \end{cases} \tag{2.5}$$

and

$$\psi_-(d_1, d_2) \leq \begin{cases} 2B_1 - B_4 & B_1 \geq B_4 + 2|B_3|, \\ 2B_1\sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}} & \text{when} \quad B_1^2 \leq 2|B_3|(B_4 + 2|B_3|), \\ 2|B_3| + \frac{B_1^2}{B_4 + 2|B_3|} & \text{otherwise}, \end{cases} \tag{2.6}$$

where $B_4 = |4B_2 + 2B_3|$. All the inequalities in (2.5) and (2.6) are sharp.

Lemma 2.4 ([17]). *Let $p \in \mathcal{P}$ be given by (2.1). Then,*

$$\begin{aligned} 2d_2 &= d_1^2 + t\xi, \\ 4d_3 &= d_1^3 + 2d_1 t\xi - d_1 t\xi^2 + 2t(1 - |\xi|^2)\eta, \\ 8d_4 &= d_1^4 + 3d_1^2 t\xi + (4 - 3d_1^2)t\xi^2 + d_1^2 t\xi^3 + 4t(1 - |\xi|^2)(1 - |\eta|^2)\gamma \\ &\quad + 4t(1 - |\xi|^2)(d_1\eta - d_1\xi\eta - \bar{\xi}\eta^2), \end{aligned} \tag{2.7}$$

for some $\xi, \eta, \ \gamma \in \bar{\mathbb{D}}$ and $t = (4 - d_1^2)$.

3. Bounds on Initial Coefficients and Hankel Determinant for the Class $S^{**}_{s,e}$

In this section, we determine the upper bounds for the first four initial coefficients, Fekete-Szegö functional $|a_3-\mu a_2^2|$ and second Hankel determinant $|a_2a_4-a_3^2|$ for our defined class $S^{**}_{s,e}$.

Theorem 3.1. *Let the function $f \in \mathcal{A}$ given by (1.1) be a member of the class $S^{**}_{s,e}$. Then,*

$$|a_2| \leq \frac{1}{2}, \quad |a_3| \leq \frac{1}{2}, \quad |a_4| \leq \frac{5}{16}, \quad |a_5| \leq \frac{78226}{196476} = 0.3981453205.$$

The first two coefficient estimates are sharp.

Proof. Assume that, $f \in S^{**}_{s,e}$. Hence, by Definition 1.1, there exists Schwarz function $w(z)$ with $w(0) = 1$ and $|w(z)| < 1$ such that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + w(z)e^{w(z)} \quad (z \in \mathbb{D}). \tag{3.1}$$

Expressing the Schwarz function w in terms of $d \in \mathcal{P}$, i.e.,

$$d(z) = \frac{1+w(z)}{1-w(z)} = 1 + d_1z + d_2z^2 + d_3z^3 + \cdots \quad (z \in \mathbb{D}),$$

or equivalently,

$$w(z) = \frac{d(z)-1}{d(z)+1} = \frac{1}{2}d_1z + \left(\frac{1}{2}d_2 - \frac{1}{4}d_1^2\right)z^2 + \left(\frac{1}{8}d_1^3 - \frac{1}{2}d_1d_2 + \frac{1}{2}d_3\right)z^3 + \cdots. \tag{3.2}$$

Making use of (3.2) in the r.h.s of (3.1) we obtain

$$\begin{aligned} 1 + w(z)e^{w(z)} = 1 + \frac{1}{2}d_1z + \frac{1}{2}d_2z^2 + \left(\frac{d_3}{2} - \frac{d_1^3}{16}\right)z^3 \\ + \left(\frac{1}{2}d_4 - \frac{3}{16}d_1^2d_2 + \frac{1}{24}d_1^4\right)z^4 + \cdots \quad (z \in \mathbb{D}). \end{aligned} \tag{3.3}$$

On the other hand, it follows from (1.1) that

$$\frac{2zf'(z)}{f(z)-f(-z)} = 1 + 2a_2z + 2a_3z^2 + (4a_4 - 2a_2a_3)z^3 + (4a_5 - 2a_3^2)z^4 + \cdots. \tag{3.4}$$

Comparing the coefficients of various powers of z from the relations (3.3) and (3.4), we obtain

$$a_2 = \frac{1}{4}d_1, \tag{3.5}$$

$$a_3 = \frac{1}{4}d_2, \tag{3.6}$$

$$a_4 = -\frac{1}{64}\left[d_1^3 - 2d_1d_2 - 8d_3\right], \tag{3.7}$$

$$a_5 = \frac{1}{4}\left[\frac{d_4}{2} - \frac{3}{16}d_1^2d_2 + \frac{1}{24}d_1^4 + \frac{d_2^2}{8}\right]. \tag{3.8}$$

Taking modulus on both sides of (3.5) and (3.6) and then using the inequality (2.2) of Lemma 2.1 we get the desired estimates bounds of $|a_2|$ and $|a_3|$ respectively. The bounds of $|a_2|$ and $|a_3|$ are sharp and the corresponding extremal function is given in (1.2).
Taking modulus on both sides of (3.7) and using (2.4) of Lemma 2.2 with $\alpha = 1, \beta = 2$ and $\gamma = -8$ we get

$$|a_4| \leq \frac{1}{64}(2)[1 + |2-2| + |1-2-8|] = \frac{5}{16}.$$

Further, rearranging the terms in (3.8) and applying (2.7) of Lemma 2.4 we get

$$\begin{aligned}
a_5 =& \frac{1}{4}\Bigg[\frac{1}{16}(d_1^4 + 3d_1^2 t\zeta + (4-3d_1^2)t\zeta^2 + d_1^2 t\zeta^3 + 4t(1-|\zeta|^2)(1-|\eta|^2)\gamma + 4t(1-|\zeta|^2)(d_1\eta - d_1\zeta\eta - \bar{\zeta}\eta^2))\\
&-\frac{3}{32}d_1^2(d_1^2 + t\zeta) + \frac{1}{24}d_1^4 + \frac{1}{32}(d_1^4 + t^2\zeta^2 + 2t\zeta d_1^2)\Bigg]\\
=& \frac{1}{4}\Bigg[\frac{1}{16}d_1^4 + \frac{3}{16}t\zeta d_1^2 + \frac{(4-3d_1^2)t\zeta^2}{16} + \frac{t\zeta^3}{16}d_1^2 + \frac{(1-|\zeta|^2)(1-|\eta|^2)\gamma}{4}t + \frac{t(1-|\zeta|^2)d_1\eta}{4} - \frac{t(1-|\zeta|^2)d_1\zeta\eta}{4}\\
&-\frac{t(1-|\zeta|^2)\bar{\zeta}\eta^2}{4} - \frac{3}{32}d_1^4 - \frac{3}{32}t\zeta d_1^2 + \frac{1}{24}d_1^4 + \frac{1}{32}d_1^4 + \frac{1}{32}\zeta^2 t^2 + \frac{1}{16}t\zeta d_1^2\Bigg]
\end{aligned} \tag{3.9}$$

Taking $t = 4 - d_1^2$ from Lemma 2.4 and without loss of any generality we can write $d_1 = d \in [0,2]$, relation (3.9) and $|\xi| = x \leq 1$ and $|\eta| = y \leq 1$. Taking modulus on both sides of (3.9) and then applying triangle inequality with $x, y \in [0,1]$ and $d \in [0,2]$, we obtain

$$\begin{aligned}
|a_5| \leq& \frac{1}{4}\Bigg[\frac{1}{16}d^4 + \frac{3}{16}(4-d^2)xd^2 + \frac{(4-3d^2)(4-d^2)x^2}{16} + \frac{(4-d^2)x^3}{16}d^2 + \frac{(1-x^2)(1-y^2)}{4}(4-d^2)\\
&+ \frac{(4-d^2)(1-x^2)dy}{4} + \frac{(4-d^2)(1-x^2)dxy}{4} + \frac{(4-d^2)(1-x^2)xy^2}{4} + \frac{3}{32}d^4\\
&+\frac{3}{32}(4-d^2)xd^2 + \frac{1}{24}d^4 + \frac{1}{32}d^4 + \frac{1}{32}x^2(4-d^2)^2 + \frac{1}{16}(4-d^2)xd^2\Bigg]\\
=& \frac{1}{4}[F(d,x,y)] \quad (say), \quad where\ \ 0 \leq d \leq 2,\ 0 \leq x \leq 1,\ 0 \leq y \leq 1.
\end{aligned}$$

Now, we have to find the maximum of $F(d,x,y)$ in the cuboid $\omega = \{(d,x,y) \in [0,2] \times [0,1] \times [0,1]\}$. Let us consider for the corner points of the cuboid ω.

$$\begin{aligned}
&F(0,0,0) = 1, \quad F(0,1,0) = \frac{3}{2}, \quad F(0,0,1) = 0, \quad F(0,1,1) = \frac{3}{2}, \quad F(2,0,0) = \frac{2}{3}, \quad F(2,1,0) = \frac{2}{3},\\
&F(2,1,1) = \frac{2}{3}, \quad F(2,0,1) = \frac{2}{3}.
\end{aligned}$$

Next, considering the following different cases as faces of the cuboid ω.

$$\begin{aligned}
F(0,x,y) =& \frac{3x^2}{2} + \left(-x^2+1\right)\left(-y^2+1\right) + \left(-x^2+1\right)x\,y^2 \leq F(0,1,1) = \frac{3}{2},\\
F(d,x,0) =& \frac{1}{24}d^4 - \frac{5}{32}d^4x + \frac{5}{8}d^2x + \frac{7}{32}d^4x^2 - d^2x^2 + \frac{1}{2}x^2 - \frac{1}{16}d^4x^3 + \frac{1}{4}d^2x^3 - \frac{1}{4}d^2 + 1 \leq F(0,1,0) = \frac{3}{2},\\
F(d,0,y) =& \frac{d^4}{24} + \frac{\left(-y^2+1\right)\left(-d^2+4\right)}{4} + \frac{\left(-d^2+4\right)dy}{4} \leq F(0,0,0) = 1, \quad F(2,x,y) \leq \frac{2}{3},\\
F(d,1,y) =& \frac{d^4}{24} + \frac{7\left(-d^2+4\right)d^2}{32} + \frac{\left(-3d^2+4\right)\left(-d^2+4\right)}{16} + \frac{\left(-d^2+4\right)^2}{32} \leq F(0,1,y) = \frac{3}{2},\\
F(d,x,1) =& \frac{1}{24}d^4 - \frac{5}{32}d^4x + \frac{3}{8}d^2x + \frac{7}{32}d^4x^2 - \frac{5}{4}d^2x^2 + \frac{3}{2}x^2 - \frac{1}{16}d^4x^3 + \frac{1}{2}d^2x^3 + \frac{1}{4}d^3x^2 - \frac{1}{4}d^3\\
&- d\,x^2 + d + \frac{1}{4}d^3x^3 - \frac{1}{4}d^3x - d\,x^3 + dx - x^3 + x \leq F\left(\frac{23552}{23159}, \frac{32786}{51627}, 1\right) = \frac{78226}{49119} = 1.59258128248404818.
\end{aligned}$$

Next is to check for the interior of the cuboid ω. Taking partial derivative of $F(d,x,y)$ with respect to d, x and y we get the following.

$$\begin{aligned}\frac{\partial F}{\partial d} &= -\frac{1}{2}d + y + \frac{7}{8}d^3x^2 + \frac{1}{2}dy^2 - x^2y - x^3y + \frac{1}{6}d^3 + \frac{5}{4}dx - 2dx^2 + \frac{1}{2}dx^3 - \frac{3}{4}d^2y - \frac{1}{2}dx^2y^2 + \frac{3}{4}d^2x^2y \\ &+ \frac{3}{4}d^2x^3y + \frac{1}{2}dx^3y^2 - \frac{3}{4}d^2xy + xy - \frac{1}{2}dxy^2 - \frac{5}{8}d^3x - \frac{1}{4}d^3x^3 \\ \frac{\partial F}{\partial x} &= x + \frac{7}{16}d^4x - \frac{3}{16}d^4x^2 + \frac{3}{4}d^2x^2 - \frac{1}{4}d^2y^2 - 3x^2y^2 - \frac{1}{4}d^3y - \frac{1}{2}d^2xy^2 - 2d^2x + \frac{3}{4}d^2x^2y^2 + \frac{3}{4}d^3x^2y \\ &- 3dx^2y + \frac{1}{2}d^3xy - \frac{5}{32}d^4 + \frac{5}{8}d^2 + y^2 - 2dxy + dy + 2xy^2 \\ \frac{\partial F}{\partial y} &= -\frac{1}{2}d^2x^2y + \frac{1}{2}d^2y + 2x^2y - 2y + \frac{1}{4}d^3x^2 - \frac{1}{4}d^3 - dx^2 + d \\ &+ \frac{1}{4}d^3x^3 - \frac{1}{4}d^3x - dx^3 + dx + \frac{1}{2}d^2x^3y - \frac{1}{2}d^2xy - 2x^3y + 2xy.\end{aligned}$$

Solving $\frac{\partial F}{\partial d} = 0$, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$ we get (0,0,0) is the only critical point which is not a interior point. Hence, it has no critical point in the interior of the cuboid. Hence, we get

$$|a_5| \le \left(\frac{1}{4}\right)\left(\frac{78226}{49119}\right) = \frac{78226}{196476} = 0.3981453205.$$

This completes the proof of Theorem 3.1. □

Next theorem gives Fekete-Szegö functional $|a_3 - \mu a_2^2|$ bounds for the class $S^{**}_{s,e}$ as follows.

Theorem 3.2. *If $f \in S^{**}_{s,e}$ has of the form (1.1), then for any complex number μ, we have*

$$|a_3 - \mu a_2^2| \le \frac{1}{2}\max\left\{1, \left|\frac{\mu-2}{2}\right|\right\}.$$

This bound is sharp.

Proof. Assume that $f \in S^{**}_{s,e}$. Making use of (3.5) and (3.6) in the functional $a_3 - \mu a_2^2$ and taking the modulus on the both sides and then applying (2.3) of Lemma 2.1, we get

$$|a_3 - \mu a_2^2| \le \frac{1}{2}\max\left\{1, \left|\frac{\mu-2}{2}\right|\right\}.$$

This bound is sharp for the schwarz function $w(z) = z^2$ and the corresponding extremal function is given in (1.3). This completes the proof of Theorem 3.2. □

Letting $\mu = 1$ in the Theorem 3.2, we obtain the following result in the form of corollary.

Corollary 3.3. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the function class $S^{**}_{s,e}$, then*

$$|a_3 - a_2^2| \le \frac{1}{2}.$$

The estimate is sharp and the extremal function is given in (1.3).

In following theorem, we investigate the upper bound of Hankel determinant of order two for the function that belongs to the class $S^{**}_{s,e}$.

Theorem 3.4. *If the function $f \in \mathcal{A}$ given by (1.1) belongs to the class $S^{**}_{s,e}$, then*

$$|a_2a_4 - a_3^2| \le \frac{1}{4}.$$

This bound is sharp.

Proof. From (3.5), (3.6) and (3.7) it follows that

$$\begin{aligned} a_2a_4 - a_3^2 &= \frac{-d_1^4}{256} + \frac{1}{128}d_1^2d_2 + \frac{d_3d_1}{32} - \frac{d_2^2}{16} \\ &= \frac{1}{256}\left[-d_1^4 + 2d_1^2d_2 + 8d_3d_1 - 16d_2^2\right]. \end{aligned}$$

By applying (2.7) of Lemma 2.4 above expression simplified into the following

$$a_2a_4 - a_3^2 = \frac{1}{256}\left[-2d_1^4 - 3d_1^2t\zeta - 2d_1^2t\zeta^2 - 4t^2\zeta^2 + 4d_1t(1-|\zeta|^2)\eta\right].$$

Taking $t = 4-d_1^2$ from Lemma 2.4 and without loss of any generality we can write $d_1 = d \in [0,2]$, in the above relation and $|\xi| = x \leq 1$ and $|\eta| \leq 1$. Taking modulus on both sides of the above relation and then, applying triangle inequality with $x \in [0,1]$ and $d \in [0,2]$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &\leq \frac{1}{256}\left[2d^4 + 3d^2(4-d^2)x + 2d^2(4-d^2)x^2 + 4(4-d^2)^2x^2 + 4d(4-d^2)(1-x^2)\right] \\ &= \frac{1}{256}[F(d,x)] \quad 0 \leq d \leq 2, 0 \leq x \leq 1. \end{aligned} \tag{3.10}$$

Now, we need to determine

$$max\{F(d,x) : (d,x) \in [0,2] \times [0,1]\}.$$

A simple computation shows

$$\frac{\partial F}{\partial x} = 3d^2(-d^2+4)x + 2d^4 + 2d^2(-d^2+4)x^2 + 4(-d^2+4)^2x^2 + 4(-d^2+4)d(-x^2+1) \geq 0 \quad ((d,x) \in [0,2] \times [0,1]).$$

As a result, $F(d,x)$ is an increasing function of x on the closed interval [0,1]. This means that the function $F(d,x)$ attains its maximum at $x = 1$. Hence

$$\max F(d,x) \leq F(d,1) = 5d^2\left(-d^2+4\right) + 2d^4 + 4\left(-d^2+4\right)^2 = G(d)(say).$$

Using the fact that $G'(d) = 4d^3 - 24d \leq 0 \; \forall d \in [0,2]$.
Hence, $G(d)$ will be decreasing on [0,2] which implies that

$$G(d) \leq G(0) = 64. \tag{3.11}$$

According to the inequalities (3.10) and (3.11) we deduce that

$$\max\{F(d,x) : (d,x) \in [0,2] \times [0,1]\} = F(0,1) = 64. \tag{3.12}$$

The desire estimates follows from (3.10) and (3.12). Here, the bound is sharp and the extremal function is given in (1.3). This completes the proof of Theorem 3.4. □

4. Bounds on Logarithmic Inverse Coefficients for the Class $S^{**}_{s,e}$

In this section, we investigate initial coefficient bounds, Hankel determinant, Toeplitz determinant and Hermitian-Toeplitz determinant of logarithmic coefficients of inverse function f^{-1} for the class $S^{**}_{s,e}$.
The following theorem gives the bounds of initial logarithmic coefficients of inverse function f^{-1} for the class $S^{**}_{s,e}$.

Theorem 4.1. *Let the function $f \in \mathcal{A}$ given in the form (1.1) be the member of the class $S^{**}_{s,e}$. Then,*

$$|\Gamma_1| \leq \frac{1}{4}, \quad |\Gamma_2| \leq \frac{1}{4}, \quad |\Gamma_3| \leq \frac{1}{6}, \quad |\Gamma_4| \leq \frac{7}{16}.$$

The first three estimates are sharp.

Proof. Assume that $f \in \mathcal{A}$ given by (1.1) is in the function class $S^{**}_{s,e}$. Then using (3.5)-(3.8) in the expression (1.7) we obtain

$$\Gamma_1 = -\frac{d_1}{8}, \tag{4.1}$$

$$\Gamma_2 = -\frac{1}{8}\left(d_2 - \frac{3}{8}d_1^2\right), \tag{4.2}$$

$$\Gamma_3 = -\frac{1}{2}\left[\frac{d_3}{8} - \frac{7}{32}d_1d_2 + \frac{7}{192}d_1^3\right] \tag{4.3}$$

and

$$\Gamma_4 = -\frac{1}{2}\left[\frac{d_4}{8} - \frac{1}{8}d_2^2 - \frac{13}{3072}d_1^4 + \frac{19}{128}d_1^2d_2 - \frac{5}{32}d_1d_3\right]. \tag{4.4}$$

Taking modulus on both sides of (4.1) and (4.2) and then using the inequalities (2.2) and (2.3) of Lemma 2.1, we get the desired estimates bounds of $|\Gamma_1|$ and $|\Gamma_2|$, respectively. The bounds of $|\Gamma_1|$ and $|\Gamma_2|$ are sharp for the function given in (1.2) and (1.3) respectively.

Applying Lemma 2.4 in the expression (4.3), we have

$$\begin{aligned}\Gamma_3 &= \frac{1}{384}\left[-7d_1^3 + 42d_1\left(\frac{d_1^2+t\zeta}{2}\right) - 24\left(\frac{d_1^3 + 2d_1t\zeta - d_1t\zeta^2 + 2t(1-|\zeta|^2)\eta}{4}\right)\right]\\ &= \frac{1}{384}\left[8d_1^3 + 9d_1t\zeta + 6d_1t\zeta^2 - 12t(1-|\zeta|^2)\eta\right].\end{aligned}$$

Taking $t = 4 - d_1^2$ from Lemma 2.4 and without loss of any generality we can write $d_1 = d \in [0,2]$, in the above relation and $|\xi| = x \le 1$ and $|\eta| \le 1$. Taking modulus on both sides of the above relation and then applying triangle inequality with $x \in [0,1]$ and $d \in [0,2]$, we obtain

$$\begin{aligned}|\Gamma_3| &\le \frac{1}{384}\left[8d^3 + 9d(4-d^2)x + 6d(4-d^2)x^2 + 12(4-d^2)(1-x^2)\right]\\ &= \frac{1}{384}[F(d,x)] \quad where\ \ 0 \le d \le 2,\ 0 \le x \le 1.\end{aligned} \tag{4.5}$$

Now, we need to determine

$$max\{F(d,x) : (d,x) \in [0,2]\times[0,1]\}.$$

Now, consider the corner point of the rectangular region

$$F(0,0) = 48, \quad F(2,0) = 64, \quad F(2,1) = 64, \quad F(0,1) = 0.$$

Next is to check for the interior of the rectangular region. Taking partial derivative of $F(d,x)$ with respect to d and x we get the following

$$\begin{aligned}\frac{\partial F}{\partial d} &= -18d^2x^2 - 27d^2x + 24dx^2 + 24d^2 + 24x^2 - 24d + 36x,\\ \frac{\partial F}{\partial x} &= -12d^3x - 9d^3 + 24d^2x + 48dx + 36d - 96x.\end{aligned}$$

Solving $\frac{\partial F}{\partial d} = 0$ and $\frac{\partial F}{\partial x} = 0$, we get (0,0) is the only critical point which is not a interior point. Hence it has no critical point in the interior of the rectangular region. Hence, we get

$$MaxF(d,x) = 64. \tag{4.6}$$

According to the (4.5) and (4.6), we deduce that

$$|\Gamma_3| \le \frac{1}{6}.$$

Here, the bound of $|\Gamma_3|$ is sharp for the schwarz function $w(z) = z$ and the corresponding extremal function given in (1.2). Rearranging the terms in the relation (4.4) and taking modulus then applying triangle inequality we have

$$|\Gamma_4| \le \frac{1}{2}\left[\frac{1}{8}|d_4 - d_2^2| + \frac{5}{32}|d_1|\left|d_3 - \frac{19}{20}d_1d_2 + \frac{13}{480}d_1^3\right|\right].$$

Applications of inequalities (2.3) of Lemma 2.1 and (2.4) of Lemma 2.2 with $\alpha = \frac{13}{480}$, $\beta = \frac{19}{20}$ and $\gamma = 1$, one may get

$$|\Gamma_4| \le \frac{1}{8} + \frac{5}{16}\left[\frac{13}{480} + \left|\frac{19}{20} - \frac{13}{240}\right| + \left|\frac{13}{480} - \frac{19}{20} + 1\right|\right] = \frac{7}{16}.$$

The proof of Theorem 4.1 is thus completed. □

Next theorem gives the upper bounds of Hankel determinant of logarithmic coefficients of inverse function for the class $S^{**}_{s,e}$.

Theorem 4.2. *Let the function $f \in \mathcal{A}$ given in the form (1.1) be the member of the function class $S^{**}_{s,e}$. Then,*

$$|H_2(1)(F_{f^{-1}}/2)| \leq \frac{1}{16}.$$

This bound is sharp.

Proof. Assume that $f \in S^{**}_{s,e}$. Making use of (3.5), (3.6) and (3.7) in (1.12) and substituting the values of d_2 and d_3 in terms of d_1 from Lemma 2.4 in the relation and after simplification

$$H_2(1)(F_{f^{-1}}/2) = \frac{1}{12288}(-35d_1^4 - 60t\xi d_1^2 - 24t\xi^2 d_1^2 - 48t^2\xi^2 + 48d_1 t(1-|\xi|^2)\eta). \tag{4.7}$$

Taking $t = 4 - d_1^2$ from Lemma 2.4 and without loss of any generality we can write $d_1 = d \in [0,2]$, relation (4.7) gives

$$\begin{aligned} H_2(1)(F_{f^{-1}}/2) = \frac{1}{12288}&[-35d^4 - 60d^2(4-d^2)\xi - 24d^2(4-d^2)\xi^2 \\ &- 48(4-d^2)^2\xi^2 + 48d(4-d^2)(1-|\xi|^2)\eta], \end{aligned} \tag{4.8}$$

where $|\xi| = x \leq 1$ and $|\eta| \leq 1$. Taking modulus on both sides of (4.8) and then applying triangle inequality with $x \in [0,1]$ and $d \in [0,2]$, we obtain

$$\begin{aligned} |H_2(1)(F_{f^{-1}}/2)| \leq &\frac{1}{12288}(35d^4 + 60d^2(4-d^2)x + 24d^2(4-d^2)x^2 + 48(4-d^2)^2x^2 \\ &+ 48d(4-d^2)(1-x^2)) =: F(d,x)(say). \end{aligned} \tag{4.9}$$

Now, we need to determine

$$max\{F(d,x) : (d,x) \in [0,2] \times [0,1]\}.$$

A simple computation shows

$$\frac{\partial F}{\partial x} = \frac{1}{1024}(4-d^2)(5d^2 + 4x(8 - d^2 - 2d) \geq 0 \quad ((d,x) \in [0,2] \times [0,1]).$$

As a result, $F(d,x)$ is an increasing function of x on the closed interval [0,1]. This means that the function $F(d,x)$ attains its maximum at $x = 1$. Hence,

$$MaxF(d,x) \leq F(d,1) = -\frac{1}{12288}[d^4 + 48d^2 - 768] = G(d)(say). \tag{4.10}$$

Using the fact that $G'(d) = -\frac{1}{3072}d(d^2+24) < 0 \; \forall d \in [0,2])$.
Hence, $G(d)$ will be decreasing on [0,2] which implies that

$$G(d) \leq G(0) = \frac{768}{12288} = \frac{1}{16}. \tag{4.11}$$

According to the inequalities (4.10) and (4.11) we deduce that

$$Max\{F(d,x) : (d,x) \in [0,2] \times [0,1]\} = F(0,1) = \frac{1}{16}. \tag{4.12}$$

The desire estimates follows from (4.9) and (4.12). Here the bound is sharp and the extremal function is given in (1.3). This completes the proof of Theorem 4.2. □

The next theorem gives the bound of Toeplitz determinant for logarithmic coefficient of inverse function for the class $S^{**}_{s,e}$.

Theorem 4.3. *Let the function $f \in \mathcal{A}$ given by (1.1) be a member of the class $S^{**}_{s,e}$. Then,*

$$|T_2(1)(F_{f^{-1}}/2)| \leq \frac{1}{16}.$$

This bound is sharp.

Proof. Assume that $f \in S^{**}_{s,e}$. Then making use of (3.5) and (3.6) in the relation (1.13) we get

$$T_2(1)(F_{f^{-1}}/2) = \frac{1}{16}\left[-\frac{9}{256}d_1^4 + \frac{d_1^2}{4} - \frac{d_2^2}{4} + \frac{3}{16}d_1^2 d_2\right]. \tag{4.13}$$

Expressing the values of d_2 and d_3 in terms of d_1 by virtue of (2.7) of Lemma 2.4 and without any loss of generality we assume that $d_1 = d \in [0,2]$, from (4.13) we have

$$T_2(1)(F_{f-1}/2) = \frac{1}{16}\left[-\frac{1}{256}d^4 + \frac{d^2}{4} - \frac{1}{16}(4-d^2)^2\xi^2 - \frac{1}{32}d^2(4-d^2)\xi\right]. \tag{4.14}$$

Application of triangle inequality to the relation (4.14) and assuming $|\xi| = x \leq 1$, we have

$$\begin{aligned} |T_2(1)(F_{f^{-1}/2})| \leq & \frac{1}{16}\left[\frac{1}{256}d^4 + \frac{d^2}{4} + \frac{1}{16}(4-d^2)^2x^2 + \frac{1}{32}d^2(4-d^2)x\right] \\ =& \psi(d,x)(say). \end{aligned} \tag{4.15}$$

Now, we need to find $Max\{\psi(d,x) : (d,x) \in [0,2] \times [0,1]\}$. A simple computation shows

$$\frac{\partial\psi}{\partial x} = \frac{1}{16}\left[\frac{1}{8}(4-d^2)^2x + \frac{1}{32}d^2(4-d^2)\right] \geq 0 \quad (d,x) \in [0,2] \times [0,1].$$

Clearly $\psi(d,x)$ is increasing on [0,1]. As a result at $x = 1$, the function $\psi(d,x)$ attains its maximum value.

$$max\{\psi(d,x)\} \leq \psi(d,1) = \frac{1}{16}\left[\frac{9}{256}d^4 - \frac{1}{8}d^2 + 1\right] = \alpha(d)(say). \tag{4.16}$$

Using the fact that $\alpha'(d) = \frac{d}{64}\left[\frac{9d^2-16}{16}\right]$. Setting $\alpha'(d) = 0$ we get either $d = 0$ or $d = \frac{4}{3}$. Further, $\alpha''(d) = \frac{1}{16}\left[\frac{27}{64}d^2 - \frac{1}{4}\right]$. At $d = 0$, $\alpha''(d) = -\frac{1}{64} < 0$ and $\alpha''(d) = \frac{1}{32} > 0$ at $d = \frac{4}{3}$.
This shows that the function $\alpha(d)$ attains its maximum value at $d = 0$. Therefore,

$$\alpha(d) \leq \alpha(0) = \frac{1}{16}. \tag{4.17}$$

It can be deduced from the inequalities (4.16) and (4.17) that

$$max\{\psi(d,x) : (d,x) \in [0,2] \times [0,1]\} = \psi(0,1) = \frac{1}{16}. \tag{4.18}$$

The result follows from (4.15) and (4.18). Here the estimation is sharp and the extremal function is given in (1.3). This completes the proof of Theorem 4.3. □

Theorem 4.4. *Let the function $f \in \mathcal{A}$ given by (1.1) be a member of the class $S^{**}_{s,e}$. Then,*

$$-\frac{1}{2} \leq T_{2,1}(F_{f^{-1}}/2) \leq \frac{15}{256}.$$

Proof. If $f \in S^{**}_{s,e}$, then by applying (1.14) we have

$$T_{2,1}(F_{f^{-1}}/2) = \frac{1}{16}(-A_2^4 + 4A_2^2 + 4A_2^2\Re A_3 - 4|A_3|^2).$$

By making use of (1.4), we obtain

$$T_{2,1}(F_{f^{-1}}/2) = \frac{1}{16}(-a_2^4 + 4a_2^2 + 4a_2^2\Re(-a_3 + 2a_2^2) - 4| - a_3 + 2a_2^2|^2).$$

Now, substituting the values of a_2 and a_3 from (3.5) and (3.6) in above expression and using the Lemma 2.4 for $p \in \mathcal{P}$ and letting $d_1 = d$, we get

$$T_{2,1}(F_{f^{-1}}/2) = \frac{1}{4096}(-d^4 + 64d^2 - 8d^2(4-d^2)\Re(\xi) - 128(4-d^2)^2|\xi|^2). \tag{4.19}$$

Using $-\Re(\xi) \leq |\xi|$ and $|\xi| = x \in [0,1]$ and $d \in [0,2]$, the above expression becomes

$$T_{2,1}(F_{f^{-1}}/2) \leq \frac{1}{4096}(-d^4 + 64d^2 + 8d^2(4-d^2)x - 128(4-d^2)^2x^2) = S(d,x)(say).$$

For the maximum value of the functions on $\omega = [0,2] \times [0,1]$; we have to consider following cases:
Case-I (Interior of ω)

$$\frac{\partial S}{\partial x} = 0 \quad \implies \frac{1}{4096}(8d^2(4-d^2) - 256(4-d^2)^2 x) = 0$$
$$\implies x = \frac{d^2}{32(4-d^2)} \in (0,1). \ when \ d < \sqrt{\frac{128}{33}} \in (0,2).$$

Now, substituting the value of x in $\frac{\partial S}{\partial d} = 0$, then it becomes

$$\implies \frac{1}{4096}(-4d^3 + 128d + 16d(4-d^2)x - 16d^3 x + 512d(4-d^2)x^2) = 0.$$

By substituting the value of x, we have following,

$$\implies 1024d - 284d^3 + 8d^5 = 0$$

which is not possible for $d \in (0,2)$ i.e. there is no zeroes inside $(0,2)$. The function S has no maximum value in the interior of ω. Hence the maximum value of the function D attained on the boundary.
Case-II (On the boundary ω)

$$S(0,x) = \frac{1}{4096}(-2048x^2) = -\frac{1}{2}x^2 \leq 0; \quad S(2,x) = \frac{15}{256};$$
$$S(d,0) = \frac{1}{4096}(-d^4 + 64d^2) \leq \frac{15}{256}; S(d,1) = \frac{1}{4096}(-137d^4 + 1120d^2 - 2048) \leq \frac{15}{256}.$$

Hence, from above cases, the upper bound on $T_{2,1}(F_{f^{-1}}/2)$ is $\frac{15}{256}$. On using the identity $-\Re(\xi) \geq -|\xi|$ and $x = |\xi| \in [0,1]$ in (4.19) we have,

$$T_{2,1}(F_{f^{-1}}/2) \geq \frac{1}{4096}(-d^4 - 8d^2(4-d^2)x + 64d^2 - 128(4-d^2)^2 x^2) = D(d,x).$$

Let $D(d,x) \in \omega$

$$\frac{\partial D}{\partial x} = 0 \implies \frac{1}{4096}(-8d^2(4-d^2) - 256(4-d^2)^2 x) = 0 \tag{4.20}$$

and

$$\frac{\partial D}{\partial d} = 0 \quad \implies \frac{1}{4096}(-4d^3 - 16d(4-d^2)x + 16d^3 x + 128d + 512d(4-d^2)x^2) = 0. \tag{4.21}$$

The solution satisfying equation (4.20) and (4.21) is only (0, 0). So, it doesn't have any solution in the interior of ω. Hence, the minimum value of the function D attained on the boundary.

$$D(0,x) = -\frac{1}{2}x^2 \geq -\frac{1}{2}, \quad D(2,x) = \frac{1}{16}, \quad D(d,0) = \frac{1}{4096}(-d^4 + 64d^2) \geq 0,$$
$$D(d,1) = -\frac{1}{4096}(137d^4 - 1120d^2 + 2048) \geq -\frac{1}{2}.$$

Hence, from the above discussed case it is concluded that

$$-\frac{1}{2} \leq T_{2,1}(F_{f^{-1}}/2) \leq \frac{15}{256}.$$

This completes the proof of Theorem 4.4. □

5. Bound of Difference of Logarithmic Inverse Coefficients for the Class $S^{**}_{s,e}$

In the present section, we investigate the upper and lower bounds of the mudulo difference of second and first logarithmic inverse coefficients for the such family.

Theorem 5.1. *Let the function $f \in S^{**}_{s,e}$. Then,*

$$-\frac{1}{2\sqrt{5}} \leq (|\Gamma_2| - |\Gamma_1|) \leq \frac{1}{4}.$$

Proof. Suppose that $f \in S^{**}_{s,e}$. From the relation (1.7) and using (3.5) and (3.6), we have

$$2(|\Gamma_2| - |\Gamma_1|) = \left(\left|\frac{1}{4}d_2 - \frac{3}{32}d_1^2\right| - \left|\frac{1}{4}d_1\right| \right) = \psi_+(d_1, d_2).$$

Now, we verify the conditions of Lemma 2.3. Here, $B_1 = \frac{1}{4} > 0,\ B_2 = -\frac{3}{32},\ B_3 = \frac{1}{4},\ |2B_2 + B_3| = \frac{1}{16}$ *and* $|B_3| + B_1 = \frac{1}{2}$. So, $|2B_2 + B_3| \ngeq |B_3| + B_1$.
Therefore, application (2.5) of Lemma 2.3 yield,

$$2(|\Gamma_2| - |\Gamma_1|) \le 2|B_3| = \frac{1}{2} \implies |\Gamma_2| - |\Gamma_1| \le \frac{1}{4}. \tag{5.1}$$

Further, $2(|\Gamma_1| - |\Gamma_2|) = -\psi_+(d_1, d_2) = \psi_-(d_1, d_2)$.
$B_1 = \frac{1}{4}$ and $B_4 + 2|B_3| = \frac{5}{8}$. Hence, $B_1 \ngeq B_4 + 2|B_3|$,
where $B_1^2 = \frac{1}{16}$ and $2|B_3|(B_4 + 2|B_3|) = \frac{5}{16}$. So, $B_1^2 \le 2|B_3|(B_4 + 2|B_3|)$. By virtue of (2.6) of Lemma 2.3 gives

$$\psi_-(d_1, d_2) \le 2B_1\sqrt{\frac{2|B_3|}{B_4 + 2|B_3|}} = \frac{1}{\sqrt{5}} \tag{5.2}$$
$$\implies 2(|\Gamma_1| - |\Gamma_2|) \le \frac{1}{\sqrt{5}}.$$

$\implies 2(|\Gamma_2| - |\Gamma_1|) \ge -\frac{1}{\sqrt{5}} \implies |\Gamma_2| - |\Gamma_1| \ge -\frac{1}{2\sqrt{5}}$.
Therefore, from (5.1) and (5.2), we obtain

$$-\frac{1}{2\sqrt{5}} \le (|\Gamma_2| - |\Gamma_1|) \le \frac{1}{4}.$$

This completes the proof of Theorem 5.1. □

6. Generalized Zalcman Functional of Logarithmic Inverse Function

Lawrence Zalcman posed the conjecture that if $f \in \mathcal{S}$ given by (1.1), then

$$|a_n^2 - a_{2n-1}| \le (n-1)^2 \quad (n \ge 2). \tag{6.1}$$

Equality in (6.1) holds for the Koebe function $k(z) = \frac{z}{(1-z)^2}$ $(z \in \mathbb{D})$ or its rotation. The area theorem shows that the conjecture is true for $n = 2$ [9]. Kruskal proved that the conjecture is true for $n = 3$ [13] and latter for $n = 4, 5, 6$. However, the Zalcaman conjecture remains an open problem for $n > 6$. For $f \in \mathcal{S}$, Ma [18] proposed the generalized Zalcman conjecture

$$J_{m,n}(f) = |a_n a_m - a_{n+m-1}| \le (n-1)(m-1) \quad (n,\ m \ge 2),$$

and has proved this conjecture for the classes $\mathcal{S}^*$ and the class of all functions in $\mathcal{S}$ with real coefficients. In 2017, Ravichandran and Verma [27] proved it for the classes of starlike and convex functions of given order and for the class of functions with bounded turning. Now, we discuss Zalcman functional bounds in terms of Γ_n instead of a_n by fixing $n = m = 2$ and $n = 2,\ m = 3$.

Theorem 6.1. *Let $f \in S^{**}_{s,e}$ be of the form (1.1) and its logarithmic inverse coefficients $F_{f^{-1}/2}$ is given by (1.8). Then,*

$$J_{2,2}(F_{f^{-1}/2}) = |\Gamma_3 - \Gamma_2^2| \le \frac{2754681}{14204928} = 0.1939243198.$$

Proof. Assume that, $f \in S^{**}_{s,e}$. Substituting the values of Γ_2 and Γ_3 from the relation (4.2) to (4.3), we have

$$\begin{aligned}\Gamma_3 - \Gamma_2^2 &= -\frac{7}{384}d_1^3 + \frac{7}{64}d_1 d_2 - \frac{1}{16}d_3 - \frac{d_2^2}{64} + \frac{3}{256}d_1^2 d_2 - \frac{9}{4096}d_1^4 \\ &= \frac{1}{12288}[-768d_3 - 224d_1^3 + 1344d_1 d_2 - 192d_2^2 + 144d_1^2 d_2 - 27d_1^4].\end{aligned} \tag{6.2}$$

Applying Lemma 2.4 in the expression (6.2), we have

$$\Gamma_3 - \Gamma_2^2 = \frac{1}{12288}[256d_1^3 + 288d_1 t\zeta + 192d_1 t\zeta^2 - 3d_1^4 - 48t^2\zeta^2 - 24d_1^2 t\zeta - 384(1 - |\zeta|^2)\eta]$$

Taking $t = 4 - d_1^2$ from Lemma 2.4 and without loss of any generality we can write $d_1 = d \in [0, 2]$, in the above relation and $|\xi| = x \leq 1$ and $|\eta| \leq 1$. Taking modulus on both sides of the above relation and then applying triangle inequality with $x \in [0, 1]$ and $d \in [0, 2]$, we obtain

$$|\Gamma_3 - \Gamma_2^2| \leq \frac{1}{12288}[256d^3 + 288d(4 - d^2)x + 192d(4 - d^2)x^2 + 3d^4 + 48(4 - d^2)^2x^2 + 24d^2(4 - d^2)x + 384(1 - x^2)]$$
$$= \frac{1}{12288}[F(d, x)], \quad where \ 0 \leq d \leq 2, \ 0 \leq x \leq 1.$$

Now, we have to find the maximum of $F(d, x)$ in the rectangular region $\omega = [0, 2] \times [0, 1]$. Let us consider for the corner points of the rectangular region ω

$$F(0, 1) = 768, \quad F(2, 1) = 2096, \quad F(2, 0) = 2480, \quad F(0, 0) = 384.$$

Next is to check for the interior of the rectangular region ω. Taking partial derivative of $F(d, x)$ with respect to d and x we get he following.

$$\frac{\partial F}{\partial d} = 192d^3x^2 - 96d^3x - 576d^2x^2 + 12d^3 - 864d^2x - 768d\,x^2 + 768d^2 + 192dx + 768x^2 + 1152x,$$
$$\frac{\partial F}{\partial x} = 96d^4x - 24d^4 - 384d^3x - 288d^3 - 768d^2x + 96d^2 + 1536dx + 1152d + 768x.$$

Solving $\frac{\partial F}{\partial d} = 0$ and $\frac{\partial F}{\partial x} = 0$, we get (0,0) is only critical point which is not a interior point. Hence, it has no critical point in the interior of the rectangle. Now further, to check on the edges of the rectangular region ω.

$$F(0, x) = 384x^2 + 384 \leq 768, \quad F(d, 0) = 3d^4 + 256d^3 + 384 \leq 2480,$$
$$F(2, x) = -384x^2 + 2480 \leq 2480, \quad F(d, 1) = 27d^4 - 224d^3 - 288d^3 + 1920d + 768 \leq \frac{2754681}{1156}.$$

Hence, we conclude that $|\Gamma_3 - \Gamma_2^2| \leq \left(\frac{1}{12288}\right)\left(\frac{2754681}{1156}\right) = \frac{2754681}{14204928} = 0.1939243198$. This completes the proof of Theorem 6.1. □

Theorem 6.2. *Let the function $f \in S^{**}_{s,e}$. Then, $|\Gamma_4 - \Gamma_2\Gamma_3| \leq \frac{917}{1536}$.*

Proof. Suppose that $f \in S^{**}_{s,e}$. Substituting values of Γ_i $(i = 2(1)4)$'s from (4.2)-(4.4) in the expression $\Gamma_4 - \Gamma_2\Gamma_3$, we get

$$\Gamma_4 - \Gamma_2\Gamma_3 = d_1\left(-\frac{19}{256}d_1d_2 + \frac{13}{6144}d_1^3 + \frac{5}{64}d_3\right) + d_1^2\left(-\frac{91}{12288}d_1d_2 + \frac{3}{1024}d_3 + \frac{7}{8192}d_1^3\right) - \frac{1}{16}(d_4 - d_2^2) - \frac{1}{128}d_2\left(d_3 - \frac{7}{4}d_1d_2\right). \tag{6.3}$$

Taking modulus on both sides of (6.3) and followed by triangle inequality and applying inequalities (2.2), (2.3) of Lemma 2.1 and (2.4) of Lemma 2.2, we have

$$|\Gamma_4 - \Gamma_2\Gamma_3| \leq 4\left\{\left|\frac{13}{6144}\right| + \left|\frac{19}{256} - \frac{13}{3072}\right| + \left|\frac{13}{6144} - \frac{19}{256} + \frac{5}{64}\right|\right\}$$
$$+ 4\left\{\left|\frac{7}{8192}\right| + \left|\frac{91}{12288} - \frac{7}{4096}\right| + \left|\frac{7}{8192} - \frac{91}{12288} + \frac{3}{1024}\right|\right\}$$
$$+ \frac{1}{8}max\left\{1, |2 - 1|\right\} + \frac{1}{64}max\left\{1, \left|\frac{7}{2} - 1\right|\right\} = \frac{917}{1536}.$$

This completes the proof of Theorem 6.2. □

7. Krushkal Inequalities for the Class $S^{**}_{s,e}$

We observe from Corollary 3.3 that the well-known inequality:

$$|a_n^p - a_2^{p(n-1)}| \leq 2^{p(n-1)-n^p} \tag{7.1}$$

holds for particular pair of values of $n = 3,\ p = 1$. We will investigate smaller upper bounds for the above inequality for the logarithmic inverse coefficient Γ_n for the class $S^{**}_{s,e}$. The inequality (6.1) was originally introduced and proved by Krushkal for the class of normalized univalent function $f \in \mathcal{S}$ and integers $n \geq 3$, $p \geq 1$, while it is sharp and equality holds for the Koebe function (see [14], Theorem 6.1, p.7).
The following theorem gives upper bounds of l.h.s of (7.1) for logarithmic inverse coefficients when $n = 4$ and $p = 1$ for the class $S^{**}_{s,e}$.

Theorem 7.1. *Assume that $f \in S^{**}_{s,e}$. Then, $|\Gamma_4 - {\Gamma_2}^3| \leq \frac{495}{1024}$.*

Proof. Suppose that $f \in S^{**}_{s,e}$. Making use of (4.2) and (4.4) in the expression $\Gamma_4 - \Gamma_2^3$, we have

$$\begin{aligned}\Gamma_4 - \Gamma_2^3 = &-\frac{1}{16}(d_4 - d_2^2) + \frac{5}{64}d_1\left(d_3 - \frac{19}{20}d_1d_2 + \frac{13}{480}d_1^3\right)\\ &+\frac{1}{512}d_2^2\left(d_2 - \frac{9}{8}d_1^2\right) + \frac{27}{32768}d_1^4(d_2 - \frac{1}{8}d_1^2).\end{aligned} \tag{7.2}$$

Application of triangle inequality on both sides of (7.2) gives

$$\begin{aligned}|\Gamma_4 - \Gamma_2^3| \leq &\frac{1}{16}|d_4 - d_2^2| + \frac{5}{64}|d_1|\left|d_3 - \frac{19}{20}d_1d_2 + \frac{13}{480}d_1^3\right| + \frac{1}{512}|d_2|^2\left|d_2 - \frac{9}{8}d_1^2\right|\\ &+ \frac{27}{32768}|d_1|^4\left|d_2 - \frac{1}{8}d_1^2\right|.\end{aligned}$$

By virtue of inequalities (2.2), (2.3) of Lemma 2.1 and (2.4) of Lemma 2.2 with $\alpha = \frac{13}{480}$, $\beta = \frac{19}{20}$ and $\gamma = 1$, we get

$$\begin{aligned}|\Gamma_4 - \Gamma_2^3| \leq &\frac{1}{16}(2) + \frac{5}{16}\left[\frac{13}{480} + \left|\frac{19}{20} - \frac{13}{240}\right| + \left|\frac{13}{480} - \frac{19}{20} + 1\right|\right]\\ &+ \frac{8}{512}max\left\{1, \left|\frac{9}{4} - 1\right|\right\} + \frac{27}{32768}(16)(2) = \frac{495}{1024}.\end{aligned}$$

This completes the proof of Theorem 7.1. ◻

Concluding Remark: In the present paper with aid of subordination between two analytic functions, the authors introduced a class of starlike functions with respect to symmetric points associated with cardioid domain. We mainly investigated the upper bounds of the first four initial coefficients, Fekete-Szegö functional and Hankel determinant of order two for Taylor-Maclaurin's coefficients of f for the class $S^{**}_{s,e}$. Further, we determine Initial bounds, Hankel determinant, Toeplitz determinant, Hermitian-Toeplitz determinants, Zalcman conjecture, Krushkal inequality and Modulo difference for logarithmic inverse coefficients for the class $S^{**}_{s,e}$. Most of the results are sharp; however, for the remaining bounds that are not, researchers can work on improving them.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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