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BOUR'S THEOREM UNDER THE CONFORMAL MAP WITH LIGHT LIKE PROFILE CURVE

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ABSTRACT

A generalized helicoid and a rotational surface have an isometric relation by Bour's theorem. It is that "A generalized helicoid is isometric to a rotational surface. Hence, helices on the helicoid can be transformed to parallel circles on the rotational surface under the isometric transformation".

In this study, we give a conformal relation between a generalized helicoid (with lightlike profile curve) and a spiral surface (with lightlike profile curve). In this sitiutation, we can say that helices on the helicoid can be transformed to spirals on the spiral surface under the conformal transformation. Also, some related examples and their figures are given.

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ÖZET

Bir genelleştirilmiş helikoid ve bir dönel yüzey arasında Bour teoremi ile izometrik bir ilişkisi vardır. Bu "bir genelleştirilmiş helikoid bir dönel yüzeye izometrik. Dolayısıyla, helikoid üzerindeki helisler izometrik dönüşüm altında dönel yüzey üzerindeki paralel çemberlere karşılık gelir." olmasıdır.

Bu çalışmada, bir genelleştirilmiş helikoid (lightlike profil eğrisi ile) ve bir spiral yüzey (lightlike profil eğrisi ile) arasında bir konformal ilişki verilmiştir. Bu durumda, bir helikoid üzerindeki helislerin konformal dönüşüm altında spiral yüzey üzerindeki spirallere karşılık geldiğini söyleyebiliriz. Ayrıca, bu konuyla ilgili bazı örnekler ve bu örneklerin şekilleri verilmiştir.

Anahtar kelimeler. *Bour teoreim, spiral yüzey, dönel yüzey, helikoid*

1. INTRODUCTION

Surface theory in 3-dimensional Euclidean space has been studied for a long time. In classical differential geometry, rotational surfaces with constant curvature and the right helicoid (resp catenoid) which is the only ruled minimal surface have been known. Also, a pair of these two surfaces has interesting properties namely, they are both members of a one parameter family of isometric minimal surfaces, and if they have the same Gauss map then they are minimal surfaces.

Moreover, Bour showed that a helicoidal surface and a rotational surface are isometric in 3-dimensional Euclidean space. In this generalization, the orginal properties such as, minimality and preservation of the Gauss map are not generaly maintained [1]. Ikawa showed that a helicoidal surface and a rotational surface are isometric according to Bour's theorem in 3-dimensional Euclidean space. He determined the pairs of surfaces with an additional condition that they have the same Gauss map by Bour's theorem [12]. Also, Ikawa gave a classification of the surfaces by types of axis and the profile curves named as (axis's type, profile curve's type)-type. For example, the (S,L)-type which means that the surface has a spacelike axis and a lightlike profile curve [11]. After these studies, Güler give some characterizations for Bour’s theorem in 3-dimensional Euclidean space[6, 8]. Also, he considered the null (lightlike) profile curves of helicoidal and rotational surfaces in Bour's theorem and he showed that Bour's theorem is true in 3-dimensional Minkowski space [7, 9, 10].

Carmo and Dajczer found that there exist a two-parameter family of helicoidal surfaces which is isometric to a given helicoidal surface by using a result of Bour's theorem. Furthermore, with the help of this parametrization, they characterized helicoidal surfaces which have constant mean curvature [5].

Spiral curves and surfaces are most fascinating objects. Because they have important properties such as the size increases without altering the shape. Their properties are seen on different objects around us in differential geometry, in science and in the nature. Let us, we mention some phenomena seen curves which are similar to the spirals. For examples, the approach of a hawk to its prey, the approach of an insect to a light source (see for details [3]), the arms of a spiral galaxy, the arms of the tropical cyclones, the nerves of the cornea and several biological structures, e.g. Romanesco broccoli, Convallaria majalis, some spiral roses, sunflower heads, Nautilus shells and so on because of the fact that these curves are named also growth spirals.

In this paper, we give the relations of between Bour's theorem and the conformal map in Lorentz-Minkowski space E_1^3 with null profile curves. We showed that a spiral surface (with null profile curve) and a generalized helicoid (with null profile curve) have a conformal relation. So, helices on the helicoid can be transformed to spirals on the spiral surface. When the conformal map is isometry we can easily see that this case of the Bour's theorem is Güler's study. So, this paper is generalizations of his study.

2. PRELIMINARIES

Let E_1^3 be the 3-dimensional pseudo-Euclidean space which endowed with the standard flat metric given by

$$g(x,y) = x_1y_1 + x_2y_2 - x_3y_3$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ are the usual coordinate system in E_1^3. Due to semi-Riemannian metric there are three different kind of vectors, namely spacelike, timelike and lightlike (null) depending on the properties $g(x,x) > 0$, $g(x,x) < 0$ and $g(x,x) = 0$, respectively for any vector x in $E_1^3 - \{0\}$. These can be generalized for curves depending on the casual character of their tangent vectors, that is, the curve α is called a spacelike (resp. timelike and lightlike) if its velocity vector $\alpha'(t)$ is spacelike (resp. timelike and lightlike) for any $t \in I$ [14].

In particular, the norm (length) of a vector x is given by $\|x\| = \sqrt{|g(x,x)|}$. Two vectors x and y are orthogonal, if $g(x,y) = 0$. Next, recall that the Lorentzian vector product is defined by

$$x \times y = (x_2y_3 - y_2x_3, y_3x_1 - x_3y_1, y_1x_2 - x_1y_2)$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3) \in E_1^3$ [14].

Let $X(u,v)$ be a parametrization of a surface in E_1^3. Then we introduce the following traditional notation for coefficients of the first and second fundamental forms with regard to the natural bases $\{X_u, X_v\}$:

$$
\begin{aligned}
E &= g(X_u, X_u) \qquad & L &= g(X_{uu}, e)\\
F &= g(X_u, X_v) \qquad & M &= g(X_{uv}, e)\\
G &= g(X_v, X_v) \qquad & N &= g(X_{vv}, e)
\end{aligned}
$$

where the line element of the surface $X(u,v)$ is defined by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

The Gauss map is given by

$$e = \frac{X_u \times X_v}{\sqrt{|g(X_u \times X_v, X_u \times X_v)|}}.$$

Let $\gamma : I \subset \mathrm{R} \to \pi$ be a curve in a plane π in E_1^3 and l be a straight line in π which has no any common point of the curve γ and the straight line l. A surface of rotational in E_1^3 is obtained by rotating a curve γ around a line l (they are called the profile curve and axis, respectively) [11, 13].

Suppose that when a profile curve γ rotates around the axis l, it simultaneously moves parallel to l so that the speed of displacement is proportional to the speed of rotational. Then the resulting surface is called the generalized helicoid with axis l [11, 13].

A spiral surface is the locus of two different positions of any curve which both rotated around an axis and subjected to a homothetic transformation with respect to a point of the axis. Moreover, the tangent vectors of the curve make a constant angle with the rotational axis at any points of the curve [18].

When the surface is non degenerate, the profile curve must be spacelike or timelike. Otherwise; if the surface is degenerate, the profile curve can be only lightlike (null). In this paper, we consider that the profile curve γ is lightlike curve while the axis l be spacelike, timelike or lightlike line in

E_1^3. We accepted that the surface is (S,L)-type (resp. $(T,L)-$type and (L,L)-type) which means that the surface has a spacelike axis and a lightlike profile curve.

3. SPIRAL, HELICOIDAL AND ROTATIONAL SURFACES WITH LIGHTLIKE PROFILE CURVE IN E_1^3

Let suppose that l is the line spanned by a spacelike vector $(1,0,0)$, timelike vector $(0,0,1)$ or lightlike vector $(0,1,1)$. Then the semi-orthogonal matrices of the above vector system can be shown by

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh v & \sinh v \\ 0 & \sinh v & \cosh v \end{bmatrix}, T_1 = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 1 & -v & v \\ v & 1-\frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2} \end{bmatrix}$$

we know that $M_i.l = l$, $M_i^t.\varepsilon.M_i = \varepsilon$, $\varepsilon =$ diag $(1,1,-1)$ and $\det M_i = +1$ $(i = 1,2)$ where $M_i = S_1$, $M_i = T_1$ or $M_i = L_i$ [11, 13].

Hence, the parametrizations of the surface $X(u,v)$ are denoted by

$$X(u,v) = M_i.\gamma.$$

In this paper, we consider that the profile curve is $\gamma(u) = (u,u,\phi(u))$. If the curve γ is a null curve, that is, $\gamma'(u) = \frac{d}{du}\gamma(u) \neq 0$ and $g(\gamma'(u),\gamma'(u)) = 0$ where the function ϕ is given by

$$\phi(u) = \sqrt{2}u + c, \; c = \text{constant}.$$

Then we have three different cases for the axis l of rotational surface in E_1^3.

Case1. Let us consider that the profile curve γ is a lightlike curve and the axis l is spacelike line. Then there exist a Lorentzian transformation which transform l into x_1 −axis and the spiral surface $S(u,v)$ can be written as

$$\begin{aligned} S(u,v) = (&f(v)u, \\ &f(v)(u\cosh v + \phi(u)\sinh v), \\ &f(v)(u\sinh v + \phi(u)\cosh v)) \end{aligned}$$

where $f : I \subset \mathrm{R} \to \mathrm{R}$ is a differentiable function for all $v \in I$, the function $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function.

The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = (u, u\cosh v + \phi(u)\sinh v, u\sinh v + \phi(u)\cosh v)$$

and

$$H(u,v) = (u + av, u\cosh v + \phi(u)\sinh v, u\sinh v + \phi(u)\cosh v)$$

where a is a non-zero constant.

Case 2. Let us consider that the profile curve γ is a lightlike curve and the axis l is timelike line. Then there exist a Lorentzian transformation which transform l into x_3 − axis and the spiral surface $S(u,v)$ can be written as

$$S(u,v) = (f(v)(u\cos v - u\sin v), f(v)(u\sin v + u\cos v), f(v)\phi(u))$$

where $f : I \subset \mathrm{R} \to \mathrm{R}$ is a differentiable function for all $v \in I$, the funtion $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function.

The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = (u\cos v - u\sin v, u\sin v + u\cos v, \phi(u))$$

and

$$H(u,v) = (u\cos v - u\sin v, u\sin v + u\cos v, \phi(u) + av)$$

where a

is a non-zero constant.

Case 3. Let us consider that the profile curve γ is lightlike curve and the axis l is a lightlike line. Then there exist Lorentzian transformation which transform l into $Sp\{\xi = (0,1,1)\}$ in E_1^3 and the spiral surface $S(u,v)$ can be written as

$$S(u,v) = \begin{pmatrix} f(v)(u - uv + \phi(u)v) \\ f(v)(uv + (1 - \frac{v^2}{2})u + \frac{v^2}{2}\phi(u)) \\ f(v)(uv - \frac{uv^2}{2} + (1 + \frac{v^2}{2})\phi(u)) \end{pmatrix}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ differentiable function for all $v \in I$, the function $f(v)$ is equal to $e^{g'(v)}$ and $g''(v) = b$ is a constant function.

The rotational surface $R(u,v)$ and the generalized helicoid $H(u,v)$ can be calculated as

$$R(u,v) = \begin{pmatrix} u - uv + \phi(u)v \\ uv + (1 - \frac{v^2}{2})u + \frac{v^2}{2}\phi(u) \\ uv - \frac{uv^2}{2} + (1 + \frac{v^2}{2})\phi(u) \end{pmatrix}$$

And

$$H(u,v) = \begin{pmatrix} u - uv + \phi(u)v \\ uv + (1 - \frac{v^2}{2})u + \frac{v^2}{2}\phi(u) + av \\ uv - \frac{uv^2}{2} + (1 + \frac{v^2}{2})\phi(u) + av \end{pmatrix}$$

Where a is a non-zero constant.

Remark. If $\phi(u)$ is a constant function then we can easily see that the surface $H(u,v)$ is a right helicoid in E_1^3.

Definition 1. Let X and Y

denote the surfaces. A mapping of surfaces

$f : X \to Y$ is conformal provided there exist a real-valued function $\lambda > 0$ on X such that

$$g(f_* v_p, f_* v_p) = \lambda(p) g(v_p, v_p)$$

for all tangent vectors to X.

i) If λ is constant function then f is named as a homothetic function.

ii) If λ is equal to 1 then we say f is an isometric function.

Where the function λ is called the *scale factor* of f [17].

Definition 2. Let X and Y denote the surfaces and $f : X \to Y$ be a mapping. Then f is a conformal map *iff*

$$E = \lambda^2 \overline{E},\ F = \lambda^2 \overline{F},\ G = \lambda^2 \overline{G}$$

Where E, F, G and $\overline{E}$, $\overline{F}$, $\overline{G}$ are the coefficients of the first fundamental form of X and Y, respectively [15, 16].

Theorem 1. (***Bour's theorem***) A generalized helicoid is isometric to a rotational surface such that helices on the helicoid can be transformed to the parallel circles on the rotational surface [1].

In this section, we give the *Bour's theorem* under the conformal map with a null profile curve and spacelike, timelike or lightlike axis l. Also, we give some related examples and their figures in Minkowski space E_1^3.

4. BOUR'S THEOREM UNDER THE CONFORMAL MAP IN MINKOWSKI SPACE E_1^3

In the following section, we obtain the *Bour's theorem* and an application of the theorem under the conformal map with a null profile curve and spacelike axis l which are given in the Case 1.

Theorem 2. Let $H(u,v)$ and $S(u,v)$ be (S,L)-type helicoidal surface and (S,L)-type spiral surface which are given in the Case 1, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equation

$$S(u,v)=\begin{pmatrix} e^{g'(v)}u_s \\ e^{g'(v)}u_s\cosh v+e^{g'(v)}(\sqrt{2}u_s+c_2)\sinh v \\ e^{g'(v)}u_s\sinh v+e^{g'(v)}(\sqrt{2}u_s+c_2)\cosh v \end{pmatrix}$$

where

$$u_s=(b^2-1)\sqrt{2}c_2\pm\sqrt{(2b^4-3b^2+1)c_2^2+a+a^2+u^2+\sqrt{2}uc_1}$$

$$c_2=\frac{a+c_1}{1-b\sqrt{2}},\ c_1,\ a,\ b=\text{const.}$$

and $a\neq 0,\ b\neq 0,\ c_1$ are constants.

Consequently, we can easily say that a helix on the (S,L)-type helicoidal surface can be transformed to a spiral on the (S,L)-type spiral surface.

Proof. The coefficients of the first fundamental form of the (S,L)-type generalized helicoid are obtained as

$$E=g(H_u,H_u)=0$$
$$F=g(H_u,H_v)=a+c_1$$
$$G=g(H_v,H_v)=a^2+u^2+2\sqrt{2}c_1u+c_1^2.$$

and so, we obtain the line element of the helicoidal surface $H(u,v)$ as

$$ds^2 = (2a + \phi - u\phi' c_1)dudv + (a^2 + u^2 + 2\sqrt{2}c_1 u + c_1^2)dv^2 \qquad \textbf{(1)}$$

On the other hand, the coefficients of the first fundamental form of the (S,L)-type spiral surface

$$S(u_s, v_s) = \begin{pmatrix} e^{g'(v_s)} u_s \\ e^{g'(v_s)}(u_s \cosh v_s + (\sqrt{2}u_s + c_2)\sinh v_s) \\ e^{g'(v_s)}(u_s \sinh v_s + (\sqrt{2}u_s + c_2)\cosh v_s) \end{pmatrix}$$

(where $g''(v_s) = b \in \mathrm{R} - \{0\}$) are obtained as

$$E_s = g(S_{u_s}, S_{u_s}) = 0$$
$$F_s = g(S_{u_s}, S_{v_s}) = \left(e^{g'(v_s)}\right)^2 (c_2 - b\sqrt{2}c_2)$$
$$G_s = g(S_{v_s}, S_{v_s}) = \left(e^{g'(v_s)}\right)^2 (2\sqrt{2}c_2 u_s + u_s^2 - 2b^2\sqrt{2}c_2 u_s + (1-b^2)c_2^2)$$

and so, we obtain the line element of the spiral surface $S(u_s, v_s)$ as

$$\begin{aligned} ds_s^2 = \left(e^{g'(v_s)}\right)^2 [&2(c_2 - b\sqrt{2}c_2)du_s dv_s + \\ &(2\sqrt{2}c_2 u_s + u_s^2 - 2b^2\sqrt{2}c_2 u_s^2 + (1-b^2)c_2^2)dv_s^2]. \end{aligned} \qquad \textbf{(2)}$$

comparing the Eq.(1) with Eq.(2), we obtain

$$u_s = (b^2 - 1)\sqrt{2}c_2 \pm \sqrt{(2b^4 - 3b^2 + 1)c_2^2 + c_1^2 + a^2 + u^2 + 2\sqrt{2}uc_1}$$

where

$$c_2 = \frac{a + c_1}{1 - b\sqrt{2}}, \; c_1, a, b = \text{const.}$$

Therefore

$$E_s = \left(e^{g'(v_s)}\right)^2 E, \; F_s = \left(e^{g'(v_s)}\right)^2 F, \; G_s = \left(e^{g'(v_s)}\right)^2 G.$$

So, we obtain a conformal relation between a generalized helicoid

$$H(u,v) = (u + av, u\cosh v + (\sqrt{2}u + c_1)\sinh v,$$
$$u\sinh v + (\sqrt{2}u + c_1)\cosh v),\ a = constant$$

and the spiral surface

$$S(u,v) = \begin{pmatrix} e^{g'(v)}u_s \\ e^{g'(v)}u_s\cosh v + e^{g'(v)}(\sqrt{2}u_s + c_2)\sinh v \\ e^{g'(v)}u_s\sinh v + e^{g'(v)}(\sqrt{2}u_s + c_2)\cosh v \end{pmatrix}$$

Since the profile curve γ is lightlike and $EG - F^2 < 0$ the helicoidal and spiral surfaces are timelike.

Also, we obtain a helix on the generalized helicoid when u is a constant. To give a curve orthogonal to helix, we consider the orthogonal case,

$$(a + c_1)du + (a^2 + u^2 + 2\sqrt{2}c_1u + c_1^2)dv = 0.$$

Then we obtain that

$$v = -\int \frac{a + c_1}{a^2 + u^2 + 2\sqrt{2}c_1u + c_1^2}du + c,\ c = \text{constant}.$$

Hence, if we consider that

$$\overline{v} = v - \int \frac{a + c_1}{a^2 + u^2 + 2\sqrt{2}c_1u + c_1^2}du$$

then the orthogonal curve is given as $\overline{v} = \text{constant}$. If we differentiate the last equation we obtain

$$d\overline{v} = dv - \frac{a + c_1}{a^2 + u^2 + 2\sqrt{2}c_1u + c_1^2}du.$$

Substituting this equation into the line element, we have

$$ds^2 = \left(-\frac{(a+c_1)^2}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}\right)du^2 + \left(a^2+u^2+2\sqrt{2}c_1u+c_1^2\right)d\bar{v}^2. \quad \textbf{(3)}$$

If we consider that

$$\bar{u} = \int\sqrt{\frac{(a+c_1)^2}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}}, f(\bar{u}) = \sqrt{a^2+u^2+2\sqrt{2}c_1u+c_1^2},$$

then the line element (in Eq.(3)) of the generalized helicoidal surface reduces to

$$ds^2 = -d\bar{u}^2 + f^2(\bar{u})d\bar{v}^2.$$

On the other hand, if we consider that $e^{g'(v_s)}$ is constant, i.e, $b=0$, then the spiral surface $S(u_s, v_s)$ reduce to a rotational surface whose line element is

$$ds_s^2 = \left(e^{g'(v_s)}\right)^2\left[-\left(\frac{c_2^2}{(2\sqrt{2}c_2u_s+u_s^2+c_2^2)}\right)d\bar{u}_s^2 + (2\sqrt{2}c_2u_s+u_s^2+c_2^2)d\bar{v}_s^2\right]. \quad \textbf{(4)}$$

if we consider that

$$\bar{u}_s = \int\sqrt{\frac{c_2^2}{(2\sqrt{2}c_2u_s+u_s^2+c_2^2)}},\ f_s(\bar{u}_s) = \sqrt{2\sqrt{2}c_2u_s+u_s^2+c_2^2}.$$

Comparing Eq.(3) with Eq.(4), and putting

$$\bar{u}_s = \bar{u},\ \bar{v}_s = \bar{v},\ f_s(\bar{u}_s) = f(\bar{u})$$

we obtain that

$$u_s = -\sqrt{2}c_2 \pm \sqrt{a+a^2+u^2+c_2^2+\sqrt{2}uc_1},\ c_2\ a = const.$$

So, we obtain a homothetic relation between a generalized helicoidal surface

$H(u,v)=\left(u+av, u\cosh v+(\sqrt{2}u+c_1)\sinh v, u\sinh v+(\sqrt{2}u+c_1)\cosh v\right)$
and the rotational surface

$$R(u,v)=\begin{pmatrix} (-u+(\sqrt{2}u_s+c_2), \\ e^{g'(v_s)}u_s\cosh(v-\int\frac{a+c_1}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}du+ \\ \int\frac{c_2^2}{2\sqrt{2}c_2u_s+u_s^2+c_2^2}du_s) \\ +e^{g'(v_s)}(\sqrt{2}u_s+c_2)\sinh(v-\int\frac{a+c_1}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}du+ \\ \int\frac{c_2^2}{2\sqrt{2}c_2u_s+u_s^2+c_2^2}du_s), \\ e^{g'(v_s)}u_s\sinh(v-\int\frac{a+c_1}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}du+ \\ \int\frac{c_2^2}{2\sqrt{2}c_2u_s+u_s^2+c_2^2}du_s) \\ +e^{g'(v_s)}(\sqrt{2}u_s+c_2)\cosh(v-\int\frac{a+c_1}{a^2+u^2+2\sqrt{2}c_1u+c_1^2}du+ \\ \int\frac{c_2^2}{2\sqrt{2}c_2u_s+u_s^2+c_2^2}du_s)) \end{pmatrix}.$$

If we consider that $e^{g'(v_s)}=1$ then we obtain an isometric relation between a generalized helicoidal surface and the rotational surface, i.e, we obtain Bour's theorem (in [9]) in Minkowski 3-space E_1^3.

Example 1. If we consider any (S,L)-type helicoidal surface with the following equation

$$H(u,v)=\left(u+v, u\cosh v+\left(\sqrt{2}u+1\right)\sinh v, u\sinh v+\left(\sqrt{2}u+1\right)\cosh v)\right)$$

then we can easily obtain that the (S,L)-type spiral surface which is image of the helicoidal surface $H(u,v)$ under the conformal map is given by

$$S(u,v)=\begin{pmatrix} e^{2v/3}\left(\dfrac{16}{9(1-\frac{2\sqrt{2}}{3})}+\dfrac{2\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}-\sqrt{2+\dfrac{40}{81(1-\frac{2\sqrt{2}}{3})^2}}+\sqrt{u}+u^2\right), \\ e^{2v/3}\left(\dfrac{16}{9(1-\frac{2\sqrt{2}}{3})}+\dfrac{2\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}-\sqrt{2+\dfrac{40}{81(1-\frac{2\sqrt{2}}{3})^2}}+\sqrt{u}+u^2\right)\cosh v+ \\ \sqrt{2}e^{2v/3}\left(\dfrac{16}{9(1-\frac{2\sqrt{2}}{3})}+\dfrac{2\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}-\sqrt{2+\dfrac{40}{81(1-\frac{2\sqrt{2}}{3})^2}}+\sqrt{u}+u^2\right)\sinh v, \\ e^{2v/3}\left(\dfrac{16}{9(1-\frac{2\sqrt{2}}{3})}+\dfrac{2\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}-\sqrt{2+\dfrac{40}{81(1-\frac{2\sqrt{2}}{3})^2}}+\sqrt{u}+u^2\right)\sinh v+ \\ \sqrt{2}e^{2v/3}\left(\dfrac{16}{9(1-\frac{2\sqrt{2}}{3})}+\dfrac{2\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}-\sqrt{2+\dfrac{40}{81(1-\frac{2\sqrt{2}}{3})^2}}+\sqrt{u}+u^2\right)\cosh v \end{pmatrix}$$

The picture of the helicoidal surface $H(u,v)$ and spiral surface $S(u,v)$ are rendered in Figure1.

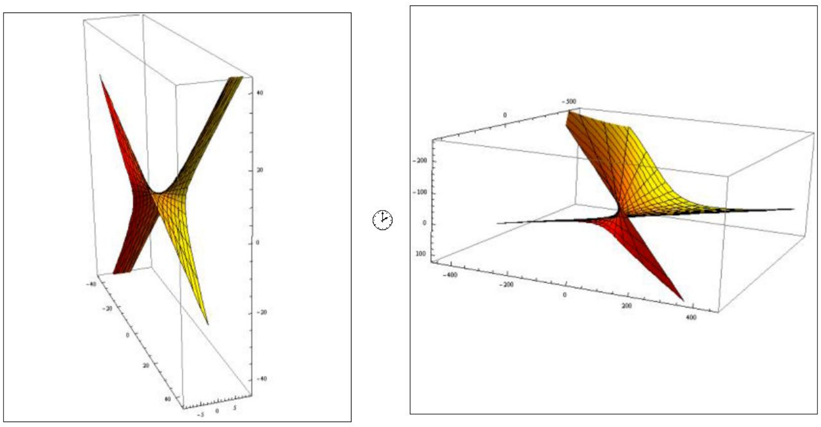


Figure 1 : A (S, L) - typehelicoidal surface and its image (S, L) - typespiral surface under the conformal map.

In the following section, we obtain the *Bour's theorem* and an application of the theorem under the conformal map with a null profile curve and timelike axis l which are given in the Case 2.

Theorem 3. Let $H(u,v)$ and $S(u,v)$ be (T,L)-type helicoidal surface and (T,L)-type spiral surface which are given in the Case 2, respectively, If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be shown by the following equation

$$S(u,v)=\begin{pmatrix} e^{g'(v)}u_s\cos v-e^{g'(v)}u_s\sin v \\ e^{g'(v)}u_s\sin v-e^{g'(v)}u_s\cos v \\ e^{g'(v)}(\sqrt{2}u_s+c_2) \end{pmatrix}$$

where

$$u_s=\frac{\sqrt{2}c_2b^2}{2}\pm\sqrt{c_2^2(b^4+b^2+2u^2-a^2)},$$

$$c_2=\frac{a}{b}$$

and $a\neq 0,\ b\neq 0$ are constants.

Consequently, we can easily say that a helix on the (T,L)-type helicoidal surface can be transformed to a spiral on the (T,L)-type spiral surface.

Proof. The coefficients of the first fundamental form of the (T,L)-type generalized helicoidal surface

$$H(u,v)=\left(u\cos v-u\sin v, u\sin v+u\cos v, \sqrt{2}u+c_1+av\right),$$
$$a,c_1=\text{constant}$$

are obtained as

$$E=g(H_u,H_u)=0$$
$$F=g(H_u,H_v)=-a\sqrt{2}$$
$$G=g(H_v,H_v)=2u^2-a^2$$

and so, we obtain the line element of the $H(u,v)$ as

$$ds^2 = 2\left(-a\sqrt{2}\right)dudv + (2u^2 - a^2)dv^2. \qquad (5)$$

On the other hand, the coefficients of the first fundamental form of the (T,L)-type spiral surface

$$S(u_s, v_s) = \begin{pmatrix} e^{g'(v_s)}u_s \cos v_s - e^{g'(v_s)}u_s \sin v_s \\ e^{g'(v_s)}u_s \sin v_s - e^{g'(v_s)}u_s \cos v_s \\ e^{g'(v_s)}(\sqrt{2}u_s + c_2) \end{pmatrix}$$

(where $g''(v_s) = b \in \mathrm{R} - \{0\}$) are obtained as

$$E_s = g(S_{u_s}, S_{u_s}) = 0$$

$$F_s = g(S_{u_s}, S_{v_s}) = \left(e^{g'(v_s)}\right)^2(-\sqrt{2}bc_2)$$

$$G_s = g(S_{v_s}, S_{v_s}) = \left(e^{g'(v_s)}\right)^2(-2\sqrt{2}c_2b^2u_s + 2u_s^2 - b^2c_2^2)$$

and so, we obtain the line element of the spiral surface $S(u_s, v_s)$ as

$$ds_s^2 = \left(e^{g'(v_s)}\right)^2[2(-2u_s - \sqrt{2}bc_2)du_sdv_s + (2\sqrt{2}c_2b^2u_s + 2u_s^2 - b^2c_2^2)dv_s^2]. \qquad (6)$$

comparing Eq.(5) with Eq.(6), we obtain

$$c_2 = \frac{a}{b}$$

and

$$u_s = \frac{\sqrt{2}c_2b^2}{2} \pm \sqrt{c_2^2(b^4 + b^2 + 2u^2 - a^2)}$$

where a, b are nonzero constant.

Therefore

$$E_s = \left(e^{g'(v_s)}\right)^2 E,\ F_s = \left(e^{g'(v_s)}\right)^2 F,\ G_s = \left(e^{g'(v_s)}\right)^2 G.$$

So, we obtain a conformal relation between a generalized helicoidal surface

$$H(u,v)=\left(u\cos v-u\sin v, u\sin v+u\cos v, \sqrt{2}u+c_1+av\right), a, c_1=\text{constant}$$

and the spiral surface

$$S(u,v)=\begin{pmatrix} e^{g'(v)}u_s\cos v-e^{g'(v)}u_s\sin v \\ e^{g'(v)}u_s\sin v-e^{g'(v)}u_s\cos v \\ e^{g'(v)}(\sqrt{2}u_s+c_2) \end{pmatrix}$$

Since the profile curve γ is lightlike and $EG-F^2<0,$ surfaces are timelike.

Also, we obtain a helix on the generalized helicoidal surface when u is a constant. To give a curve orthogonal to helix, we consider the orthogonal case,

$$(-a\sqrt{2})du+(2u^2-a^2)dv=0.$$

Then, we obtain

$$v=\int\frac{a\sqrt{2}}{2u^2-a^2}du+c,\ c=\text{constant}.$$

Hence, if we consider that

$$\bar{v}=v-\int\frac{-a\phi'}{2u^2-a^2}du$$

then the orthogonal curve is given by $\bar{v}=$ constant. If we differentiate the last equation we obtain

$$d\bar{v}=dv+\frac{a\sqrt{2}}{2u^2-a^2}du.$$

Substituting this equation into the line element, we have

$$ds^2 = \left(-\frac{\left(a\sqrt{2}\right)^2}{2u^2 - a^2}\right)du^2 + \left(2u^2 - a^2\right)d\bar{v}^2. \tag{7}$$

If we consider that

$$\bar{u} = \int \sqrt{\frac{\left(a\sqrt{2}\right)^2}{2u^2 - a^2}},\ f(\bar{u}) = \sqrt{2u^2 - a^2}$$

then the line element (in Eq.(7)) of the generalized helicoidal surface reduces to

$$ds^2 = -d\bar{u}^2 + f^2(\bar{u})d\bar{v}^2.$$

On the other hand, if we consider that $e^{g'(v_s)}$ is constant, i.e, $b = 0$, then the spiral surface $S(u_s, v_s)$ reduces to rotational surface has the line element

$$ds_s^2 = \left(e^{g'(v_s)}\right)^2[-du_s^2 + 2u_s^2 d\bar{v}_s^2]. \tag{8}$$

where if we consider that

$$\bar{u}_s = 0,\ f_s(\bar{u}_s) = \sqrt{2u_s^2}.$$

then the line element (in Eq.(8)) of the rotational surface reduces to

$$ds_s^2 = -d\bar{u}_s^2 + f_s^2(\bar{u}_s)d\bar{v}_s^2.$$

Comparing Eq.(7) with Eq.(8), and putting

$$\bar{u}_s = \bar{u},\ \bar{v}_s = \bar{v},\ f_s(\bar{u}_s) = f(\bar{u}).$$

we obtain

$$u_s = \int \sqrt{u^2 - \frac{a^2}{2}}du,\ a = const.$$

and

$$c_2 = c_1 - \frac{2a^2}{2u^2 - u^2}$$

So, we obtain a homothetic relation between a generalized helicoidal surface

$$H(u,v) = \left(u\cos v - u\sin v, u\sin v + u\cos v, \sqrt{2}u + c_1 + av\right),$$
$a,\ c_1 = \text{constant}$

and the rotational surface

$$R(u,v) = \begin{pmatrix} e^{g'(v)}\sqrt{u^2 - \frac{a^2}{2}}\cos v - e^{g'(v)}\sqrt{u^2 - \frac{a^2}{2}}\sin v \\ e^{g'(v)}\sqrt{u^2 - \frac{a^2}{2}}\sin v - e^{g'(v)}\sqrt{u^2 - \frac{a^2}{2}}\cos v \\ e^{g'(v)}(\sqrt{2u^2 - a^2} + c_2) \end{pmatrix}.$$

If we consider that $e^{g'(v_s)} = 1$ then we obtain a isometric relation between a generalized helicoidal surface and the rotational surface, i.e, we obtain Bour's theorem (in [9]) in Minkowski 3-space E_1^3.

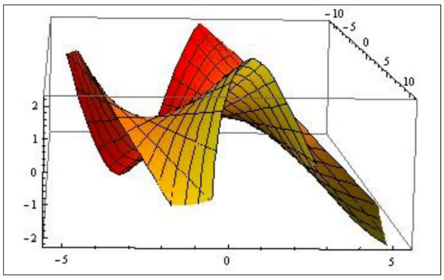
Example 2. If we consider any (T,L)-type helicoidal surface with the following equation

$$H(u,v) = \left(u\cos v - u\sin v, u\sin v + u\cos v, \sqrt{2}u + 1 + v\right)$$

then we can easily obtain that the (T,L)-type spiral surface which is image of the helicoidal surface $H(u,v)$ under the conformal map is given by

$$S(u,v) = \begin{pmatrix} e^{2v/3}(\frac{\sqrt{2}}{3} - \frac{3}{2}\sqrt{2u^2 - \frac{29}{81}})\cos v - e^{2v/3}(\frac{\sqrt{2}}{3} - \frac{3}{2}\sqrt{2u^2 - \frac{29}{81}})\sin v \\ e^{2v/3}(\frac{\sqrt{2}}{3} - \frac{3}{2}\sqrt{2u^2 - \frac{29}{81}})\sin v + e^{2v/3}(\frac{\sqrt{2}}{3} - \frac{3}{2}\sqrt{2u^2 - \frac{29}{81}})\cos v \\ e^{2v/3}(\sqrt{2}(\frac{\sqrt{2}}{3} - \frac{3}{2}\sqrt{2u^2 - \frac{29}{81}}) + \frac{3}{2}) \end{pmatrix}.$$

The picture of the helicoidal surface $H(u,v)$ and spiral surface curve $S(u,v)$ are rendered in Figure 2.



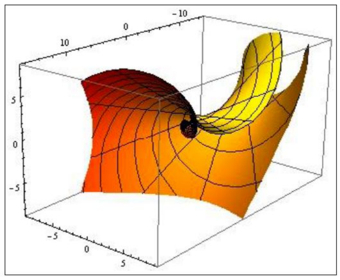


Figure 2 : A (T, L) - typehelicoidal surface and its image (T, L) typespiral surface under the conformal map

In the following section, we obtain the *Bour's theorem* and we give an application of the theorem under the conformal map with a null profile curve and lightlike axis l which are given in the Case 3.

Theorem 4. Let $H(u,v)$ and $S(u,v)$ be (L,L)-type helicoidal surface and (L,L)-type spiral surface which are given in the Case 3, respectively. If $H(u,v)$ and $S(u,v)$ have a conformal relation then the spiral surface must be given by the following equatio

$$S(u,v)=\begin{pmatrix} e^{g'(v)}(u_s(1-v)+(\sqrt{2}u_s+c_2)v) \\ e^{g'(v)}(u_s(v-\frac{v^2}{2}+1)+\frac{v^2}{2}(\sqrt{2}u_s+c_2)) \\ e^{g'(v)}(u_s(v-\frac{v^2}{2})+(1+\frac{v^2}{2})(\sqrt{2}u_s+c_2)) \end{pmatrix}$$

where

$$u_s=\frac{-c_2-\sqrt{2}c_2b^2\pm\sqrt{(c_2-\sqrt{2}c_2b^2)^2+(1+2\sqrt{2})(b^2c_2^2)+(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}},$$

$$c_2=\frac{a-\sqrt{2}a+c_1}{1-\sqrt{2}b}$$

and $a\neq 0,\ b\neq 0,\ c_1$ are constants.

Consequently, we can easily say that a helix on the (T,L)-type helicoidal surface can be transformed to a spiral on the (T,L)-type spiral surface.

Proof. The coefficients of the first fundamental form of the (L,L)-type generalized helicoidal surface

$$H(u,v)=\begin{pmatrix} u-uv-(\sqrt{2}u+c_1)v \\ uv+(1-\frac{v^2}{2})u+\frac{v^2}{2}(\sqrt{2}u+c_1)+av \\ uv-\frac{v^2}{2}u+(1+\frac{v^2}{2})(\sqrt{2}u+c_1)+av) \end{pmatrix}, a=\text{constant}$$

are obtained as

$$E=g(H_u,H_u)=0$$
$$F=g(H_u,H_v)=a+c_1-\sqrt{2}a$$
$$G=g(H_v,H_v)=\left(u-\sqrt{2}u-c_1\right)^2$$

and so, we obtain the line element of $H(u,v)$ as

$$ds^2=2(a+c_1-\sqrt{2}a)dudv+\left(u-\sqrt{2}u-c_1\right)^2dv^2. \tag{9}$$

On the other hand, the coefficients of the first fundamental form of (L,L) -type spiral surface

$$S(u_s,v_s)=\begin{pmatrix} e^{g'(v_s)}(u-uv+(\sqrt{2}u_s+c_2)v) \\ e^{g'(v_s)}(uv+(1-\frac{v^2}{2})u+\frac{v^2}{2}(\sqrt{2}u_s+c_2)) \\ e^{g'(v_s)}(uv+-\frac{v^2}{2}u+(1+\frac{v^2}{2})(\sqrt{2}u_s+c_2)) \end{pmatrix}$$

where $g''(v_s)=b\in\mathbb{R}-\{0\})$ are obtained as

$$E_s=g(S_{u_s},S_{u_s})=0$$
$$F_s=g(S_{u_s},S_{v_s})=\left(e^{g'(v_s)}\right)^2(c_2-b\sqrt{2}c_2)$$
$$G_s=g(S_{v_s},S_{v_s})=\left(e^{g'(v_s)}\right)^2(2b^2u_s^2+(\sqrt{2}u_s+c_2-u_s)^2-b^2(\sqrt{2}u_s+c_2)^2)$$

and so, we obtain the line element of $S(u_s,v_s)$ as

$$ds_s^2 = \left(e^{g'(v_s)}\right)^2 [2(c_2 - b\sqrt{2}c_2)du_s dv_s + (2b^2u_s^2 + (\sqrt{2}u_s + c_2 - u_s)^2 - b^2(\sqrt{2}u_s + c_2)^2)dv_s^2]. \qquad \textbf{(10)}$$

comparing Eq.(9) with Eq.(10), we obtain

$$u_s = \frac{-c_2 - \sqrt{2}c_2b^2 \pm \sqrt{(2b^4 + b^2 + 1)^2 c_2^2 + (u - \sqrt{2}u - c_1)^2}}{1 + 2\sqrt{2}}.$$

where

$$c_2 = \frac{a - \sqrt{2}a + c_1}{1 - \sqrt{2}b}$$

Therefore

$$E_s = \left(e^{g'(v_s)}\right)^2 E,\ F_s = \left(e^{g'(v_s)}\right)^2 F,\ G_s = \left(e^{g'(v_s)}\right)^2 G.$$

So, we obtain a conformal relation between a generalized helicoidal surface

$$H(u,v) = \begin{pmatrix} u - uv - (\sqrt{2}u + c_1)v \\ uv + (1 - \frac{v^2}{2})u + \frac{v^2}{2}(\sqrt{2}u + c_1) + av \\ uv - \frac{v^2}{2}u + (1 + \frac{v^2}{2})(\sqrt{2}u + c_1) + av) \end{pmatrix}, a = \text{constant}$$

and the spiral surface

$$S(u,v) = \begin{pmatrix} e^{g'(v)}(u_s(1 - v) + (\sqrt{2}u_s + c_2)v) \\ e^{g'(v)}(u_s(v - \frac{v^2}{2} + 1) + \frac{v^2}{2}(\sqrt{2}u_s + c_2)) \\ e^{g'(v)}(u_s(v - \frac{v^2}{2}) + (1 + \frac{v^2}{2})(\sqrt{2}u_s + c_2)) \end{pmatrix}$$

Since the profile curve γ is lightlike and $EG - F^2 < 0$, surfaces are timelike.

Also, we obtain a helix on the generalized helicoidal surface when u is a constant. To give an orthogonal curve to helix, we consider the orthogonal case,

$$(a + c_1 - \sqrt{2}a)du + \left(u - \sqrt{2}u - c_1\right)^2 dv = 0.$$

Then we obtain

$$v = \int \frac{a + c_1 - \sqrt{2}a}{\left(u - \sqrt{2}u - c_1\right)^2} du + c, \ c = \text{constant}.$$

Hence, if we consider that

$$\overline{v} = v - \int \frac{a + c_1 - \sqrt{2}a}{\left(u - \sqrt{2}u - c_1\right)^2} du$$

then the orthogonal curve is given by $\overline{v} =$ constant. If we differentiate the last equation we obtain

$$d\overline{v} = dv - \frac{a + c_1 - \sqrt{2}a}{\left(u - \sqrt{2}u - c_1\right)^2} du.$$

Substituting this equation into the line element, we have

$$ds^2 = (-\frac{\left(a + c_1 - \sqrt{2}a\right)^2}{\left(u - \sqrt{2}u - c_1\right)^2})du^2 + \left(u - \sqrt{2}u - c_1\right)^2 d\overline{v}^2. \qquad \textbf{(11)}$$

If we consider that

$$\overline{u} = \int \sqrt{\frac{\left(a + c_1 - \sqrt{2}a\right)^2}{\left(u - \sqrt{2}u - c_1\right)^2}}, f(\overline{u}) = u - \sqrt{2}u - c_1,$$

then the line element (in Eq.(11)) of the generalized helicoid reduces to

$$ds^2 = -d\overline{u}^2 + f^2(\overline{u})d\overline{v}^2.$$

On the other hand, if we consider that $e^{g'(v_s)}$ is constant, i.e, $b=0$, then the spiral surface $S(u_s,v_s)$ reduce to the rotational surface has the line element

$$ds_s^2=\left(e^{g'(v_s)}\right)^2[-\left(\frac{c_2{}^2}{(\sqrt{2}u_s+c_2-u_s)^2}\right)du_s^2+(\phi_s-u_s)^2 d\overline{v}_s^2]. \qquad \textbf{(12)}$$

where if we consider that

$$\overline{u}_s=\int\sqrt{\frac{c_2{}^2}{(\sqrt{2}u_s+c_2-u_s)^2}},\ f_s(\overline{u}_s)=\sqrt{2}u_s+c_2-u_s.$$

then the line element (in Eq.(12)) of the rotational surface reduces to

$$ds_s^2=-d\overline{u}_s^2+f_s^2(\overline{u}_s)d\overline{v}_s^2.$$

Comparing Eq.(11) with Eq.(12), and putting

$$\overline{u}_s=\overline{u},\ \overline{v}_s=\overline{v},\ f_s(\overline{u}_s)=f(\overline{u}).$$

we obtain

$$u_s=\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}},$$

$$v_s=v-\int\frac{a+c_1-\sqrt{2}a}{\left(u-\sqrt{2}u-c_1\right)^2}du-\int\frac{c_2{}^2}{(\sqrt{2}u_s+c_2-u_s)^2}du_s,\ c_1=\text{const.}$$

and

$$c_2=\frac{a+c_1-a\sqrt{2}}{1-b\sqrt{2}}$$

where a,b are constants and $b\neq\frac{1}{\sqrt{2}}$. So, we obtain a homothetic relation between a generalized helicoid

$$H(u,v)=\begin{pmatrix} u-uv-(\sqrt{2}u+c_1)v \\ uv+(1-\frac{v^2}{2})u+\frac{v^2}{2}(\sqrt{2}u+c_1)+av \\ uv-\frac{v^2}{2}u+(1+\frac{v^2}{2})(\sqrt{2}u+c_1)+av) \end{pmatrix}, a=\text{const.}$$

and the rotational surface

$$R(u,v)=\begin{pmatrix} e^{g'(v_s)}(\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}(1-v_s)+ \\ (\sqrt{2}\,\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}+c_2)v_s), \\ e^{g'(v_s)}(\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}(v_s-\frac{v_s^2}{2}+1)+ \\ \frac{v_s^2}{2}(\sqrt{2}\,\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}+c_2)), \\ e^{g'(v_s)}(\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}(v_s-\frac{v_s^2}{2})+ \\ (1+\frac{v_s^2}{2})(\sqrt{2}\,\frac{-c_2\pm\sqrt{c_2{}^2+(1+2\sqrt{2})(u-\sqrt{2}u-c_1)^2}}{1+2\sqrt{2}}+c_2)) \end{pmatrix}$$

If we consider that $e^{g'(v_s)}=1$ then we obtain an isometric relation between a generalized helicoidal and the rotational surface, that is, we obtain Bour's theorem (in [9]) in Minkowski 3-space in E_1^3.

Example 3. If we consider any (L,L)-type helicoidal surface with the following equation

$$H(u,v)=\begin{pmatrix} (u-uv-(\sqrt{2}u+1)v, uv+(1-\frac{v^2}{2})u+\frac{v^2}{2}(\sqrt{2}u+1)+av, \\ uv-\frac{v^2}{2}u+(1+\frac{v^2}{2})(\sqrt{2}u+1)+v)) \end{pmatrix}$$

then we can easily obtain that the (L,L)-type spiral surface $S(u,v)$ which is image of the helicoidal surface $H(u,v)$ under the conformal map is given by

$$S(u,v)=(S_1(u,v),S_2(u,v),S_3(u,v))$$

where

$$S_1(u,v)=\begin{pmatrix} e^{2v/3}\frac{(1-v)}{1+2\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2u}-1)^2}\,)+ \\ e^{2v/3}(\frac{\sqrt{2}v}{1+2\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2u}-1)^2}\,)+ \\ \frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}v) \end{pmatrix},$$

$$S_2(u,v)=\begin{pmatrix} e^{2v/3}\frac{1+v-\frac{v^2}{2}}{1+2\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2}u-1)^2}\,)+ \\ e^{2v/3}\frac{\sqrt{2}v^2}{2+4\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2}u-1)^2}\,) \\ +\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}\frac{v^2}{2} \end{pmatrix},$$

$$S_3(u,v)=\begin{pmatrix} e^{2v/3}\frac{(v-\frac{v^2}{2})}{1+2\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2u}-1)^2}\,)+ \\ e^{2v/3}(\frac{\sqrt{2}(1+\frac{v^2}{2})}{1+2\sqrt{2}}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{4\sqrt{2}}{9})- \\ \sqrt{\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1-\frac{4\sqrt{2}}{9})+(1+2\sqrt{2})(\frac{4}{9}(\frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}})^2+u-\sqrt{2u}-1)^2}\,)+ \\ \frac{2-\sqrt{2}}{1-\frac{2\sqrt{2}}{3}}(1+\frac{v^2}{2})) \end{pmatrix}.$$

The picture of the helicoidal surface $H(u,v)$ and spiral surface $S(u,v)$ are rendered in

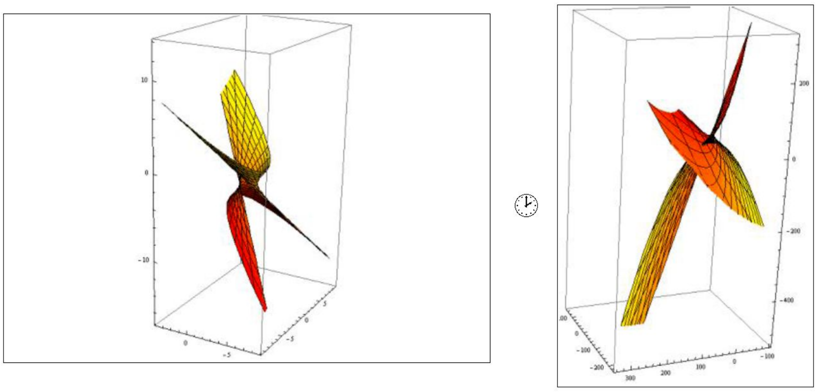


Figure 3 : A (L, L) - typehelicoidal surface and its image (L, L) - typespiral surface under the conformal map

Corollary 1. A spiral surface and a rotational surface have a conformal relation. So, a spiral on the spiral surface can be transformed to a circle on the rotational surface. Consequently; helicoidal, rotational and spiral surface have following diagram which is comutative.

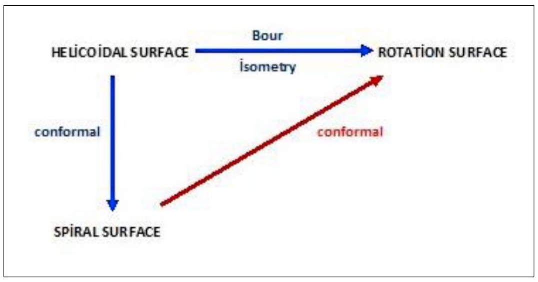


Figure 4 : Relation among the helicoidal, rotational and spiral surface.

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