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ON A VARIATIONAL APPROACH TO NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

Afgan ASLANOV
Beykent University, Department of Mathematics and Computing,
Ayazaga-Maslak Campus, 34396, Istanbul, Turkey
afganaslanov@beykent.edu.tr

ABSTRACT

The main purpose of this paper is to analyze the variational iteration method for solving the two-point boundary value problems. We show that Wang [Shu-Qjang Wang, A variational approach to nonlinear two-point boundary value problems, Comput. Math. Appl. 58 (2009) 2452-2455] demonstrated some incorrect solutions for two-point boundary value problems studied by means of the variational approach. We verify our assertion by direct calculation and in addition, we represent the correct results using different approaches.

Keywords: *Two-point boundry value problem, approximate solution, methotds of variation*

ÖZET

Bu makalenin esas amacı iki-noktalı sınır değer probleminin çözümü için varyasyon iterasyon metodunu analiz etmektir. Biz Wang'ın [Shu-Qjang Wang, A variational approach to nonlinear two-point boundary value problems, Comput. Math. Appl. 58 (2009) 2452-2455] iki-noktalı sınır değer problemi için uyguladığı varyasyon yaklaşımının yanlışlarını gösteriyoruz. Farklı yaklaşımlarla doğru çözümlerin nasıl bulunması gerektiğini örneklerle gösteriyoruz.

Anahtar kelimeler: *iki-noktalı sınır-değer problemi, yaklaşık çözüm, varyasyon metodu*

1. INTRODUCTION

The two-point boundary value problems of the type

$$u'' = f(x, u, u'),\ 0 \le x \le 1,\ u(0) = 0,\ u(1) = 0 \quad (1)$$

were studied by using different methods. In [1] Wazwaz used the Adomian decomposition method (ADM) for solving this type of problems. Applying the variational method Wang [2] attempted to demonstrate more effective method to overcome the difficulties in ADM. However, as we observe, the approach in [2] is not true in general. In what follows we first show that the proposed method (approach) is incorrect and then we show the correct results by using different approaches.

2. ON THE WRONG IMPLEMENTATION OF THE VARIATIONAL APPROACH IN [2]

The main mistake in [2] is that the author assumes the solution in the form

$$u(x) = x(x-1)(ax+b) \quad (2)$$

where a and b are constants. In fact, the variational method is used to find the most appropriate polynomial of degree three (among all polynomials of degree three) which can be taken as an approximate solution. It is easily seen that this approach has serious defects. If the solution of the problem (accidentally) is a polynomial of degree three, then the method usually gives an exact solution. If the solution is a polynomial of degree greater than three, then the method fails to find any exact solution. For example the proposed method can not be applied for solving the problem

$$u'' = 12x^2, u(0) = u(1) = 0$$

since its solution is x^4-x and the proposed approximate solution is the polynomial of degree 3. On the other hand, even if you suppose that the solution has a form (2) then you do not need any method: the solution can be obtained by a simple substitution as in (2). Let us demonstrate this in the first example in [2].

Example 1 [2]. Consider the problem

$$u'' + \frac{2}{x}u' + u^3 + 3xu^2 + 3ux^2 + x^3 + \frac{2}{x} - 6 - x^6 = 0 \quad (3)$$

with the boundary conditions

$$u(0) = u(1) = 0. \quad (4)$$

If we assume (?!) that the solution has a form (2) then immediately we obtain

$$2b - 2a + 6xa + \frac{2}{x}(3ax^2 - b - 2ax + 2bx)$$
$$+x^3(x-1)^3(ax+b)^3 + 3x^3(x-1)^2(ax+b)^2$$
$$+3x(x-1)(ax+b)x^2 + x^3 + \frac{2}{x} - 6 - x^6 = 0$$

and equating the same powers of x we obtain easily $a = 0$ and $b = 1$. Note that the author [2] attained this solution by solving very complicated system of (nonlinear) equations. And we do not know yet that the (simultaneous) solution of this system of equations corresponds to the minimum error, in general (this may correspond to the saddle point and maximum point as well).

Probably the series representation method is most easy and effective for solving the problem (3) and every first year student can solve this problem by using the series method (usually, the "classical" series solution method becomes most effective for singular type of problems). But the ADM can also be used in more appropriate form.

The same is true for the Example 2 [2]:

Example 2 [2]. The problem

$$u'' + \frac{4}{x}(u' + 1) + (u + x + 2)^2 - 4 - 18x - 4x^3 - x^6 = 0 \quad (5)$$

with boundary conditions

$$u(0) = u(1) = 0$$

has the exact solution $u = x^3 - x$ and after making the substitution (2) we do not need any method: Simply replace u in (5) to find a and b.

The serious error obtained in the solution of next example in [2].

Example 3 [2]. The problem

$$u'' + \pi^3 \frac{u^2}{sin\pi x} = 0 \tag{6}$$

with boundary conditions

$$u(0) = u(1) = 0$$

has no polynomial type of solution (with finite order). That is the method used in [2] leads a serious mistake: The exact solution is

$$u = \frac{1}{\pi}\sin(\pi x) = \frac{1}{\pi}\left(\pi x - \frac{\pi^3 x^3}{6} + \cdots\right) \approx x - 1.6449x^3 + \cdots$$

and approximate solution in [2] is

$$u_{appr} = -1.2463x^3 + 1.2463x.$$

The method used in [2] for solving the Example 3 can not be improved for higher order polynomial approximations.

The Example 3 can be solved by using ADM: Take the initial approximation as $u_0 = \alpha\sin(\pi x)$ and then apply the standard ADM. Note that this type of initial approximation may be very useful for the problems with boundary conditions like (4)-that is not only for the equations with $\sin(\pi x)$.

Note that the method of successive approximation can be used easily for solving Example 3: Take

$$u = a\sin(\pi x),$$

then

$$u_1'' = -\pi^3 \frac{u_0^2}{sin\pi x}, \; u_1(0) = u_1(1) = 0$$

gives

$$u_1'' = -\pi^3 \frac{a^2 sin^2(\pi x)}{sin\pi x}$$

and

$$u_1 = \pi a^2 sin(\pi x)$$

and so on, we obtain

$$u_n = \pi^{2^n - 1} a^{2^n} \sin(\pi x) = \frac{1}{\pi}(\pi a)^{2^n} \sin(\pi x).$$

$(\pi a)^{2^n} \to 0$ for $a < 1/\pi$ and $(\pi a)^{2^n} \to 1$for $a = 1/\pi$. That is both solutions $u = 0$ and $u = (1/\pi)\sin(\pi x)$ can be received easily by using the method of successive approximation.

3. CONCLUSION

We proved that Wang who intended to develop in [2] a new approach to apply variational approach for solving two-point boundary value problems, included in the paper some mistakes and misinterpretations. After making the substitution in the form of cubic polynomial, which seems strange for the problems of such type, the author used variational functional for finding unknown constants. We demonstrated that the variational approach is not effective and even is not necessary for finding these constants. We demonstrated more effective ways for solving similar problems.

REFERENCES

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