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Research Article

The minimal f^g-statistical convergence and Cauchy degree of a sequence

Tamim Aziz, Sanjoy Ghosal*

Department of Mathematics, University of North Bengal, Raja Rammohunpur, Darjeeling-734013, West Bengal, India

Abstract

In this paper, we introduce and characterize the rough f^g-statistical limit set, minimal f^g-statistical convergence degree, and minimal f^g-statistical Cauchy degree of a sequence in an arbitrary normed space. We clarify these concepts for normed spaces of any dimension and explore their properties and relationships. Our findings offer a new perspective that differs from some established results.

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1. Introduction

The concept of statistical convergence was separately introduced by Fast [12] and Steinhaus [32] in the year 1951. Following some outstanding works by Fridy [13] and Šalát [31], several generalizations and applications of this concept have been examined over a variety of spaces. Recently, in the year 2015, Balcerzak et al. [8] modified the natural density of subsets of $\mathbb{N}$ by introducing the density of the weight $g : \mathbb{N} \to [0, \infty)$, where $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$ which they referred to as g-density. Additionally, for a given such weight g, they created the associated ideals $\mathcal{Z}_g$ with g-density zero sets of $\mathbb{N}$ and thoroughly investigated the characteristics of such ideals.

Now, we delineate the concept of density of weight of subsets of $\mathbb{N}$.

Definition 1.1 ([8])**.** Let A be a subset of $\mathbb{N}$ and $g : \mathbb{N} \to [0, \infty)$, where $g(n) \to \infty$ and $n/g(n) \nrightarrow 0$. Then the density of the weight g of A is given by

$$d_g(A) = \lim_{n\to\infty} \frac{|A \cap [1, n]|}{g(n)}, \text{ provided that the limit exists}$$

and $|A \cap [1, n]|$ denotes the cardinality of the set $\{k \in A : 1 \le k \le n\}$.

At this stage, "ideals", a significant class of set theoretical objects, come into consideration. A family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal [21] on $\mathbb{N}$ if it satisfies the following conditions:

*Corresponding Author.
Email addresses: tamimaziz99@gmail.com (T. Aziz), sanjoyghosalju@gmail.com (S. Ghosal)
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- $\phi \in \mathfrak{I}$,
- if $A, B \in \mathfrak{I}$ then $A \cup B \in \mathfrak{I}$,
- if $A \subset B$ and $B \in \mathfrak{I}$ then $A \in \mathfrak{I}$.

Let $\mathbb{G} = \{g : \mathbb{N} \to [0,\infty) : g(n) \to \infty \text{ and } n/g(n) \nrightarrow 0\}$, then for each $g \in \mathbb{G}$, the set $\mathcal{Z}_g = \{A \subset \mathbb{N} : d_g(A) = 0\}$ forms an ideal of subsets of $\mathbb{N}$. We will write $\mathcal{Z}_g = \mathcal{Z}$ when g is the identity function.
Let us recall an important result that is relevant to this literature.

Theorem 1.2. *[8, Theorem 2.7] There exists a family $\mathbb{G}_0 \subset \mathbb{G}$ of cardinality $\mathfrak{c}$ such that $\mathcal{Z}_g$ is incomparable with $\mathcal{Z}$ for every $f \in \mathbb{G}_0$, and $\mathcal{Z}_f$ and $\mathcal{Z}_g$ are incomparable for any distinct $f, g \in \mathbb{G}_0$.*

Following the idea of Balcerzak et al. [8], in the year 2018, Bose et al. [9] extended the idea of weighted density of [8] to weighted f-density of subsets of $\mathbb{N}$ (where f is an unbounded modulus function) which parallelly broadens the notion of f-density [1]. We now recall the definition of a modulus function: A non-negative function f defined on $[0,\infty)$ is called modulus function if (i) $f(x) = 0$ if and only if $x = 0$; (ii) f is sub-additive, i.e., $f(x+y) \leq f(x) + f(y)$, for any $x, y \geq 0$; (iii) f is increasing, and (iv) f is right continuous at 0. A modulus function is continuous everywhere on $[0,\infty)$ thanks to the properties (1)-(4). Some well-known examples of modulus functions are $\frac{x}{1+x}$, $\log(1+x)$, and x^p for $p \in (0,1]$.

Let us now recall the notion of f-density of weight g of subsets of $\mathbb{N}$ as proposed by Bose et al. [9].

Definition 1.3 ([9])**.** Let f be an unbounded modulus function and $g \in \mathbb{G}$. Then for a subset A of $\mathbb{N}$ the f-density of weight g (or, f^g-density) of A is denoted by $d_g^f(A)$ and is given by

$$d_g^f(A) = \lim_{n\to\infty} \frac{f(|A \cap [1,n]|)}{f(g(n))}$$

provided the above limit exists.

Clearly, d_g^f simply coincides with d_g if f is the identity function, and the f-density is obtained if $f(n) = g(n)$ except for finitely many $n \in \mathbb{N}$. For an unbounded modulus function f and for each $g \in \mathbb{G}$, Bose et al. [9] constructed the ideal $\mathcal{Z}_g(f)$ generated by the density function d_g^f, i.e., $\mathcal{Z}_g(f) = \{A \subset \mathbb{N} : d_g^f(A) = 0\}$ which generalizes the notion of the weighted density ideal $\mathcal{Z}_g$.
Let us recall some results from [9] concerning the ideals $\mathcal{Z}_g(f)$ that are relevant to this literature.

Theorem 1.4. *[9, Proposition 2.6] For any modulus function f and $g \in \mathbb{G}, \mathcal{Z}_g(f) \subset \mathcal{Z}_g$.*

Theorem 1.5. *[9, Proposition 3.1] For any modulus function f and $g \in \mathbb{G}$, the ideal $\mathcal{Z}_g(f)$ is a P-ideal. In fact $\mathcal{Z}_g(f)$ is equal to $Exh(\varphi)$, where φ is a lower semicontinuous sub-measure on $\mathbb{N}$ given by*

$$\varphi(A) = \sup_{n\in\mathbb{N}} \frac{f(|A \cap [1,n]|)}{f(g(n))}$$

for $A \subset \mathbb{N}$.

Bose et al. [9] also observed that $\mathcal{Z}_g(f)$ is a density ideal and hence is an $F_{\sigma\delta}$ P-ideal. The subsequent theorem demonstrates that there is a set $A \subset \mathbb{N}$ such that $d_f(A) = 0$ (where d_f stands for f-density) but $d_g^f(A) \neq 0$.

Theorem 1.6. *[9, Proposition 3.3] Let f be an unbounded modulus function and let $g \in \mathbb{G}$ be such that $f(n)/f(g(n)) \to \infty$. Then there exists a set $A \subset \mathbb{N}$ such that the sequence $(f(|A \cap [1,n])/f(g(n)))$ is bounded but not convergent to 0.*

In the other direction, a non-trivial and interesting generalization of classical convergence is rough convergence which was first pioneered by Phu [27] over finite dimensional normed linear spaces and obtained several results associated with the rough limit set $LIM^r x_i$ of a sequence $\{x_n\}$. Then, Phu [28] extended this concept over infinite dimensional normed spaces and obtained some generalized results. Finally, he established a connection between rough convergence sequences and rough Cauchy sequences using the notion of Jung constant of a normed space.

We now recall the concept of rough convergence of a normed space-valued sequence in a formal manner.

Definition 1.7 ([27, 28])**.** Let r be a non-negative real number. A sequence $\{x_n\}$ in a normed space $(X, \|\cdot\|)$ is said to be r-convergent or rough convergent to x_* w.r.t the degree of roughness r, denoted by $x_n \xrightarrow{r} x_*$ provided that

$$\text{for any } \varepsilon > 0, \text{ there exists } n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \ \Rightarrow \ ||x_n - x_*|| \leq r + \varepsilon.$$

The non-negative real number r stands for the degree of roughness and the collection $LIM^r x_i = \left\{ x_* \in X : x_n \xrightarrow{r} x_* \right\}$ is called the r-limit set of the sequence $\{x_n\}$.

Phu [27] proposed the concept of rough Cauchy sequences as an extension of Cauchy sequences over normed spaces in the following way:

Definition 1.8 ([27, 28])**.** Let ρ be a non-negative real number. A sequence $x = \{x_n\}$ in a normed space X is said to be rough Cauchy sequence w.r.t the Cauchy degree ρ or ρ-Cauchy sequence,

$$\text{if for any } \varepsilon > 0, \text{ there exists } k_\varepsilon \in \mathbb{N} : n, m \geq k_\varepsilon \ \Rightarrow \ ||x_n - x_m|| \leq \rho + \varepsilon.$$

For a comprehensive overview of established results on rough convergence, and reference therein, visit [2, 4–6, 11, 16–18, 22, 25, 27–30].

Let us present our main definition of rough g-weighted f-statistical convergence (briefly, rough f^g-statistical convergence).

Definition 1.9. Suppose $r \geq 0$, $g \in \mathbb{G}$ and f is an unbounded modulus function. Then the sequence $\{x_n\}$ in a normed space X is said to be rough g-weighted f-statistically convergent to x_*, w.r.t the degree of roughness r, (briefly, $r - f^g$-statistically convergent) denoted by $x_n \xrightarrow[r]{f^g st} x_*$, provided that the set $\{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon\}$ has f^g-density zero, for every positive ε. For the sequence $\{x_n\}$, we denote $f^g st\text{-}LIM^r x_i = \left\{ x_* \in X : x_n \xrightarrow[r]{f^g st} x_* \right\}$ as the rough f^g-statistical limit set with degree of roughness r. A sequence $\{x_n\}$ is referred to as rough f^g-statistically convergent if there exists $r \geq 0$ such that $f^g st\text{-}LIM^r x_i \neq \varnothing$.

Note 1.10. Observe that the definition of f^g-statistical convergence [10] is obtained if we set $r = 0$ in the above definition. From Theorem 1.2 and Theorem 1.4, it follows that f^g-statistical convergence and statistical convergence [12, 13, 31–33] are incomparable in general. Also from Theorem 1.6, we conclude that f^g-statistical convergence is quite different from f-statistical convergence [1, 23]. Since for any $A \subseteq \mathbb{N}$, we have $\frac{f(|A\cap[1,n]|)}{f(n)} = \frac{f(|A\cap[1,n]|)}{f(g(n))} \frac{f(g(n))}{f(n)}$, it follows that f^g-statistical convergence implies f-statistical convergence if the sequence $\{f(g(n))/f(n)\}_{n\in\mathbb{N}}$ is bounded above. Furthermore, if there exist $m, M > 0$ such that $m \leq f(g(n))/f(n) \leq M$ for all $n \in \mathbb{N}$, then f^g-statistical convergence coincides with f-statistical convergence. For recent works along those lines, visit [3, 7].

As we proceed on to the primary objective of this article, it is noteworthy that the rough limit point set at the minimal degree of roughness has not received much attention in the literature on rough convergence. The rough limit set $LIM^r x_i$ and the rough statistical limit set $st\text{-}LIM^r x_i$ of a sequence $x = \{x_n\}$ in normed spaces with the minimal convergent degree $\tilde{r}(x)$ were initially described over normed spaces in [22,27,28]. The primary goal of this article is to investigate the minimal f^g-statistical convergence degree and f^g-statistical Cauchy degree of a sequence $x = \{x_n\}$ over any dimensional normed spaces using the ideas of Chebyshev radius and diameter of the f^g-statistical cluster point set $\Gamma_x^{f^g}$ of x. Furthermore, in infinite dimensional settings, we provide a number of noteworthy examples demonstrating that some of the previously mentioned results are do not hold true. In this article, we establish what follows:

(i) $f^g st\text{-}LIM^r x_i$ is an $F_{\sigma\delta}$ (hence a Borel) subset X (Theorem 2.1);

(ii) $f^g st\text{-}LIM^r x_i$ is strictly convex, provided that X is uniformly convex (Theorem 2.6) and the uniform convexity condition on X cannot be relaxed in general (Example 2.7);

(iii) For any $\{x_n\}$ lies in some totally bounded subset of X and $y+\sigma B_X \subseteq f^g st\text{-}LIM^r x_i$, then $r \geq \sigma$ and $y \in f^g st\text{-}LIM^{r-\sigma} x_i$ (Theorem 2.8). Moreover $int(f^g st\text{-}LIM^{\tilde{r}(x)} x_i) = \varnothing$ (Corollary 2.9);

(iv) $int(f^g st\text{-}LIM^{\tilde{r}(x)} x_i) = \varnothing$, provided that X is uniformly convex in some direction $z \in S_X$, where $\tilde{r}(x)$ denotes the minimal f^g-statistical convergence degree of $x = \{x_n\}$ (Theorem 2.11);

(v) If $r > \tilde{r}(x)$, then $int(f^g st\text{-}LIM^r x_i) \neq \varnothing$ (Corollary 2.13);

(vi) $f^g st\text{-}LIM^{\tilde{r}(x)} x_i \neq \varnothing$, for reflexive normed spaces X (Theorem 2.17). In non-reflexive normed spaces the result may not hold (Example 2.18);

We also analyze some results concerning minimal f^g-statistical Cauchy and minimal f^g-statistical convergence degree of the sequence $x = \{x_n\}$ which contained in a compact subset of X, respectively in terms of the diameter $D_X(\Gamma_x^{f^g})$ and Chebyshev radius $r_X(\Gamma_x^{f^g})$ of the f^g-statistical cluster point set $\Gamma_x^{f^g}$ of x. Furthermore, we give some relationship between the set of rough f^g-statistical limit points and the set of f^g-statistical cluster points of x.

(vii) $f^g st\text{-}LIM^r x_i = \bigcap_{\gamma \in \Gamma_x^{f^g}} (\gamma + rB_X)$ (Theorem 3.7);

(viii) $D_X(\Gamma_x^{f^g})$ is the minimal f^g-statistical Cauchy degree and $r_X(\Gamma_x^{f^g})$ is the minimal f^g-statistical convergence degree of x (Theorem 4.1, 4.2). In general, it is not possible to loosen the compactness constraint (Example 4.3);

(ix) If x is $\rho - f^g$-statistically Cauchy, then x is $r - f^g$-statistically convergence for every $r \geq \frac{J(X)\rho}{2}$, where $J(X)$ denotes the Jung constant of X (Theorem 4.4) and the constraint that the sequence is contained in a compact set cannot be relaxed in general (Example 4.5).

Additionally, it is shown that:

(x) For any f^g-nonthin subsequence of x which lies in some compact subset of X, we have $\Gamma_x^{f^g} \neq \varnothing$ (Theorem 3.4) and the compactness restriction is generally unrelaxable (Example 3.5);

(xi) A sequence $\{x_n\}$ is f^g-statistically Cauchy if and only if for each $\varepsilon > 0$ and each f^g-nonthin set $\Omega \subseteq \mathbb{N}$, there exists $m(\varepsilon) \in \Omega$ such that $d_g^f(\{k \in \mathbb{N} : \|x_k - x_{m(\varepsilon)}\| > \varepsilon\}) = 0$ (Theorem 3.9).

We conclude by exhibiting a relationship between the rough f^g-statistically convergent sequences and rough f^g-statistically Cauchy sequences that says without the compactness restriction on the $\rho - f^g$-statistically Cauchy sequence x, $J(X)\rho$ is the minimal f^g-statistical convergence degree (Theorem 4.6).

2. Characterization of the rough f^g-statistical limit set $f^g st\text{-}LIM^r x_i$

We now present a characterization of the limit set $f^g st\text{-}LIM^r x_i$ corresponding to a sequence $\{x_n\}$ which infers that it is a Borel set.

Theorem 2.1. *Suppose $r \geq 0$, and $\{x_n\}$ is a sequence in a normed space X. Then the limit set $f^g st\text{-}LIM^r x_i$ is an $F_{\sigma\delta}$ subset X.*

Proof. For $r, i \in \mathbb{N}$, we consider the open set

$$U_{r,i} = \left\{ x_* \in X : \|x_r - x_*\| > r + \frac{1}{i} \right\}.$$

Then from the definition of rough f^g-statistical convergence, we have

$$\begin{aligned}
f^g st\text{-}LIM^r x_i &= \left\{ x_* \in X : (\forall i \in \mathbb{N})\ d_g^f(\{r \in \mathbb{N} : x_* \in U_{r,i}\}) = 0 \right\} \\
&= \bigcap_{i=1}^{\infty} \left\{ x_* \in X : d_g^f(\{r \in \mathbb{N} : x_* \in U_{r,i}\}) = 0 \right\} \\
&= \bigcap_{i=1}^{\infty} \left\{ x_* \in X : \lim_{l\to\infty} \frac{f(|\{r \in \mathbb{N} : x_* \in U_{r,i}\} \cap [1,l]|)}{f(g(l))} = 0 \right\} \\
&= \bigcap_{i=1}^{\infty} \left\{ x_* \in X : (\forall j \in \mathbb{N})\ (\exists k \in \mathbb{N}) \text{ such that } \frac{f(|\{r \in \mathbb{N} : x_* \in U_{r,i}\} \cap [1,l]|)}{f(g(l))} \leq \frac{1}{j}, \forall l \geq k \right\} \\
&= \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{l=k}^{\infty} \left\{ x_* \in X : \frac{f(|\{r \in \mathbb{N} : x_* \in U_{r,i}\} \cap [1,l]|)}{f(g(l))} \leq \frac{1}{j} \right\}
\end{aligned}$$

Now, for $j, l \in \mathbb{N}$, consider the family

$$F_{l,j} = \{F \subseteq [1,l] : \frac{f(|F|)}{f(g(l))} > \frac{1}{j}\}.$$

Therefore we have

$$\begin{aligned}
&\left\{ x_* \in X : \frac{f(|\{r \in \mathbb{N} : x \in U_{r,i}\} \cap [1,l]|)}{f(g(l))} \leq \frac{1}{j} \right\} \\
&= \{x_* \in X : (\forall F \in F_{l,j})\ (\exists r \in F) \text{ such that } x \notin U_{r,i}\} \\
&= \bigcap_{F \in F_{n,j}} \bigcup_{i \in F} X \setminus U_{r,i}.
\end{aligned}$$

Since $F, F_{n,j}$ are finite sets and $X \setminus U_{r,i}$ is a closed set for any $r, i \in \mathbb{N}$, we conclude that the set $\left\{ x_* \in X : \frac{f(|\{r \in \mathbb{N} : x \in U_{r,i}\} \cap [1,l]|)}{f(g(l))} \leq \frac{1}{j} \right\}$ is closed in X. Thus we deduce that $f^g st\text{-}LIM^r x_i$ is an $F_{\sigma\delta}$ set. □

Definition 2.2. A sequence $\{x_n\}$ in a normed space is called f^g-statistically bounded if there exists $M > 0$ such that $d_g^f(\{n \in \mathbb{N} : \|x_n\| \geq M\}) = 0$.

Next, we present a necessary and sufficient condition for a sequence to be rough f^g-statistical convergent.

Proposition 2.3. *A sequence $\{x_n\}$ in a normed space X is rough f^g-statistically convergent if and only if it is f^g-statistically bounded.*

Proof. First suppose that $\{x_n\}$ is f^g-statistically bounded. Then there exists $M > 0$ such that $d_g^f(A) = 0$, where $A = \{n \in \mathbb{N} : \|x_n\| \geq M\}$. Therefore $0 \in f^g st\text{-}LIM^r x_i$ where $r := \sup_{n \in \mathbb{N} \setminus A} \|x_n\|$.

Conversely, let us assume that there exists $r \geq 0$ such that $y \in f^g st\text{-}LIM^r x_i$. Therefore, we have

$$d_g^f(\{n \in \mathbb{N} : \|x_n - y\| \geq r + \|y\| + 1\}) = 0 \Rightarrow d_g^f(\{n \in \mathbb{N} : \|x_n\| \geq r + 1\}) = 0.$$

This ensures that $\{x_n\}$ is f^g-statistically bounded. □

Proposition 2.4. *Suppose $r, \sigma \geq 0$. Then for any $\{x_n\}$ in X, $f^g st\text{-}LIM^r x_i + \sigma B_X \subseteq f^g st\text{-}LIM^{r+\sigma} x_i$.*

Proof. Let us take $y = u + v$, where $u \in f^g st\text{-}LIM^r x_i$ and $\|v\| \leq \sigma$. Then for each $\varepsilon > 0$, we have

$$\{n \in \mathbb{N} : \|x_n - y\| \geq r + \sigma + \varepsilon\} \subseteq \{n \in \mathbb{N} : \|x_n - u\| \geq r + \varepsilon\}.$$

Since $u \in f^g st\text{-}LIM^r x_i$, it follows that $d_g^f(\{n \in \mathbb{N} : \|x_n - y\| \geq r + \sigma + \varepsilon\}) = 0$. Consequently, we have $y \in f^g st\text{-}LIM^{r+\sigma} x_i$. □

Before moving forward, let us recall an important result concerning uniform convex normed spaces, which will be employed in Theorem 2.6.

Theorem 2.5. *[28, Lemma 2.3] X is uniformly convex if and only if for each $r > 0$, and all $\varepsilon \in (0, 2r]$ there exists $\delta(\varepsilon) > 0$ such that, for arbitrary sequences $\{z_{0n}\}$ and $\{z_{1n}\}$ in X,*

$$\limsup_{n\to\infty} \|z_{0n}\| \leq r, \limsup_{n\to\infty} \|z_{1n}\| \leq r, \|z_{0n} - z_{1n}\| \geq \varepsilon, n = 1, 2, ...$$

implies $\limsup_{n\to\infty} \frac{1}{2}\|z_{0n} + z_{1n}\| \leq r - \delta(\varepsilon)$.

The subsequent result reveals that in uniformly convex normed spaces, $f^g st\text{-}LIM^r x_i$ is strictly convex.

Theorem 2.6. *If X is a uniformly convex normed space, then for any sequence $\{x_n\}$ in X the limit set $f^g st\text{-}LIM^r x_i$ is strictly convex.*

Proof. To show $f^g st\text{-}LIM^r x_i$ is strictly convex, it is enough to prove that for any two distinct elements $y_1, y_2 \in f^g st\text{-}LIM^r x_i$ implies $\hat{y} = \frac{1}{2}(y_1 + y_2) \in int(f^g st\text{-}LIM^r x_i)$. Since $y_1, y_2 \in f^g st\text{-}LIM^r x_i$, there exists $A = \{n_1 < n_2 < n_3 < ...\} \subseteq \mathbb{N}$ with $d_g^f(\mathbb{N} \setminus A) = 0$ such that

$$\limsup_{k\to\infty} \|u_{1,n_k}\| \leq r \text{ and } \limsup_{k\to\infty} \|u_{2,n_k}\| \leq r,$$

where $u_{1,k} = y_1 - x_k$ and $u_{2,k} = y_2 - x_k$, for each $k \in \mathbb{N}$.
Since $\|u_{1,n_k} - u_{2,n_k}\| = \|y_1 - y_2\| = \varepsilon_0 > 0$ (say). Then there exists $\delta(\varepsilon_0) > 0$ such that

$$\limsup_{k\to\infty} \|\frac{1}{2}(y_1 + y_2) - x_{n_k}\| = \limsup_{k\to\infty} \frac{1}{2}\|u_{1,n_k} - u_{2,n_k}\| \leq r - \delta. \tag{2.1}$$

From the definition of lim sup, it follows that for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that for all $k > m_0$, we have

$$\|\frac{1}{2}(y_1 + y_2) - x_{n_k}\| < r - \delta + \varepsilon.$$

Thus $A \setminus \{n_1, n_2, ..., n_{m_0}\} \subseteq B$, where $B = \{k \in \mathbb{N} : \|\frac{1}{2}(y_1 + y_2) - x_k\| < r - \delta + \varepsilon\}$. Let us choose ζ such that $0 < \zeta < \delta$ and then pick arbitrary $y \in B(\hat{y}, \zeta)$. Now for each $k \in B$, we have

$$\|x_k - y\| \leq \|\hat{y} - x_k\| + \|y - \hat{y}\| \leq r - \delta + \varepsilon + \zeta < r + \varepsilon.$$

This gives $\{k \in \mathbb{N} : \|x_k - y\| > r + \varepsilon\} \subseteq \mathbb{N} \setminus B$. Since $d_g^f(\mathbb{N} \setminus B) = d_g^f(\mathbb{N} \setminus A) = 0$, we get that $B(\hat{y}, \zeta) \subseteq f^g st\text{-}LIM^r x_i$. Hence we deduce that $f^g st\text{-}LIM^r x_i$ is strictly convex. □

The uniform convexity condition cannot generally be relaxed from the previous Theorem 2.6. We emphasize our claim with an example.

Example 2.7. Consider the sequence $\{e_n\}$ in the normed space $\ell^\infty(\mathbb{R})$, where e_n represents the sequence $(0, ..., 0, \overbrace{1}^{n^{\text{th}}\text{place}}, 0, ...)$. Let us choose $f(x) = \log(1 + x), x \in [0, \infty)$ and $g(n) = \sqrt[3]{n}, n \in \mathbb{N}$. If we set $r = 1$, then it is evident that $\{e_n : n \in \mathbb{N}\} \subseteq f^g st\text{-}LIM^r e_i$. We intend to show that for any fixed $r, s \in \mathbb{N}$, $e_* = \frac{1}{2}(e_r + e_s)$ is not an interior point of $f^g st\text{-}LIM^r e_i$. So for any $\zeta > 0$, we define $y_* = \{y_n\} \in \ell^\infty(\mathbb{R})$ such that

$$y_n := \begin{cases} \frac{1}{2}, & \text{if } n \in \{r, s\}, \\ -\frac{\zeta}{2}, & \text{otherwise.} \end{cases}$$

Since for each $n > \max\{r, s\}$, we have $\|e_n - y_*\|_\infty = 1 + \frac{\zeta}{2}$, therefore it follows that $d_g^f(\{n \in \mathbb{N} : \|e_n - y_*\|_\infty > 1 + \frac{\zeta}{4}\}) \neq 0$, but $y_* \in B_\zeta(e_*)$. Since $\zeta > 0$ was arbitrary, we conclude that $e_* \notin int(f^g st\text{-}LIM^r e_i)$. Consequently, $f^g st\text{-}LIM^r e_i$ is not strictly convex.

Our next result displays that for a sequence with totally bounded-range, if the associated rough f^g-statistical limit set includes an interior point, that interior point resides within a certain f^g-statistical limit set with a smaller degree of roughness.

Theorem 2.8. *If $\{x_n\}$ lies in a totally bounded subset of X and $y + \sigma B_X \subseteq f^g st\text{-}LIM^r x_i$, then $r \geq \sigma$ and $y \in f^g st\text{-}LIM^{r-\sigma} x_i$.*

Proof. The inequality $r \geq \sigma$ follows directly from the fact that $diam(f^g st\text{-}LIM^r x_i) \leq 2r$. Since translation of a totally bounded set is totally bounded, we can find a totally bounded set T such that $\{x_n - y : n \in \mathbb{N}\} \subseteq T$. Then for each $\varepsilon > 0$, there exists $m \in \mathbb{N}$ and $c_1, c_2, ..., c_m \in X \setminus \{0\}$, such that $T \subseteq \bigcup_{i=1}^{m} \left(c_i + \frac{\varepsilon}{3} B_X\right)$. Let us define

$$C := \left\{-\frac{\sigma c_i}{\|c_i\|} : i = 1, 2, ..., m\right\} \subseteq \sigma + B_X.$$

Since $y + \sigma B_X \subseteq f^g st\text{-}LIM^r x_i$, then for any $z \in C$ we have $d_g^f(\mathbb{N} \setminus D_{z,\varepsilon}) = 0$, where $D_{z,\varepsilon} = \{n \in \mathbb{N} : \|x_n - y - z\| < r + \varepsilon/3\}$. Therefore $D_\varepsilon = \bigcap_{z \in C} D_{z,\varepsilon}$ satisfies $d_g^f(\mathbb{N} \setminus D_\varepsilon) = 0$, since C is finite.
Let us take arbitrary $n \in D_\varepsilon$. So there exists $i \in \{1, 2, ..., m\}$ such that $\|x_n - y - c_i\| \leq \varepsilon/3$. Therefore, we have

$$\begin{aligned} \sigma + \|c_i\| &= \|c_i + \frac{\sigma c_i}{\|c_i\|}\| \\ &\leq \sup_{z \in C} \|c_i - z\| \\ &\leq \sup_{z \in C} (\|x_n - y - z\| + \|x_n - y - c_i\|) \\ &\leq r + 2\varepsilon/3. \\ \Rightarrow \|c_i\| &\leq r - \sigma + 2\varepsilon/3. \end{aligned}$$

This gives

$$\|x_n - y\| \leq \|x_n - y - c_i\| + \|c_i\| \leq r - \sigma + \varepsilon.$$

Thus we have $\{n \in \mathbb{N} : \|x_n - y\| > r - \sigma + \varepsilon\} \subseteq \mathbb{N} \setminus D_\varepsilon$. Consequently, $y \in f^g st\text{-}LIM^{r-\sigma} x_i$. □

Given the aforementioned Theorem 2.8, it is reasonable to wonder what the minimum value of the degree of roughness 'r' can be in order to maintain the non-empty status of the rough f^g-statistical limit set. This observation leads us to introduce the concept of minimal f^g-statistical convergence degree of sequence. Given a sequence $x = \{x_n\}$ taking values in a normed space X, the minimal f^g-statistical convergence degree $\tilde{r}(x)$ (briefly, $\tilde{r}$) is defined as follows:

$$\tilde{r}(x) := \inf\{r \in \mathbb{R}_+ : f^g st\text{-}LIM^r x_i \neq \varnothing\}. \tag{2.2}$$

At this point, we include some noteworthy findings regarding the minimal f^g-statistical convergence degree.

Corollary 2.9. *If $\{x_n\}$ is contained in a totally bounded subset of X, then $f^g st\text{-}LIM^{\tilde{r}} x_i$ has empty interior.*

Proof. Assume, on the contrary, that there exists $y \in X$ and $\sigma > 0$ such that

$$y + \sigma B_X \subseteq f^g st\text{-}LIM^{\tilde{r}} x_i.$$

Therefore, Theorem 2.8 ensures that there exists $r < \tilde{r}$ such that $f^g st\text{-}LIM^r x_i \neq \varnothing$, which contradicts the minimality of $\tilde{r}$. □

Let us now recall from [21] that an ideal $\mathcal{I}$ is a P-ideal if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets in $\mathcal{I}$ there is a single set A_∞ in $\mathcal{I}$ such that $A_n \subseteq^* A_\infty$ (i.e., $A_n \setminus A_\infty$ is finite) for all $n \in \mathbb{N}$. In ([21, Theorem 3.2]) it was also proved that for any P-ideal $\mathcal{I}$ on $\mathbb{N}$, a sequence $\{x_n\}$ is $\mathcal{I}$ convergent to x if and only if there exists $A \subseteq \mathbb{N}$ with $\mathbb{N} \setminus A \in \mathcal{I}$ such that $\lim\limits_{n\in A} x_n = x$.

In view of the above result and Theorem 1.5, it is easy to realize that if $x_n \xrightarrow[r]{f^g st} x_*$, then there exists $A \subseteq \mathbb{N}$ with $d_g^f(\mathbb{N} \setminus A) = 0$ such that $\limsup\limits_{n\in A} \|x_n - x_*\| \leq r$.

At this point, let us recall a well-known concept in the geometry of normed spaces called "uniform convexity" which is based on the geometric condition that if two members of the unit ball are far apart, then their midpoint is well inside the unit ball.

Definition 2.10 ([15])**.** Let X be normed space and $\varepsilon > 0$ be arbitrary.
(a) X is said to be uniformly convex in the direction $z(\neq 0)$ if there exists a $\delta_z > 0$ such that if $\|x\| = \|y\| = 1,\ x - y \in span(\{z\})$, and $\|x - y\| \geq \varepsilon$, then $\frac{1}{2}\|x + y\| < 1 - \delta_z$.
(b) X is said to be uniformly convex in every direction if for any non-zero z in X, there exists a $\delta > 0$ such that if $\|x\| = \|y\| = 1,\ x - y \in span(\{z\})$, and $\|x - y\| \geq \varepsilon$, then $\frac{1}{2}\|x + y\| < 1 - \delta$.

According to [15], if X is uniformly convex in some direction $z \in S_X$ if and only if whenever $\{x_n\}, \{y_n\}$ are sequences in X with $x_n - y_n \in span(\{z\})$ for each n, $\|x_n\| \to 1, \|y_n\| \to 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

Theorem 2.11. *If X is uniformly convex in some direction $z \in S_X$ then for every $x = \{x_n\}$ in X, $int(f^g st\text{-}LIM^{\tilde{r}(x)} x_i) = \varnothing$.*

Proof. Assume, on the contrary, that $y \in int(f^g st\text{-}LIM^{\tilde{r}(x)} x_i)$. Then there exists $a > 0$ such that $y + aB_X \subseteq f^g st\text{-}LIM^{\tilde{r}(x)} x_i$. Now for each $k \in \mathbb{N}$, we consider the sets

$$\begin{aligned} A_k &:= \left\{n \in \mathbb{N} : \|x_n - y - az\| \leq \tilde{r} + \tfrac{1}{k}\right\} \\ B_k &:= \left\{n \in \mathbb{N} : \|x_n - y + az\| \leq \tilde{r} + \tfrac{1}{k}\right\} \\ C_k &:= \left\{n \in \mathbb{N} : \|x_n - y\| > \tilde{r} - \tfrac{1}{k}\right\}. \end{aligned}$$

Now, observe that $d_g^f(\mathbb{N} \setminus A_k) = 0$, $d_g^f(\mathbb{N} \setminus B_k) = 0$. Also from the minimality of $\tilde{r}$, we get $d_g^f(C_k) \neq 0$. Consequently, $d_g^f(A_k \cap B_k \cap C_k) \neq 0$. Now choose a sequence of positive integers $\{n_k\}_{k\in\mathbb{N}}$ such that $n_k \in A_k \cap B_k \cap C_k$, i.e.,

$$\|x_{n_k} - y - az\| \leq \tilde{r} + \frac{1}{k}, \|x_{n_k} - y + az\| \leq \tilde{r} + \frac{1}{k} \text{ and } \|x_{n_k} - y\| > \tilde{r} - \frac{1}{k}.$$

Let us fix $\alpha_k = \dfrac{x_{n_k} - y - az}{\tilde{r} + 1/k}$ and $\beta_k = \dfrac{x_{n_k} - y + az}{\tilde{r} + 1/k}$. Therefore we have

$$\|\alpha_k\| \leq 1, \|\beta_k\| \leq 1 \text{ and } 1 - \frac{2}{\tilde{r}k + 1} \leq \left\| \frac{\alpha_k + \beta_k}{2} \right\| \leq 1.$$

This shows that $\left\| \frac{\alpha_k+\beta_k}{2} \right\| \to 1$ as $k \to \infty$ but $\|\alpha_k - \beta_k\| = \frac{2ak}{1+k\tilde{r}} \nrightarrow 0$ as $k \to \infty$ which contradicts the uniform convexity of X in the direction $z \in S_X$. Hence we deduce that $f^g st\text{-}LIM^{\tilde{r}(x)} x_i$ has no interior point. □

Corollary 2.12. *Suppose X is uniformly convex in some direction $z \in S_X$, and y is an interior point of $f^g st\text{-}LIM^r x_i$, then there exists $r' \in (0, r)$ such that $y \in f^g st\text{-}LIM^{r'} x_i$.*

Proof. If no such r' exists, then for each $k \in \mathbb{N}$, we have

$$d_g^f(A_k \cap B_k \cap C_k) \neq 0$$

where

$$\begin{aligned} A_k &:= \left\{ n \in \mathbb{N} : \|x_n - y - az\| \leq r + \tfrac{1}{k} \right\} \\ B_k &:= \left\{ n \in \mathbb{N} : \|x_n - y + az\| \leq r + \tfrac{1}{k} \right\} \\ C_k &:= \left\{ n \in \mathbb{N} : \|x_n - y\| > r - \tfrac{1}{k} \right\}. \end{aligned}$$

Now proceeding similarly as Theorem 2.11, we arrive at a contradiction that X is not uniformly convex in the direction $z \in S_X$. Thus we conclude that $y \in f^g st\text{-}LIM^{r'} x_i$, for some $r' \in (0, r)$. □

Corollary 2.13. *For any $\{x_n\}$ in X, $f^g st\text{-}LIM^r x_i$ has non-empty interior whenever $r > \tilde{r}$.*

Proof. Since $r - \frac{r-\tilde{r}}{2} > \tilde{r}$, therefore, by Equation 2.2, we obtain that $f^g st\text{-}LIM^{(r-\frac{r-\tilde{r}}{2})} x_i \neq \varnothing$. Now Proposition 2.4 ensures that

$$f^g st\text{-}LIM^{(r-\frac{r-\tilde{r}}{2})} x_i + \frac{r - \tilde{r}}{2} \sigma B(0, 1) \subseteq f^g st\text{-}LIM^r x_i.$$

Thus we conclude that $int(f^g st\text{-}LIM^r x_i) \neq \varnothing$. □

Proposition 2.14. *For any sequence $\{x_n\}$ in a normed space X, we have*

$$f^g st\text{-}LIM^r x_i = \bigcap_{s>r} f^g st\text{-}LIM^s x_i.$$

Proof. From the definition of rough f^g-statistical convergence it follows that $f^g st\text{-}LIM^r x_i \subseteq f^g st\text{-}LIM^s x_i$, whenever $r < s$. Therefore we have

$$f^g st\text{-}LIM^r x_i \subseteq \bigcap_{s>r} f^g st\text{-}LIM^s x_i.$$

Now suppose $y \in X \setminus f^g st\text{-}LIM^r x_i$. Then there exists $\varepsilon_0 > 0$ such that

$$d_g^f(\{k \in \mathbb{N} : t_k\|x_k - y\| > r + \varepsilon_0\}) \neq 0.$$

Let s be such that $r < s < r + \varepsilon_0$, i.e., $\varepsilon := r + \varepsilon_0 - s > 0$.
Therefore, it is evident that

$$d_g^f(\{k \in \mathbb{N} : t_k\|x_k - y\| > s + \varepsilon\}) \neq 0.$$

This shows that $y \notin f^g st\text{-}LIM^s x_i$. Consequently, $\bigcap_{s>r} f^g st\text{-}LIM^s x_i \subseteq f^g st\text{-}LIM^r x_i$. Hence we deduce that $f^g st\text{-}LIM^r x_i = \bigcap_{s>r} f^g st\text{-}LIM^s x_i$. □

The following corollary is evident from Proposition 2.14. So we present it without proof.

Corollary 2.15. *For any sequence $\{x_n\}$ in a normed space X, we have*

$$f^g st\text{-}LIM^r x_i = \bigcap_{n\in\mathbb{N}} f^g st\text{-}LIM^{(r+\frac{1}{n})} x_i \quad \text{for all} \quad r \geq 0.$$

Note 2.16. It is easy to realize that, for each $\{x_n\}$ in X, $f^g st\text{-}LIM^r x_i$ is closed, bounded, and convex subset of X.

The subsequent result shows that in a reflexive normed space X, the limit set $f^g st\text{-}LIM^r x_i$ with minimal f^g-statistical convergent degree $\tilde{r}$ includes at least one element.

Theorem 2.17. *For any sequence $\{x_n\}$ in a reflexive normed space X, $f^g st\text{-}LIM^{\tilde{r}} x_i \neq \varnothing$.*

Proof. We define

$$C_n := f^g st\text{-}LIM^{(\tilde{r}+\frac{1}{n})} x_i.$$

Evidently, for each $n \in \mathbb{N}$, we have $C_n \neq \varnothing$. Now, the definition of rough f^g-statistical convergence ensures that $C_n \supseteq C_{n+1}$. Thus $\{C_n\}$ is a sequence of nonempty closed bounded convex subsets of X such that $C_n \supseteq C_{n+1}$ for each n. Therefore, we obtain that

$$f^g st\text{-}LIM^{\tilde{r}} x_i = \bigcap_{n\in\mathbb{N}} C_n \neq \varnothing.$$

This completes the proof. □

In non-reflexive normed spaces, the preceding Theorem 2.17 may not lead to the result. The subsequent example demonstrates this fact.

Example 2.18. Consider the non-reflexive normed space $(C[0,1], \|.\|)$, where $\|x\| = \int_0^1 |x(t)|dt$.
Assume that the modulus is $f(x) = \sqrt{x}, x \in [0,\infty)$, and the weight $g \in \mathbb{G}$ is $g = \sqrt[4]{n}, n \in \mathbb{N}$.
Since $\lim_{n\in\mathbb{N}} \frac{\sqrt{n^{1/5}}}{\sqrt{n^{1/4}}} = 0$, it follows that $d_g^f(A) = 0$, where $A = \{n^5 : n \in \mathbb{N}\}$.
The sequence $\{x_n\}$ in $C[0,1]$ is now set up as follows:

$$x_n(t) = \begin{cases} y_n(t), & \text{if } n \in A, \\ z_n(t), & \text{if } n \in \mathbb{N} \setminus A. \end{cases}$$

The sequences $\{y_n\}, \{z_n\} \in C[0,1]$ are defined, respectively, as follows:

$$y_n(t) = 4nt^2, \quad \text{where} \quad t \in [0,1] \quad \text{and} \quad n \in \mathbb{N}.$$

Also, for each $n \in \mathbb{N}$,

$$z_n(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ n(t-\frac{1}{2}), & \text{if } \frac{1}{2} < t < \frac{1}{2}+\frac{1}{n}, \\ 1, & \text{if } \frac{1}{2}+\frac{1}{n} \leq t \leq 1. \end{cases}$$

Since $\{x_n\}_{n\in\mathbb{N}\setminus A}$ has no sub-sequential limits and $\mathcal{Z}_g(f)$ is a P-ideal, $f^g st\text{-}LIM^0 x_i = \varnothing$.
Note that for each $r > 0$, we can write

$$\|x_m - x_n\| < r, \text{ whenever } m,n \geq \left[\frac{1}{r}\right] + 1 \text{ and } m,n \notin A.$$

Let $\varepsilon > 0$ be given. Let us set $p_r = \left[\frac{1}{r}\right] + 1$. Then we have

$$\{k \in \mathbb{N} : \|x_k(t) - x_{p_r}(t)\| > r + \varepsilon\} \subseteq A \setminus \{1,2,...,[1/r]\}.$$

This ensures that $d_g^f(\{k \in \mathbb{N} : \|x_k(t) - x_{p_r}(t)\| > r+\varepsilon\}) = 0$ i.e., $x_{p_r} \in f^g st\text{-}LIM^r x_i$.
Consequently, $f^g st\text{-}LIM^r x_i \neq \varnothing$ for every $r > 0$. Thereon $\tilde{r} = 0$ but $f^g st\text{-}LIM^0 x_i = \varnothing$.

Corollary 2.19. *If X is uniformly convex, then $f^g st\text{-}LIM^{\tilde{r}} x_i$ includes at most one element.*

Proof. Since uniform convexity implies uniform convexity in every direction, Theorem 2.11 ensures that $f^g st\text{-}LIM^{\tilde{r}} x_i$ has no interior point. Now according to Theorem 2.6, it follows that $f^g st\text{-}LIM^{\tilde{r}} x_i$ is strictly convex. Therefore, it cannot contain more than one element, otherwise $int(f^g st\text{-}LIM^{\tilde{r}} x_i)$ would be non-empty. □

3. f^g-statistical cluster points and f^g-statistically Cauchy sequences

Here, we present the concepts of f^g-statistical Cauchy sequences and f^g-statistical cluster point sets. The set of f^g-statistical cluster points of a compact-range sequence in normed spaces is classified in a few ways. Furthermore, we use the notion of f^g-nonthin subsets of $\mathbb{N}$ to give a characterization of f^g-statistically Cauchy sequences.

Definition 3.1. Let $x = \{x_n\}$ be a sequence in a normed space X, and let $x_\Omega = \{x_n\}_{n\in\Omega}$ be any f^g-nonthin subsequence of x, i.e., $d_g^f(\Omega) \neq 0$. Then $\gamma \in X$ is an f^g-statistical cluster point of x_Ω, if for each $\varepsilon > 0$, $d_g^f(\{k \in \Omega : \|x_k - \gamma\| < \varepsilon\}) \neq 0$. The set $\Gamma_{x_\Omega}^{f^g}$ denotes the assortment of f^g-statistical cluster points of x_Ω (visit [14, 19, 20, 26] for references related to statistical cluster point and statistical limit point).

Note 3.2. It is easy to verify that the set of f^g-statistical cluster points $\Gamma_x^{f^g}$ of a sequence x in a normed space X forms a closed set.

In the next result, we give an association between the f^g-statistical cluster point of a sequence and its ordinary limit points.

Theorem 3.3. *If the f^g-nonthin subsequence $x_\Omega = \{x_n\}_{n\in\Omega}$ of a sequence $x = \{x_n\}$ lies in a compact subset of X, then there exists a sequence $y = \{y_n\}$ in X such that $L_{y_\Omega} = \Gamma_{x_\Omega}^{f^g}$ and $d_g^f(\{k \in \Omega : x_k = y_k\}) \neq 0$, where L_{y_Ω} denotes the set of ordinary limit points of $\{y_n\}_{n\in\Omega}$.*

Proof. If $\Gamma_{x_\Omega}^{f^g} = L_{x_\Omega}$, then we are done. So we assume that $\Gamma_{x_\Omega}^{f^g}$ is a proper subset of L_{x_Ω}. Then for each $\eta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}$ there is an open ball B_η having center at η such that $d_g^f(\{k \in \Omega : x_k \in B_\eta\}) = 0$. It is obvious that $\{B_\eta : \eta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}\}$ is an open cover for $L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}$. Since $L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}$ is included in a compact set and a metric space, compactness implies separability and subspace of a separable space is separable and hence Lindelöf. Consequently, a countable subcover is generated by $\{B_\eta : \eta \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}\}$, say it is $\{B_{\eta_i} : i \in \mathbb{N}\}$. Note that each open ball B_{η_i} contains a convergent f^g-thin subsequence of x_Ω. Since for each $i \in \mathbb{N}$, $d_g^f(\{k \in \Omega : x_k \in B_{\eta_i}\}) = 0$, and $\mathcal{Z}_g(f)$ is a P-ideal, it follows that there exists a set $K(\subsetneq \mathbb{N})$ such that $d_g^f(K) = 0$ and for each $i \in \mathbb{N}$, $\{k \in \Omega : x_k \in B_{\zeta_i}\} \setminus K$ is finite. Suppose $\Omega \setminus K = \{j(1) < j(2) < ... < j(n) < ...\}$, and construct the sequence $y = \{y_n\}$ as follows:

$$y_k = \begin{cases} x_{j(k)}, & \text{for } \; k \in K, \\ x_k, & \text{for } \; k \in \Omega \setminus K, \\ x_1, & \text{otherwise.} \end{cases}$$

Since $\{n \in \Omega : x_n \neq y_n\} \subseteq \Omega \cap K$, i.e., $d_g^f(\{n \in \Omega : x_n \neq y_n\}) = 0$, we conclude that $\Gamma_{x_\Omega}^{f^g} = \Gamma_{y_\Omega}^{f^g}$. Observe that $\{y_n\}_{n\in K}$ is an f^g-thin subsequence of y since $d_g^f(K) = 0$. We intend to show that $\{y_n\}_{n\in K}$ has no ordinary limit points. Assume, on the contrary, that $\{y_n\}_{n\in K} = \{x_{j(k)}\}_{k\in K}$ has a convergent subsequence which converges to $p \in X$. Since $d_g^f(K) = 0$ and $\{j(k)\}_{k\in\mathbb{N}}$ is increasing, we conclude that $d_g^f(\{j(k) : k \in K\}) = 0$. Therefore, we get that $p \in L_{x_\Omega} \setminus \Gamma_{x_\Omega}^{f^g}$. Then there exists $i_0 \in \mathbb{N}$ such that $p \in B_{\zeta_{i_0}}$, i.e., the infinite set $\{j(k) \in \Omega : x_{j(k)} \in B_{\zeta_{i_0}}\}$ is included in $\Omega \setminus K$. As a result, $\{j(k) \in \Omega : x_{j(k)} \in B_{\zeta_{i_0}}\} \setminus K$ is infinite, which leads to a contradiction. Likewise, we conclude that $\{y_n\}_{n\in\Omega\setminus K}$ has no convergent f^g-thin subsequences. Therefore, any convergent subsequence of y_Ω

must be f^g-nonthin, i.e., $L_{y\Omega} \subseteq \Gamma^{f^g}_{y\Omega}$. Consequently, $L_{y\Omega} = \Gamma^{f^g}_{y\Omega}$. Thus we deduce that $L_{y\Omega} = \Gamma^{f^g}_{x\Omega}$.

□

Our next step is to present a sufficient condition for a sequence to have f^g-statistical cluster points.

Corollary 3.4. *For any f^g-nonthin subsequence of a sequence $x = \{x_n\}$ that is contained in a compact subset of X, then $\Gamma^{f^g}_x \neq \varnothing$.*

Proof. Suppose the f^g-nonthin subsequence $x_\Omega = \{x_n\}_{n\in\Omega}$ of x is contained in a compact subset of X. Now Theorem 3.3 ensures the existence of a sequence $y = \{y_n\}$ in X such that $L_{y\Omega} = \Gamma^{f^g}_{x\Omega}$ and $d^f_g(\{k \in \Omega : x_k = y_k\}) \neq 0$. Since $x_\Omega = \{x_n\}_{n\in\Omega}$ lies in a compact set and the set $\{k \in \Omega : x_k = y_k\}$ is infinite, we have $L_{y\Omega} \neq \varnothing$. Hence we conclude that $\Gamma^{f^g}_x \supseteq \Gamma^{f^g}_{x\Omega} \neq \varnothing$. □

The compactness condition in the above theorem cannot generally be relaxed; even a bounded sequence in a normed space need not always possess an f^g-statistical cluster point. The following example substantiates our claim.

Example 3.5. Suppose $\ell^\infty(\mathbb{R})$ denotes the infinite dimensional normed space consisting of bounded sequences $\{x_n\}$ of real numbers endowed with *sup* norm, i.e., $\|(x_1, x_2, ...)\|_\infty := \sup_{k\in\mathbb{N}} |x_k|$. Consider the sequence $e = \{e_n\}$ in the normed space where e_n represents the sequence $(0, ..., 0, \overbrace{1}^{n^{\text{th}} \text{ place}}, 0, ...)$. We choose the modulus function $f(x) = \sqrt{x},\ x \in [0, \infty)$ and $g(n) = \log(1 + n^2), n \in \mathbb{N}$. Then we obtain $\Gamma^{f^g}_e = \varnothing$.

Corollary 3.6. *For any sequence $x = \{x_n\}$ that is included in a compact subset of X, there exists a sequence $y = \{y_n\}$ in X such that $L_y = \Gamma^{f^g}_x$ and $d^f_g(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$.*

Next, we give a relationship between rough f^g-statistical limit points and f^g-statistical cluster points of a sequence.

Theorem 3.7. *For any sequence $x = \{x_n\}$ in a compact subset of a normed space X, we have*

$$f^g st\text{-}LIM^r x_i = \bigcap_{\gamma\in\Gamma^{f^g}_x} (\gamma + rB_X).$$

Proof. Since x is contained in a compact set, Corollary 3.4 ensures that $\Gamma^{f^g}_x \neq \varnothing$. Firstly, we claim that $f^g st\text{-}LIM^r x_i \subseteq \bigcap_{\gamma\in\Gamma^{f^g}_x} (\gamma + rB_X)$. Assume, on the contrary, that there exist $\gamma \in \Gamma^{f^g}_x$ and $x_* \in f^g st\text{-}LIM^r x_i$ such that $\|x_* - \gamma\| > r$. Let us set $\varepsilon_0 = \frac{\|x_*-\gamma\|-r}{3} > 0$. Then for each $\gamma \in \Gamma^{f^g}_x$, we have

$$d^f_g(\{k \in \mathbb{N} : \|x_k - \gamma\| \leq \varepsilon_0\}) \neq 0.$$

Therefore, it follows immediately that

$$\{k \in \mathbb{N} : \|x_k - \gamma\| \leq \varepsilon_0\} \subseteq \{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon_0\}.$$

Consequently, we have $d^f_g(\{k \in \mathbb{N} : \|x_k - x_*\| \geq r + \varepsilon_0\}) \neq 0$ which is a contradiction, since $x_* \in f^g st\text{-}LIM^r x_i$. Thus we obtain

$$f^g st\text{-}LIM^r x_i \subseteq \bigcap_{\gamma\in\Gamma^f_x} (\gamma + rB_X).$$

Secondly, we take $y \in \bigcap_{\gamma\in\Gamma_x^{f^g}} (\gamma + rB_X)$. Now, let $y \notin f^g st\text{-}LIM^r x_i$. Then there exists $\varepsilon_0 > 0$ such that

$$d_g^f(K) \neq 0 \text{ where } K = \{k \in \mathbb{N} : \|x_k - y\| \geq r + \varepsilon_0\}.$$

From Corollary 3.6, it follows that there exists a sequence $y = \{y_n\}$ such that $L_y = \Gamma_x^{f^g}$ and $d_g^f(A) = 0$, where $A = \{k \in \mathbb{N} : x_k \neq y_k\}$. Therefore, it follows that $d_g^f(K \cap (\mathbb{N} \setminus A)) \neq 0$; otherwise $d_g^f(K) = 0$. Now for each $i \in K \cap (\mathbb{N} \setminus A)$, we have $\|y_i - y\| \geq r + \varepsilon_0$. Evidently, $\{y_n\}_{n\in K\cap(\mathbb{N}\setminus A)}$ lies in a compact set and this ensures the existence of a $\gamma' \in L_y$ such that $\|y - \gamma'\| \geq r + \varepsilon_0 > r$, which is a contradiction. As a consequence, we obtain $\bigcap_{\gamma\in\Gamma_x^{f^g}} (\gamma + rB_X) = f^g st\text{-}LIM^r x_i$. □

Let us now propose the concept of rough f^g-statistically Cauchy sequences, which extends the notion statistical Cauchy sequences.

Definition 3.8. A sequence $\{x_n\}$ in a normed space $(X, \|.\|)$ is said to be rough f^g-statistically Cauchy with degree of roughness $\rho \geq 0$ (or, shortly $\rho-f^g$-statistically Cauchy) if for each $\varepsilon > 0$, there exists $n(\varepsilon) \in \mathbb{N}$ such that $\{n \in \mathbb{N} : \|x_n - x_{n(\varepsilon)}\| > \rho + \varepsilon\}$ has f^g-density zero.

The sequence $\{x_n\}$ becomes f^g-statistically Cauchy if we let $\rho = 0$ in the above definition. The following theorem establishes a conncetion between f^g-statistical Cauchy sequences and f^g-nonthin subsets of $\mathbb{N}$.

Theorem 3.9. *Let $\Omega \subseteq \mathbb{N}$ be such that $d_g^f(\Omega) \neq 0$. A sequence $\{x_n\}$ is f^g-statistically Cauchy if and only if for each $\varepsilon > 0$ there exists $m(\varepsilon) \in \Omega$ such that $d_g^f(\{k \in \mathbb{N} : \|x_k - x_{m(\varepsilon)}\| > \varepsilon\}) = 0$.*

Proof. Let us assume that $\{x_n\}$ is f^g-statistically Cauchy sequence. Then for each $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $d_g^f(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_{n(\varepsilon)}\| > \frac{\varepsilon}{2}\}$. Therefore it follows that $d_g^f(\Omega \cap (\mathbb{N} \setminus A(\varepsilon))) \neq 0$, otherwise $d_g^f(\Omega) = 0$. Now pick $m(\varepsilon) \in \Omega \cap (\mathbb{N} \setminus A(\varepsilon))$ be arbitrary and consider the set $B(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x_{m(\varepsilon)}\| > \varepsilon\}$. Then for each $k \in B(\varepsilon)$ we have

$$\|x_k - x_{n(\varepsilon)}\| \geq \|x_k - x_{m(\varepsilon)}\| - \|x_{n(\varepsilon)} - x_{m(\varepsilon)}\| > \frac{\varepsilon}{2},$$

i.e., $B(\varepsilon) \subseteq A(\varepsilon)$. Consequently, we obtain $d_g^f(\{k \in \mathbb{N} : \|x_k - x_{m(\varepsilon)}\| > \varepsilon\}) = 0$. □

4. Minimal f^g-statistical convergence and Cauchy degree of a sequence via f^g-statistical cluster point set

In this section, we continue our investigation on minimal f^g-statistical convergence as well as minimal f^g-statistical Cauchy degree of a sequence in terms of Chebyshev radius and diameter of the f^g-statistical cluster point set. The Chebyshev radius and the diameter of the f^g-ststistical cluster point set are, in fact, the minimal f^g-statistical Cauchy degree and the minimal f^g-statistical convergence degree, respectively, under certain circumstances. Finally, the notion of Jung constant is employed to draw a relationship between f^g-statistical convergence and Cauchy degree of a sequence.

Before moving forward, let us recall the concepts of diametere, Chebyshev radius and Jung constant. The diameter and Chebyshev radius of a bounded subset F of a normed space X is given by

$$D_X(F) = \sup_{x,y\in F} \|x - y\| \text{ and } r_X(F) = \inf_{x\in X} \sup_{f\in F} \|x - f\|. \tag{4.1}$$

Also, the Jung constant [28, Equation 5.3] of a normed space X is defined by

$$J(X) = \sup\{2r_X(F) : F \subseteq X,\ D_X(F) \leq 1\} \tag{4.2}$$

where $D_X(F)$ is the diameter of the set F. Note that $1 \leq J(X) \leq 2$, for any normed space X. We direct readers to [24, 27, 28] for a more thorough investigation along those lines.

Theorem 4.1. *For any sequence $x = \{x_n\}$ in a compact subset of X, $D_X(\Gamma_x^{f^g})$ is the mimimal f^g-statistical Cauchy degree of x.*

Proof. It is evident that $\Gamma_x^{f^g}$ is nonempty and compact since $\{x_n\}$ lies in a compact set. We now show that for any ρ with $0 \leq \rho < D_X(\Gamma_x^{f^g})$, $\{x_n\}$ cannot be $\rho - f^g$-statistically Cauchy.
Let us first set $\varepsilon_0 = \frac{D(\Gamma_x^{f^g})-\rho}{3}$. The definition of $D(\Gamma_x^{f^g})$ entails that there exist $\gamma_1, \gamma_2 \in \Gamma_x^{f^g}$ such that

$$\|\gamma_1 - \gamma_2\| > \rho + 2\varepsilon_0.$$

Since $\gamma_1, \gamma_2 \in \Gamma_x^{f^g}$, we have that the sets A and B are defined as:

$$A := \{k \in \mathbb{N} : \|x_k - \gamma_1\| < \frac{\varepsilon_0}{2}\} \ \text{ and } \ B := \{k \in \mathbb{N} : \|x_k - \gamma_2\| < \frac{\varepsilon_0}{2}\}$$

have nonzero f^g-density.
Now let us pick arbitrary $k \in A$. Then for each $m \in B$, we have

$$\begin{aligned}\|x_k - x_m\| &\geq \|\gamma_1 - \gamma_2\| - \|(x_k - \gamma_1) - (x_m - \gamma_2)\| \\ &\geq \|\gamma_1 - \gamma_2\| - (\|x_k - \gamma_1\| + \|x_m - \gamma_2\|) \\ &> \rho + 2\varepsilon_0 - \frac{\varepsilon_0}{2} - \frac{\varepsilon_0}{2} = \rho + \varepsilon_0.\end{aligned}$$

Thus we have

$$\begin{aligned}&\{k \in \mathbb{N} : \|x_k - x_m\| > \rho + \varepsilon_0\} \supseteq A \\ &\Rightarrow d_g^f(\{k \in \mathbb{N} : \|x_k - x_m\| > \rho + \varepsilon_0\}) \neq 0,\ \text{for each } m \in B \\ &\Rightarrow d_g^f(\{m \in \mathbb{N} : d_g^f(\{k \in \mathbb{N} : \|x_k - x_m\| > \rho + \varepsilon_0\}) \neq 0\})(\geq d_g^f(B)) \neq 0.\end{aligned}$$

Consequently, $\{x_n\}$ is not $\rho - f^g$-statistically Cauchy.

Let us now prove that $\{x_n\}$ is $\rho - f^g$-statistically Cauchy for $\rho = D_X(\Gamma_x^{f^g})$. If this does not happen, then there exists $\varepsilon_0 > 0$ and for each $j \in \mathbb{N}$ such that the set

$$B_j = \{k \in \mathbb{N} : \|x_k - x_j\| > D_X(\Gamma_x^{f^g}) + \varepsilon_0\}$$

has nonzero f^g-density.
Let $\varepsilon > 0$ be such that $2\varepsilon < \varepsilon_0$. Compactness of $\Gamma_x^{f^g}$ ensures that there exist $\gamma_1, \gamma_2 \in \Gamma_x^{f^g}$ such that $D_X(\Gamma_x^{f^g}) = \|\gamma_1 - \gamma_2\|$. Let us define

$$U := \{k \in \mathbb{N} : x_k \notin \Gamma_x^{f^g} + \varepsilon B_X\}.$$

We claim that $d_g^f(U) = 0$. Assume, on the contrary, that $\{x_n\}_{n \in U}$ is an f^g-nonthin subsequence of x. Since x lies in a compact set, Corollary 3.6 ensures that there exists a sequence $y = \{y_n\}$ in X such that $L_y = \Gamma_x^{f^g}$ and $d_g^f(\{k \in \mathbb{N} : y_k \neq x_k\}) = 0$. Therefore the set $B = \{k \in \mathbb{N} : y_k = x_k\} \cap U$ must be infinite; otherwise $d_g^f(U) = 0$. This shows that the subsequence $\{y_n\}_{n \in B}$ is also contained in a compact subset of X as well as in the closed set $X \setminus (\Gamma_x^{f^g} + \varepsilon B_X)$. Consequently $\{y_n\}_{n \in B}$ has a convergent subsequence converging to some point in $X \setminus (\Gamma_x^{f^g} + \varepsilon B_X)$. Since $\gamma \in L_y (= \Gamma_x^{f^g})$, we have $\gamma \in X \setminus (\Gamma_x^{f^g} + \varepsilon B_X) \cap \Gamma_x^{f^g} (= \varnothing)$ which is a contradiction. Hence we must have

$$d_g^f(\{k \in \mathbb{N} : x_k \notin \Gamma_x^{f^g} + \varepsilon B_X\}) = 0.$$

Let us choose $t \in \mathbb{N} \setminus U$, then there exists $\gamma' \in \Gamma_x^{f^g}$ such that $\|x_t - \gamma'\| < \varepsilon$. Also for any $k \in B_t$, it follows that $\|x_k - x_t\| > \|\gamma_1 - \gamma_2\| + \varepsilon_0$. Now for arbitrary $\gamma \in \Gamma_x^{f^g}$ we have,

$$\begin{aligned}\|x_k - \gamma\| &\geq \|x_k - x_t\| - \|x_t - \gamma\| \\ &> \|\gamma_1 - \gamma_2\| + \varepsilon_0 - \|x_t - \gamma'\| - \|\gamma - \gamma'\| \\ &> \|\gamma_1 - \gamma_2\| + \varepsilon_0 - \varepsilon - \|\gamma_1 - \gamma_2\| > \varepsilon.\end{aligned}$$

This shows that $x_k \notin \Gamma_x^{f^g} + \varepsilon B_X$ i.e., $B_t \subseteq U$. Therefore we must have $d_g^f(B_t) = 0$, which is a contradiction. This ensures that $\{x_n\}$ is $D_X(\Gamma_x^{f^g}) - f^g$-statistically Cauchy.
Now combining both the parts, we conclude that $D_X(\Gamma_x^f)$ is the minimal f^g-statistical Cauchy degree of $\{x_n\}_{n\in\mathbb{N}}$. □

Theorem 4.2. *For any sequence $x = \{x_n\}$ in a compact subset of X, $r_X(\Gamma_x^{f^g})$ is the minimal f^g-statistical convergence degree.*

Proof. Suppose $0 \leq r < r_X(\Gamma_x^{f^g})$. Then from the definition of $r_X(\Gamma_x^f)$, it follows that for any $y \in X$ there exists $\gamma \in \Gamma_x^{f^g}$ such that $\|y - \gamma\| > r$. Now, Theorem 3.7 ensures that for any $y \in X$ such that $y \notin f^g st\text{-}LIM^r x_i$, i.e., $f^g st\text{-}LIM^r x_i = \varnothing$. If $r \geq r_X(\Gamma_x^{f^g})$, then there exists $y \in X$ such that

$$\begin{aligned}&\|y - \gamma\| \leq r \ \text{ for all } \ \gamma \in \Gamma_x^{f^g}. \\ \Rightarrow\ & y \in \bigcap_{\gamma \in \Gamma_x^{f^g}} (\gamma + rB_X) \\ \Rightarrow\ & y \in f^g st\text{-}LIM^r x_i \ \text{ (by Theorem 3.7).}\end{aligned}$$

Consequently, we have $f^g st\text{-}LIM^r x_i \neq \varnothing$ whenever $r \geq r_X(\Gamma_x^{f^g})$. Hence we conclude that $r_X(\Gamma_x^{f^g})$ is the minimal f^g-statistical convergence degree of $\{x_n\}$. □

Without the compactness condition, the conclusion of Theorem 4.1 and Theorem 4.2 may not hold. To highlight this fact, we provide the example below.

Example 4.3. Let $f(x) = \log(1 + x),\ x \in [0, \infty)$ be the modulus function and $g \in \mathbb{G}$ be such that $g(n) = \sqrt{n+1}, n \in \mathbb{N}$. Now fix any $A \subseteq \mathbb{N}$ in such a way that both $d_g^f(A)$ and $d_g^f(\mathbb{N} \setminus A)$ are non-zero (note that existence of such A can always be assured by taking $A = 2\mathbb{N}$). Let us now define the sequence $x = \{x_n\}$ in $\ell^\infty(\mathbb{R})$ as follows:

$$x_n := \begin{cases} e_1, & \text{if } n \in A \\ e_n, & \text{otherwise.} \end{cases}$$

where the sequence $\{e_n\}$ is given in Example 2.7. It is evident that $\{x_n\}$ is not contained in any compact subset of $\ell^\infty(\mathbb{R})$. Since $d_g^f(A) \neq 0$, it follows that $e_1 \in \Gamma_x^{f^g}$. For any $z \in \ell^\infty(\mathbb{R})$ with $e_1 \neq z$, we choose $\varepsilon_* = \min\{\frac{1}{4}, \|z - e_1\|\}$. We now prove that $d_g^f(\{k \in \mathbb{N} : \|x_k - z\| < \varepsilon_*\}) = 0$. Assume, on the contrary, that there exist $t_1, t_2 \in \mathbb{N} \setminus A$ with $1 < t_1 < t_2$ such that

$$\|x_{t_1} - z\| < \varepsilon_* \ \text{ and } \ \|x_{t_2} - z\| < \varepsilon_*.$$

This leads to the contradiction that $1 = \|x_{t_1} - x_{t_2}\| \leq \frac{1}{2}$. Consequently, we must have $\Gamma_x^{f^g} = \{e_1\}$. So in this case $r_{\ell^\infty(\mathbb{R})}(\Gamma_x^{f^g}) = D_{\ell^\infty(\mathbb{R})}(\Gamma_x^{f^g}) = 0$, but it is easy to observe that the sequence is neither f^g-statistically Cauchy nor f^g-statistically convergence.

The two consecutive results connote the relationship between rough f^g-statistically convergence and rough f^g-statistically Cauchy sequences via Jung constant.

Theorem 4.4. *Suppose the sequence $x = \{x_n\}$ is in a compact subset of X. If $\{x_n\}$ is $\rho - f^g$-statistically Cauchy, then $\{x_n\}$ is $r - f^g$-statistically convergence for every $r \geq \frac{J(X)\rho}{2}$.*

Proof. Since $\{x_n\}$ is a $\rho - f^g$-statistically Cauchy sequence, Theorem 4.1 entails that $\rho \geq D_X(\Gamma_x^{f^g})$. Note that we can write

$$r_X(\Gamma_x^{f^g}) \leq \frac{J(X)D_X(\Gamma_x^{f^g})}{2} \quad \text{(by definition of } J(X)).$$

Then for any $r \geq \frac{J(X)\rho}{2}$, we have $r \geq \frac{J(X)D_X(\Gamma_x^{f^g})}{2} \geq r_X(\Gamma_x^{f^g})$. Now, Theorem 4.2 ensures that $\{x_n\}$ is $r - f^g$-statistically convergence. □

In the above Theorem 4.4, the constraint that the sequence is contained in a compact set cannot generally be relaxed. We now select an example to highlight our claim.

Example 4.5. Setting the weight function $g \in \mathbb{G}$ and the modulus function f are respectively as follows:

$$g(n) = n, \; n \in \mathbb{N} \;\; \text{and} \;\; f(x) = \log(1 + x), \; x \in [0, \infty).$$

We now consider $D \subset \mathbb{N}$ be such that $d_g^f(D) = 0$ (note that such set exists by [9, Example 2.3 (Case 2)]). Let us consider $\{x_n\}$ in $(C[0,1], \|.\|)$ (where $\|x\| = \int_0^1 |x(t)|dt$) as follows:

$$x_k(t) = \begin{cases} a_k(t) & \text{if } k \in D \\ b_k(t) & \text{if } k \in \mathbb{N} \setminus D. \end{cases}$$

The sequences $\{a_n\}$ and $\{b_n\}$ in $C[0,1]$ are defined in the following manner:

$$a_k(t) = kt, \;\; \text{and} \;\; b_k(t) = \begin{cases} k & \text{if } 0 \leq t \leq \frac{1}{k^2}, \\ \frac{1}{\sqrt{t}} & \text{if } \frac{1}{k^2} \leq t \leq 1, \end{cases} \;\; \text{for} \;\; k \in \mathbb{N} \;\; \text{and} \;\; t \in [0,1].$$

From the construction, it follows that $\{x_n\}_{n \in \mathbb{N} \setminus D}$ does not have any convergent subsequences since $\{x_n\}_{n \in \mathbb{N} \setminus D} \xrightarrow{\|.\|} x$ gives

$$x(t) = \frac{1}{\sqrt{t}}, \;\; \text{for } 0 < t \leq 1, \;\; \text{which is a discontinuous function in } [0,1].$$

Thus the cluster point set of $\{x_n\}$ is empty. Consequently, $\{x_n\}$ is not contained in any compact subset of X. Since $\mathcal{Z}_g(f)$ is a P-ideal, it follows that $\{x_n\}$ is not f^g-statistically convergent to any continuous function in $[0,1]$. Therefore, we must have $f^g st\text{-}LIM^0 x_k = \varnothing$.

Now, observe that $\|b_m - b_n\| = \frac{1}{n} - \frac{1}{m}$ whenever $m > n$. Let $\varepsilon > 0$ be given. We set $n(\varepsilon) = \left[\frac{1}{\varepsilon}\right] + 1$. Therefore, we have

$$\|x_k - x_{n(\varepsilon)}\| < \varepsilon \text{ whenever } k > \left[\frac{1}{\varepsilon}\right] + 1 \text{ and } k \in \mathbb{N} \setminus D,$$

i.e., $\{k \in \mathbb{N} : \|x_k - x_{n(\varepsilon)}\| \geq \varepsilon\} \subseteq D \cup \{1, 2, ..., \left[\frac{1}{\varepsilon}\right] + 1\}$. Since $d_g^f(D) = 0$, it follows that $d_g^f(\{k \in \mathbb{N} : \|x_k - x_{n(\varepsilon)}\| \geq \varepsilon\}) = 0$. This ensures that $\{x_n\}$ is $\rho - f^g$-statistically Cauchy with $\rho = 0$. However, the sequence $\{x_n\}$ is not $r - f^g$-statistically convergent for $r = \frac{J(C[0,1])\rho}{2} = 0$.

Theorem 4.6. *If $\{x_n\}$ is $\rho - f^g$-statistically Cauchy in X, then $\{x_n\}$ is $r - f^g$-statistically convergent for every $r > J(X)\rho$.*

Proof. Let us take any $r > J(X)\rho$ and set $\varepsilon = (r - J(X)\rho)/(2^{-1}J(X) + 1) > 0$. Since $\{x_n\}$ is $\rho - f^g$-statistically Cauchy, there exists $m_\varepsilon \in \mathbb{N}$ such that

$$d_g^f(A) = 0, \;\; \text{where} \;\; A = \{k \in \mathbb{N} : \|x_k - x_{m_\varepsilon}\| \geq \rho + \frac{\varepsilon}{2}\}.$$

Let us consider $S = \{x_k : k \in \mathbb{N} \setminus A\}$. Therefore it is evident that $D_X(S) < 2\rho + \varepsilon$. Now Equation 4.1 yields that there exists $x_* \in X$ such that

$$\|x - x_*\| \leq r_X(S) + \varepsilon \quad \text{for all} \quad x \in S.$$

and also from Equation 4.2 we have

$$r_X(S) \leq \frac{J(X)D_X(S)}{2}.$$

Thus for all $x \in S$, we obtain

$$\begin{aligned}\|x - x_*\| &\leq \frac{J(X)D_X(S)}{2} + \varepsilon \\ &< \frac{J(X)(2\rho + \varepsilon)}{2} + \varepsilon \\ &= J(X)\rho + \left(\frac{1}{2}J(X) + 1\right)\varepsilon \\ &= J(X)\rho + r - J(X)\rho = r\end{aligned}$$

Therefore, we must have

$$\begin{aligned}&\mathbb{N} \setminus A \subseteq \{k \in \mathbb{N} : \|x_k - x_*\| < r\} \\ \Rightarrow &\{k \in \mathbb{N} : \|x_k - x_*\| \geq r\} \subseteq A \\ \Rightarrow &d_g^f(\{k \in \mathbb{N} : \|x_k - x_*\| \geq r\}) = 0.\end{aligned}$$

Hence we deduce that $f^g st\text{-}LIM^r x_i \neq \varnothing$. □

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References

[1] A. Aizpuru, M.C. Listán-García and F. Rambla-Barreno, *Density by moduli and statistical convergence*, Quaest. Math. **37** (4), 525–530, 2014.

[2] M. Arslan and E. Dündar, *On rough convergence in 2-normed spaces and some properties*, Filomat **33** (16), 5077–5086, 2019.

[3] A. Aydın, *Statistically order compact operators on Riesz spaces*, Hacet. J. Math. Stat. **53** (3), 1–9, 2024.

[4] S. Aytar, *The rough limit set and the core of a real sequence*, Numer. Funct. Anal. Optim. **29** (3-4), 283-290, 2008.

[5] S. Aytar, *Rough statistical convergence*, Numer. Funct. Anal. Optim. **29** (3-4), 291–303, 2008.

[6] S. Aytar, *Rough statistical cluster points*, Filomat **31** (16), 5295-5304, 2017.

[7] T. Aziz and S. Ghosal, *f-rough Cauchy sequences*, Quaest. Math. **48** (1), 2025, DOI: 10.2989/16073606.2025.2464951.

[8] M. Balcerzak, P. Das, M. Filipczak and J. Swaczyna, *Generalized kinds of density and the associated ideals*, Acta Math. Hungar. **147** (1), 97–115, 2015.

[9] K. Bose, P. Das and A. Kwela, *Generating new ideals using weighted density via modulus functions*, Indag. Math. **29** (5), 1196–1209, 2018.

[10] P. Das and A. Ghosh, *Generating subgroups of the circle using a generalized class of density functions*, Indag. Math. **32** (3), 598618, 2021.
[11] E. Dündar, *On rough $\mathcal{I}_2$-convergence of double sequences*, Numer. Funct. Anal. Optim. **37** (4), 480-491, 2016.
[12] H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (3-4), 241–244, 1951.
[13] J. A. Fridy, *On statistical convergence*, Analysis **5** (4), 301–314, 1985.
[14] J. A. Fridy, *Statistical limit points*, Proc. Am. Math. Soc. **118** (4), 1187–1192, 1993.
[15] A. L. Garkavi, *On the optimal net and best cross-section of a set in a normed space*, Izv. Akad. Nauk SSSR Ser. Mat. **26** (1), 87–106, 1962.
[16] S. Ghosal and M. Banerjee, *Rough weighted statistical convergence on locally solid Riesz spaces*, Positivity **25** (5), 1789-1804, 2021.
[17] S. Ghosal and S. Mandal, *The degree of roughness*, Topol. Appl. **307**, 107944, 2022.
[18] S. Ghosal and S. Mandal, *Rough weighted $\mathcal{I}$-$\alpha\beta$-statistical convergence in locally solid Riesz spaces*, J. Math. Anal. Appl. **506** (2), 125681, 2022.
[19] H. I. Miller and L. Miller-Van Wieren, *Statistical cluster point and statistical limit point sets of subsequences of a given sequence*, Hacet. J. Math. Stat. **49** (2), 494–497, 2020.
[20] L. Miller-Van Wieren, E. Taş and T. Yurdakadim, *Some new insights into ideal convergence and subsequences*, Hacet. J. Math. Stat. **51** (5), 1379–1384, 2022.
[21] P. Kostyrko, T. Šalát and W. Wilczyński, *$\mathcal{I}$-convergence*, Real Anal. Exchange **26**, 669–685, 2000.
[22] M. C. Listán-García, *A characterization of uniform rotundity in every direction in terms of rough convergence*, Numer. Funct. Anal. Optim. **22** (11), 1166–1174, 2011.
[23] M. C. Listán-García, *f-statistical convergence, completeness and f-cluster points*, Bull. Belg. Math. Soc. Simon Stevin **23** (2), 235–245, 2016.
[24] M. C. Listán-García and F. Rambla-Barreno, *Rough convergence and Chebyshev centers in Banach spaces*, Numer. Funct. Anal. Optim. **35** (4), 432-442, 2014.
[25] S. K. Pal, D. Chandra and S. Dutta, *Rough ideal convergence*, Hacet. J. Math. Stat. **42** (6), 633–640, 2013.
[26] S. Pehlivan, A. Güncan and M. Mamedov, *Statistical cluster points of sequences in finite dimensional spaces*, Czechosl. Math. J. **54** (1), 95–102, 2004.
[27] H. X. Phu, *Rough convergence in normed linear spaces*, Numer. Funct. Anal. Optim. **22** (1-2), 199-222, 2001 .
[28] H. X. Phu, *Rough convergence in infinite dimensional normed space*, Numer. Funct. Anal. Optim. **24** (3-4), 285-301, 2003.
[29] S. K. A. Rahaman and M. Mursaleen, *On rough deferred statistical convergence of difference sequences in L-fuzzy normed spaces*, J. Math. Anal. Appl. **530** (2), 127684, 2024.
[30] M. H. M. Rashida, *Rough statistical convergence and rough ideal convergence in random 2-normed spaces*, Filomat **38** (3), 979–996, 2024.
[31] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (2), 139-150, 1980.
[32] H. Steinhaus, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. **2** (1), 73-74, 1951.

[33] B. C. Tripathy, *On statistically convergent and statistically bounded sequences*, Bull. Malays. Math. Soc. (Second Series) **20** (1), 31-33, 1997.