Proceedings of International Mathematical Sciences
ISSN: 2717-6355, URL: https://dergipark.org.tr/tr/pub/pims
Volume 7 Issue 1 (2025), Pages 10-15.
Doi: https://doi.org/ 10.47086/pims.1589091

HIGHER-ORDER EQUATIONS WITH ROBIN BOUNDARY CONDITIONS IN THE UPPER HALF COMPLEX PLANE

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Abstract. This study explores the solvability and solution of a Robin problem for a higher-order differential equation in the upper half-plane. Using the framework of the higher-order Cauchy-Riemann operator, we extend classical techniques to handle complex boundary interactions.

Necessary conditions are established by analyzing the boundary operator's structure. To construct solutions, we apply an integral method that reduces the problem to a more manageable form, employing higher-order Cauchy-type transforms and kernel functions suited to the upper half-plane.

1. Introduction

Different boundary value issues are investigated in [1-5] for various domains with explicit solutions. Boundary value problems play a fundamental role in mathematical analysis and its applications, particularly in understanding physical phenomena modeled by partial differential equations. Among these, Robin boundary value problems represent a versatile class that interpolates between Dirichlet and Neumann conditions, offering a rich framework for both theoretical exploration and practical applications. While significant progress has been made in the study of classical boundary value problems in the complex plane, the investigation of higher-order equations in the upper half-plane, especially under Robin boundary conditions, remains an open and challenging area of research. The current work focuses on a Robin boundary value problem associated with a higher-order partial differential equation in the upper half-plane. These problems are of particular interest due to their intricate coupling of boundary and interior conditions, as well as their connection to advanced operators such as the higher-order Cauchy-Riemann operator. Such equations naturally arise in a variety of contexts, including fluid dynamics, elasticity, and electromagnetic theory, where higher-order derivatives and mixed boundary conditions are integral to the models. Our primary objective is to establish the conditions under which the Robin boundary value problem admits a solution and to provide an explicit integral representation of the solution. By employing tools from complex analysis, we develop a framework that reduces the complexity of the higher-order equation and reveals the interplay between

2020 *Mathematics Subject Classification.* Primary: 30E25; 35F15; 35G15.
Key words and phrases. Robin boundary value problem, upper half plane, higher order equation.

Submitted on 21.11.2024, Accepted on 12.04.2025.
Communicated by Metin Basarir.

the boundary conditions and the solution's analytic structure. The integral representation not only provides a practical method for constructing solutions but also offers deeper insights into the properties of the underlying operator. This study contributes to the broader theory of boundary value problems in complex analysis and lays the groundwork for further investigations into higher-order equations in more general domains. Additionally, the techniques and results presented here may find applications in mathematical physics and engineering, where the study of such equations is both theoretical and practical.

2. Preliminaries

Let $\mathbb{H} := \{z \in \mathbb{C} : Imz > 0\}$ denotes the upper half plane and $\mathbb{R}$ its boundary (the x-axis). Here $L_{p,2}(\mathbb{H};\mathbb{C})$ means the space of complex-valued functions f defined in $\mathbb{H}$. T is Pompeiu integral operator. The Cauchy–Riemann operator is a fundamental tool in complex analysis and is defined as

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

where $z = x + iy$. A complex-valued function $\omega : \mathbb{H} \to \mathbb{C}$ is said to be analytic if and only if $\omega_{\bar{z}} = 0$, that is, if it lies in the kernel of the Cauchy–Riemann operator. The inhomogeneous Cauchy–Riemann equation $\omega_{\bar{z}} = f$, where f is a given function, arises naturally in the context of boundary value problems involving non-analytic functions and serves as a starting point for constructing solutions using integral representations.

In this study, we extend the classical theory by considering higher-order versions of the Cauchy–Riemann operator. These operators are crucial for analyzing more intricate boundary conditions, such as those appearing in Robin-type problems for higher-order partial differential equations.

In this study, we formulate the problem in the upper half-plane $\mathbb{H}$ due to its analytical convenience and compatibility with classical integral operators such as the Cauchy and Pompeiu transforms. The upper half-plane provides a natural setting where the boundary $\mathbb{R}$ allows explicit use of well-known kernels and boundary value techniques. Although the analysis is carried out in $\mathbb{H}$, the approach can be extended to the lower half-plane $\mathbb{L} := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ by symmetry or by adapting the kernel functions accordingly.

Theorem 2.1. [3] *The Robin problem $\omega_{\bar{z}} = 0$ in $\mathbb{H}$, $\alpha\omega + \partial_\nu\omega = \gamma$ on $\mathbb{R}, \omega(i) = C_0$ for $\alpha \in O(\mathbb{H}) \cap C\left(\overline{\mathbb{H}};\mathbb{C}\right), \gamma \in L^2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ and $C_0 \in \mathbb{C}$ is uniquely solvable if and only if*

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\gamma(t)}{t - \bar{z}}\, dt = 0 \tag{2.1}$$

for $z \in \mathbb{H}$. The solution is given by

$$\omega(z) = C_0 e^{-\int_i^z i\alpha(\varsigma)d\varsigma} - \frac{i}{\pi}\int_{-\infty}^{\infty} \gamma(t) \int_i^z \frac{Im(s)}{|t-s|^2} e^{\int_s^z i\alpha(\varsigma)d\varsigma}\, ds\, dt. \tag{2.2}$$

Here $O(\mathbb{H})$ denotes the set of analytic functions in $\mathbb{H}$.

Proposition 2.2. [5] *The Robin problem for the inhomogeneous Cauchy–Riemann equation*

$$\omega_{\bar{z}} = f \text{ in } \mathbb{H},$$
$$\omega - \partial_y\omega = \gamma \text{ on } \mathbb{R},$$
$$\omega(0) = c + Tf(0)$$

for given $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C})$, $p > 2$ *and* $\gamma \in L^2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$ *is uniquely solvable if and only if for* $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma(t)+f(t)}{t-\overline{z}}\,dt + iTf(\overline{z}) + T\partial_\zeta f(\overline{z}) = 0. \tag{2.3}$$

The solution is given by

$$\omega(z) = ce^{-iz} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)-2if(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt + Tf(z) \tag{2.4}$$

3. Robin Boundary Value Problem For Higher Order Equation

This section explores the Robin Boundary value problem associated with higher order partial differential equations in the upper half complex plane. By generalizing the classical techniques, we aim to establish conditions fort he existence and uniqueness of the solutions. Additionally, explicit representations are derived, highlighting the interplay between boundary conditions and solution's analytical structure.

Theorem 3.1. *Let* $\gamma_i \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $p > 2$, $w \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^n(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, *and* $c_i \in \mathbb{C}$ *for* $i = 1,\dots,n$. *The Robin boundary value problem for the homogeneous Cauchy-Riemann equation is:*

$$\partial_{\bar{z}}^n w = 0 \ \text{in} \ \mathbb{H} \tag{3.1}$$

$$\partial_{\bar{z}}^k w - \partial_y \partial_{\bar{z}}^k w = \gamma_{k+1} \ \text{on} \ \mathbb{R}, \tag{3.2}$$

$$\partial_{\bar{z}}^{n-1} w + \partial_y \partial_{\bar{z}}^{n-1} w = \gamma_n \ \text{on} \ \mathbb{R}, \tag{3.3}$$

$$\partial_{\bar{z}}^k w(0) = c_{k+1} + T\partial_{\bar{z}}^{k+1} w(0), \tag{3.4}$$

$$\partial_{\bar{z}}^{n-1} w(i) = c_n, \tag{3.5}$$

is uniquely solvable if and only if for $z \in \mathbb{H}$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{i\gamma_{k+1}(t)+u_{k+1}(t)}{t-\overline{z}}\,dt + iTu_{k+1}(\overline{z}) + T\partial_\zeta u_{k+1}(\overline{z}) = 0 \quad \text{for } 0 \le k \le n-2, \tag{3.6}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\gamma_n(t)}{t-\overline{z}}\,dt = 0 \quad \text{for } k = n-1, \tag{3.7}$$

and the solution is given by

$$w(z) = c_1 e^{-iz} + Tu_1(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_1(t)-2iu_1(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt, \tag{3.8}$$

where the auxiliary functions $u_{k+1}(t)$ *and* $u_{k+1}(z)$ *are defined as:*

$$u_{k+1}(t) = c_{k+2}e^{-it} + \int_0^t (i\gamma_{k+2}(\varsigma) + u_{k+2}(\varsigma))\,e^{i(\varsigma-t)}\,d\varsigma, \tag{3.9}$$

$$u_{k+1}(z) = c_{k+2}e^{-iz} + Tu_{k+2}(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_{k+2}(t)-2iu_{k+2}(t))\,e^{i(t-z)}\int_{-t}^{z-t}\frac{e^{i\zeta}}{i\zeta}\,d\zeta\,dt \tag{3.10}$$

for $0 \le k \le n-3$,
and

$$u_{n-1}(t) = c_n e^{it} - \int_0^t i\gamma_n(\varsigma)e^{-i(\varsigma-t)}\,d\varsigma, \tag{3.11}$$

$$u_{n-1}(z) = c_n e^{1+iz} - \frac{i}{\pi}\int_{-\infty}^{\infty}\gamma(t)\int_i^z e^{i(z-s)}\frac{Im(s)}{|t-s|^2}\,ds\,dt \tag{3.12}$$

for $k = n-1$.

Proof. For $n = 1$, we consider $w(z)$ as

$$w(z) = u_1(z) : \tag{3.13}$$

$$w(z) = c_1 e^{1+iz} - \frac{i}{\pi}\int_{-\infty}^{\infty} \gamma(t) \int_i^z e^{i(z-s)} \frac{\mathrm{Im}(s)}{|t-s|^2}\, ds\, dt. \tag{3.14}$$

For $n > 1$, the problem reduces to the following sequence of equations:

(1) $w_{\bar z} = u_1$, $w - \partial_y w = \gamma_1$ on $\mathbb{R}$,

$$w(0) = c_1 + Tu_1(0);$$

(2) $u_{i_{\bar z}} = u_{i+1}$, $u_i - \partial_y u_i = \gamma_{i+1}$ on $\mathbb{R}$,

$$u_i(0) = c_{i+1} + Tu_{i+1}(0), \quad 1 \le i \le n-2;$$

(3) $(u_{n-1})_{\bar z} = 0$ in H, $u_{n-1} + \partial_y u_{n-1} = \gamma_n$ on $\mathbb{R}$,

$$u_{n-1}(i) = c_n.$$

The condition $u_i - \partial_y u_i = \gamma_{i+1}$ implies

$$u_i'(t) + iu_i(t) = i\gamma_{i+1}(t) + u_{i+1}(t) \quad \text{on } \mathbb{R}.$$

Solving this differential equation, we obtain

$$u_i(t) = c_{i+1}e^{it} + \int_0^t (i\gamma_{i+1}(\varsigma) + u_{i+1}(\varsigma))\, e^{i(\varsigma - t)}\, d\varsigma.$$

Combining the solutions and solvability conditions completes the proof. □

Theorem 3.2. *Let $\gamma_i \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, $p > 2$, $w \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^n(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C})$, and $c_i \in \mathbb{C}$ for $i = 1,\dots,n$. The Robin boundary value problem for the higher-order inhomogeneous Cauchy-Riemann equation is formulated as*

$$\partial_{\bar z}^n w = f(z) \quad \text{in } \mathbb{H}, \tag{3.15}$$

with the following boundary and point conditions:

$$\partial_{\bar z}^k w - \partial_y \partial_{\bar z}^k w = \gamma_{k+1} \text{ on } \mathbb{R},\ 0 \le k \le n-1, \tag{3.16}$$

$$\partial_{\bar z}^k w(0) = c_{k+1} + Tu_{k+1}(0), \quad 0 \le k \le n-2, \tag{3.17}$$

and

$$\partial_{\bar z}^{n-1} w(0) = c_n + Tf(0). \tag{3.18}$$

The problem is uniquely solvable if and only if the following solvability conditions hold for $z \in \mathbb{H}$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{i\gamma_k(t) + u_k(t)}{t - \bar z}\, dt + iTu_k(\bar z) + T\partial_\zeta u_k(\bar z) = 0, \quad \text{for } 1 \le k \le n-1, \tag{3.19}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{i\gamma_n(t) + f(t)}{t - \bar z}\, dt + iTf(\bar z) + T\partial_\zeta f(\bar z) = 0, \quad \text{for } k = n-1. \tag{3.20}$$

The solution $w(z)$ is given by

$$w(z) = c_1 e^{-iz} + Tu_1(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\gamma_1(t) - 2iu_1(t))\, e^{i(t-z)} \int_{-t}^{z-t} \frac{e^{i\zeta}}{i\zeta}\, d\zeta\, dt, \tag{3.21}$$

where $u_k(z)$ and $u_k(t)$ are defined recursively:

$$u_k(z) = c_{k+1}e^{-iz} + Tu_{k+1}(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\gamma_{k+1}(t) - 2iu_{k+1}(t))\, e^{i(t-z)} \int_{-t}^{z-t} \frac{e^{i\zeta}}{i\zeta}\, d\zeta\, dt, \tag{3.22}$$

$$u_k(t) = c_{k+1}e^{-it} + \int_0^t (i\gamma_{k+1}(\varsigma) + u_{k+1}(\varsigma))\, e^{i(\varsigma - t)}\, d\varsigma \tag{3.23}$$

for $1 \leq k \leq n-2$.

For $k = n-1$*:*

$$u_{n-1}(z) = c_n e^{-iz} + Tf(z) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} (\gamma_1(t) - 2if(t))\, e^{i(t-z)} \int_{-t}^{z-t} \frac{e^{i\zeta}}{i\zeta}\, d\zeta\, dt, \tag{3.24}$$

$$u_{n-1}(t) = c_n e^{-it} + \int_0^t (i\gamma_n(\varsigma) + f(\varsigma))\, e^{i(\varsigma - t)}\, d\varsigma. \tag{3.25}$$

Proof. For $n = 1$, take $u_1(t) = f(t)$ and $u_1(z) = f(z)$. Substituting these into $w(z)$ verifies the result. Assume the theorem holds for n-1 and consider the case for n. The problem decomposes into two systems

$$\partial_{\bar{z}}^{n-1} w(z) = u(z) \text{ in } \mathbb{H}, \tag{3.26}$$

$$\partial_{\bar{z}}^{k} w - \partial_y \partial_{\bar{z}}^{k} w = \gamma_{k+1} \quad \text{on } \mathbb{R},\ 0 \leq k \leq n-2, \tag{3.27}$$

$$\partial_{\bar{z}}^{k} w(0) = c_{k+1} + Tu_{k+1}(0), \quad 0 \leq k \leq n-3, \tag{3.28}$$

$$\partial_{\bar{z}}^{n-2} w(0) = c_{n-1} + Tf(0); \tag{3.29}$$

and

$$u_{\bar{z}} = f(z) \text{ in } \mathbb{H},\ u - \partial_y u = \gamma_n \quad \text{on } \mathbb{R}, \tag{3.30}$$

$$u(0) = c_n + Tf(0). \tag{3.31}$$

Solving these systems recursively using the solvability conditions and integrating the solutions yields the desired result.

□

4. Conclusion

In this study, we analyzed the Robin boundary value problem for higher-order equations in the upper half complex plane. Through a systematic exploration of existence and uniqueness conditions, we provided explicit integral representations for solutions. Our findings demonstrate the feasibility of generalizing classical boundary value problem techniques to higher-order equations, with particular emphasis on the complex interaction between boundary conditions and the operator's structure.The results have potential applications in mathematical physics and engineering, especially in problems involving mixed boundary conditions and higher-order derivatives. Future work may extend these methods to more complex domains and investigate numerical approaches for practical implementations. The theoretical results obtained from the Robin boundary value problem for higher-order equations, particularly in the context of the Cauchy-Riemann equations, can be applied in various fields, including signal processing in communication systems.

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