Journal of Science and Technology
4 (1), 2011, 51 – 59

ON THE SERIES SOLUTION OF POLYTROPIC GAS SPHERES EQUATION FOR POLYTROPIC INDEX n=3

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ABSTRACT

In this article, a power series solution for the Lane-Emden equation for polytropic index $n = 3$ has been developed. We constructed a recurrence relation for the components of the series solution and established that the convergence radius is at least $\sqrt{6.5}$.

Keywords: *Singular IVP; polytropic gas sphere; Lane-Emden equation; approximate solution*

ÖZET

Bu makalede $n = 3$ politropik indeksli Lane-Emden denkleminin seri çözümü geliştirilmiştir. Seri çözümün bileşenleri için rekürens ilişki kurarak serinin yakınsaklık yarıçapının en az $\sqrt{6.5}$ kadar olduğu kanıtlanmıştır.

Anahtar kelimeler: *tekil başlangıç değer problemleri, polytropic gaz küresi, Lane-Emden denklemi, yaklaşık çözüm*

1. INTRODUCTION

Polytropes and isothermal spheres provide simple models for stars (Kippenhahn & Weigert 1990) and for spherical galaxies (Binney & Tremain 1987). The basic equation of this study is the

Lane-Emden equation (Chandrasekhar 1939; Kipenhahn & Weigert 1990)

$$y''+\frac{2}{x}y'+y^n=0 \tag{1}$$

Here y is the negative of the scaled gravitational potential and x is a scaled radius. The physically relevant solution is that for which $y = 1$ and $dy/dx = 0$ at $x = 0$. Exact solutions are known only for three values of the polytropic index n = 0, 1 and 5. For other values of n, the Lane-Emden equation should be integrated numerically.

There are different (approximate) methods to solve Lane-Emden equation (Shawagfeh, 1993, Seidov, 2000, Adomian et al., 1995 Wazwaz, 2001). Series solutions for the Lane-Emden equation have been used by many authors. Roxburgh and Stockman (1999) computed large numbers of series coefficients for many n values, but they found that the series cease to converge before the surface of the star is reached when $2 \leq n \leq 5$. Hunter (2001) used an Euler transformation to make the series convergent all the way to the outer radius. Nouh (2004) used accelerating methods to improve the radius of convergence of the series solution.

In this article, a convergence radius of the series solution has been found for the polytropic index $n = 3$. The previous result on the radius of convergence of the series solution for the polytropic index $n = 3$ was $\sqrt{6.45}$ (Nouh, 2004). Here we showed that it is a bit larger value, i.e. larger than $\sqrt{6.5}$.

Additionally, this result can be used to obtain the radius of convergence of the series solution for the generalized polytropic model of the sun. Especially, we obtained the radius of convergence for the following equation:

$$y''+\frac{p+1}{x}y'+p^2x^{2p-2}y^3=0,\ y(0)=0,\ y'(0)=0. \qquad (2)$$

where p = 1 corresponds to the standard polytropic model for the sum.

2. A RADIUS OF CONVERGENCE FOR THE POWER SERIES SOLUTION

The solution of the Lane-Emden equation (1) in a series form is given by

$$y=1-\frac{x^2}{2\cdot 3}+\frac{nx^4}{120}-\frac{n(8n-5)x^6}{15120}+... \qquad (3)$$

(see, for example, Wazwaz 2001). We use the next relationship between the components of Adomian decomposition solution:

$$y_0(x)=y(0);\ y_1(x)=-\frac{x^2}{6}a_0;\quad y_2(x)=-\frac{x^2}{2\cdot 2\cdot 5}a_1y_1;$$

$$y_3(x)=-\frac{x^2}{3\cdot 2\cdot 7}(a_1y_2+\frac{y_1^2}{2}a_2);$$

$$y_4(x)=-\frac{x^2}{4\cdot 2\cdot 9}(a_1y_3+a_2y_1y_2+\frac{y_1^3}{6}a_3);... \qquad (4)$$

where $a_0=1,\ a_1=n,\ a_2=n(n-1),...,\ a_k=f^{(k)}(1),$ with $f(x)=x^n$ (for the proof of (4) in more general case see, Aslanov, 2008). For n = 3 we have $y_0(x)=1,...$

$$y_r=\frac{x^2(3y_{r-1}+6y_1y_{r-2}+...+6\alpha y_{N_1}y_{N_2}+3y_1^2y_{r-3}+6y_1y_2y_{r-4}+...)}{2r(2r+1)} \qquad (5)$$

where $N_1 = (r-1)/2 = N_2 - 1$ and $\alpha = 1$ if r is an even integer and $N_1 = r/2 = N_2$ and $\alpha = 0.5$ if r is an odd integer. Using the relationships (5) we shall improve the results on the radius of convergence of the series (3). In other words we shall prove the next inequality for the components y_r in (5):

$$|y_r| \le \frac{x^{2r}}{6{,}5^r}, \text{ for } r = 5,6,... \tag{6}$$

Proof of the inequality (6).

We use mathematical induction method. We checked the inequality (6) for r = 5,6,…,52. Table 1 demonstrates the values of α_r in $|y_r| = a_r x^{2r}$. We shall prove that if (6) holds for some $k \ge 50$, then it holds for k+1. Let k be an odd integer. Using induction we have

$$\left|3y_k + 6y_1 y_{k-1} + ... + 6y_{k/2-1} y_{k/2} + \frac{6}{2} y_1^2 y_{k-2} + 6y_1 y_2 y_{k-3} + ...\right|$$

$$\le \left| \frac{3x^{2k}}{6{,}5^k} + 6\left(\frac{x^2}{6}\frac{x^{2k-2}}{6{,}5^{k-1}} + \frac{x^4}{40}\frac{x^{2k-4}}{6{,}5^{k-2}} + \frac{x^6}{265\cdot 2}\frac{x^{2k-6}}{6{,}5^{k-3}} + \frac{x^8}{1758}\frac{x^{2k-8}}{6{,}5^{k-4}} + \frac{x^{2k}}{6{,}5^k} + \frac{x^{2k}}{6{,}5^k} + ...\right)\right.$$

$$+ 6y_1\left(\frac{1}{2} y_1 y_{k-2} + y_2 y_{k-3} + ...\right) + 6y_2\left(\frac{1}{2} y_2 y_{k-4} + y_3 y_{k-5} + ...\right)$$

$$\left. + 6y_3\left(\frac{1}{2} y_3 y_{k-6} + y_4 y_{k-7} + ...\right) + ... \right|. \tag{7}$$

Since

$$|\frac{1}{2}y_1y_{k-2}+y_2y_{k-3}+y_3y_{k-4}+y_4y_{k-5}|$$

$$\leq |x^{k-1}|\left|\frac{1}{2}\frac{1}{6}\frac{1}{6,5^{k-2}}+\frac{1}{40}\frac{1}{6,5^{k-3}}+\frac{1}{265}\frac{1}{6,5^{k-4}}+\frac{1}{1758}\frac{1}{6,5^{k-5}}\right|$$

$$\leq |x^{k-1}|\frac{4}{6,5^{k-1}};$$

$$\left|\frac{1}{2}y_2y_{k-4}+y_3y_{k-5}+y_4y_{k-5}\right|\leq |x^{k-2}|\frac{3}{6,5^{k-2}} \quad \text{and}$$

$$\frac{1}{2}y_3y_{k-6}+y_4y_{k-7}\leq |x^{k-3}|\frac{2}{6,5^{k-2}}$$

for $k\geq 5$ we have

$$\left|3y_k+6y_1y_{k-1}+\ldots+6y_{k/2-1}y_{k/2}+\frac{6}{2}y_1^2y_{k-2}+6y_1y_2y_{k-3}+\ldots\right|$$

$$\leq |x^{2k}|\frac{3}{6,5^k}+\frac{1}{6,5^{k-1}}+\frac{6}{40}\frac{1}{6,5^{k-2}}+\frac{6}{265}\frac{1}{6,5^{k-3}}+\frac{6}{1758}\frac{1}{6,5^{k-4}}$$

$$+\frac{6}{2}\frac{k-5}{6,5^k}+\frac{6}{2}\frac{k-5}{6,5^k}+\frac{1}{2}\frac{k-2}{6,5^{k-1}}+\frac{6}{40}\frac{k-5}{2\cdot 6,5^{k-2}}+\frac{6}{265}\frac{k-8}{2\cdot 6,5^{k-3}}$$

$$+\frac{6}{1758}\frac{k-11}{2\cdot 6,5^{k-4}}+\frac{6}{6,5^k}\left(\frac{k-15}{2}+\frac{k-18}{2}+\frac{k-21}{2}+\ldots\right)|$$

$$\leq\frac{x^{2k}}{6,5^k}\{3+6,5+\frac{6}{40}6,5^2+\frac{6}{265}6,5^3+\frac{6,5^4}{293}+3k-15+\frac{k-2}{2}6,5$$

$$+\frac{3(k-5)}{40}6,5^2+\frac{3(k-8)}{265}6,5^3+\frac{(k-11)}{546}6,5^4+\frac{(k-15)(k-12)}{2}\}$$

$$=\frac{|x^{2k}|}{6,5^k}\{\frac{(k-15)(k-12)}{2}+\frac{3(k-9)}{1758}6,5^4+\frac{3(k-6)}{265}6,5^3$$

$$\frac{3(k-3)}{40}6,5^2+\frac{k}{2}6,5+3(k-4)\}$$

$$= \frac{|x^{2k}|}{6{,}5^k}(2{,}0739k + \frac{k^2}{2} + 22{,}424) \qquad (8)$$

Now, substituting $r=k+1$ in (5) we get

$$|y_{k+1}| \le \frac{|x^{2k+2}|(4{,}1478k + k^2 + 44{,}848)}{4(k+1)(2k+3)6{,}5^k}$$

We can easily show that

$$\frac{4{,}1478k + k^2 + 44{,}848}{4(k+1)(2k+3)} = \frac{1}{8}\frac{k^2 + 4{,}1478k + 44{,}848}{k^2 + 2{,}5k + 1{,}5} \le \frac{1}{6{,}5}$$

for k≥17. Thus we get $|y_{k+1}| \le |x^{2k+2}| 6{,}5^{k+1}$ and therefore, (5) holds for $k \ge 5$.

Note that the values for the solution y in equation (1) for x close to 6,5 can be easily calculated using the Table 1. Since the series solution is alternating series it will be more convenient to use the approximation in the form

$$y_{approx} = y_1 + y_2 + \ldots + y_{k-1} + \frac{1}{2}y_k. \qquad (9)$$

For example, using the first 52 terms of the series solution we find

$$y(6{,}45) \approx 1 + \sum_{k=1}^{51} y_k(6{,}45) + \frac{1}{2}y_{52}(6{,}45) = 0{,}452 \text{ or}$$

$$y(6{,}45) \approx 0{,}45015.$$

Now let us show that the radius of convergence for equation (2) can be found. Equation (2) can be rewritten as

$$y''+\frac{\varphi'}{\varphi^k}(\frac{\varphi^k}{\varphi'})'y'+(\varphi')^2\varphi^m y^3=0,\ \ y(0)=1,\ \ y'(0)=0. \qquad (10)$$

where $\varphi = x^p$, $m = 0$, $k = 2$. It is possible to show that (Aslanov 2008), the series solution of (10) can be received from the series solution of (1) by writing x^p instead of x. That is the series solution of the problem (2) has a radius of convergence $\geq 6{,}5^{p/2}$.

Table 1. Values of a_n in $|y_n| = a_n x^{2n}$

n	1	2	3	4	5	6
a_n	0,17	0,025	0,0038	0,00057	8,58E-05	1,3E-05
7	8	9	10	11	12	13
1,9E-06	2,9E-07	4,4E-08	6,7E-09	1,0E-09	1,5E-10	2,3E-11
14	15	16	17	18	19	20
3,5E-12	5,2E-13	7,9E-14	1,2E-14	1,8E-15	2,7E-16	4,1E-17
21	22	23	24	25	26	27
6,2E-18	9,3E-19	1,4E-19	2,1E-20	3,2E-21	4,8E-22	7,2E-23
28	29	30	31	32	33	34
1,1E-23	1,6E-24	2,5E-25	3,8E-26	5,7E-27	8,5E-28	1,3E-28
35	36	37	38	39	40	41
1,9E-29	2,9E-30	4,4E-31	6,7E-32	1,0E-32	1,5E-33	2,3E-34
42	43	44	45	46	47	48
3,4E-35	5,2E-36	7,8E-37	1,2E-37	1,8E-38	2,7E-39	4,1E-40
49	50	51	52			
6,1E-41	9,2E-42	1,4E-42	2,1E-43			

3. CONCLUSION

The most recent (numerical) result for the radius of convergence of the power series solution of the polytropic gas spheres equation with polytropic index $n = 3$ is $\sqrt{6,45}$ (see, for example, Nouh, 2004). Here, we obtained a bit larger value, $\sqrt{6,5}$ by using another method. This result can be extended to investigate the convergence of the series solutions for more general polytropic models. At the same time the result would be interesting to estimating approximate values of the solution around the movable singular point of the series solution.

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