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ON BEHAVIOR OF SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS AT UNBOUNDED INCREASING OF BOUNDARY FUNCTION

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ABSTRACT

The aim of this paper is to study the behavior of solutions of initial boundary problem for nonlinear parabolic equation with boundary regime. From physical point of view the rule of the above intensity of a boundary regime, the speed is more of movement in the arising temperature of a wave.

Keywords: *Nonlinear parabolic equations, unbounded increasing solution, boundary regime of peaking, blowing up of solutions*

ÖZET

Makalenin amacı nonlineer parabolic denklem için sınır rejimli başlangıç sınır probleminin çözümünü incelemektir. Fiziksel açıdan sınır rejiminin yoğunluk kuralı ısı dalgasının hareketinin yükselmesinde önemlidir.

Anahtar Kelimeler: *Nonlineer parabolik denklemler, sınırsız büyüyen çözüm, yükselen sınır rejimi, çözümün patlaması*

1. INTRODUCTION

In this paper the unbounded increasing solution of the nonlinear parabolic type equation for the finite times is considered. These type equations describe the processes of electrical and ionic heat conductivity in plasma, diffusion of neutrons and α - particles and etc. Investigation of unbounded solution or regime of peaking solutions occurs in the theory of nonlinear equations where one of the essential ideas is the representation called eigen-function of nonlinear dissipative surroundings. It is well known that even a simple nonlinearity, subject to critical of exponent, the solution of nonlinear parabolic type equation for the finite time may increase unboundedly, i.e., there is a number $T > 0$ such that

$$\|u(x,t)\|_{L_\infty(R^n)} \to \infty, \quad t \to T < \infty .$$

In [1] the existence of unbounded solution for finite time with a simple nonlinearity has been proved. In [2] has been shown that any nonnegative solution subject to critical exponent is unbounded increasing for the finite time. Similar results were obtained in [3] and corresponding theorems are called Fujita-Hayakawa`s theorems. More detailed reviews can be found in [4], [5] and [6].

2. MAIN RESULTS

Let us consider the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} \frac{d}{dx_j}\left(\left| \frac{\partial u}{\partial x_i} \right|^{p-2} \frac{\partial u}{\partial x_i} \right) - F(x,t,u) \qquad (1)$$

in bounded domain $\Omega \subset R^n$, $n \geq 2$, with no smooth boundary, and mainly the boundary $\partial\Omega$ contains the conical points with span of the corner $\omega \in (0,\pi)$. Denote

$$\Pi_{a,b} = \{(x,t) | x \in \Omega,\ a < t < b\},\ \Pi_a = \Pi_{a,\infty},\ \Gamma_a = \Gamma_{a,\infty} .$$

$$\Gamma_{a,b} = \{(x,t) | x \in \partial\Omega, a < t < b\},$$

The functions $F(x,t,u)$, $\frac{\partial F(x,t,u)}{\partial u}$ are continuous with respect to u uniformly in $\overline{\Pi_0} \times \{u : |u| \le M\}$ for any $M < \infty$, $F(x,t,0) \equiv 0$, $\frac{\partial F}{\partial u}\Big|_{u=0} \equiv 0$. Besides, the function F is measurable on whole arguments and doesn't decrease with respect to u. Let the Dirichlet boundary condition

$$u = f(x,t) \quad x \in \partial\, \Omega\,, \tag{2}$$

the initial condition

$$u\big|_{t=0} = \varphi(x) \tag{3}$$

fulfilled on some domain $\Pi_{0,a}$, where $\varphi(x)$ is a smooth function. The condition on $f(x,t)$ is following

$$f_t' \in L_1(0,T_0,L_{\sigma+1}(\Omega)), \ ,$$

$$f \in L_\infty((0,T_0)\times\Omega) \cap L_{\sigma+1}\ (0,T_0,\ L_{\sigma+1}(\Omega))\,, \quad T_0 < T\,. \tag{4}$$

Furthermore,

$$f(x,t) \to \infty, \ \ t \to T\ . \tag{5}$$

Condition (5) is called boundary regime with sharpening at the blow-up time T. As a solution we understand the generalized solution from the Sobolev space $W_m^1(\Pi_{0,a'})$ for all $a < a'$, and the solution of problem (1)-(3) either exists in Π_0 or

$$\lim_{t \to T-0} \ \max_{\Omega} |u(x,t)| = +\infty \tag{6}$$

at some $T = const$.

Assume that

$$\sup f \in [0,T] \times B_r, \quad r < R \tag{8}$$

$$f(0,x) - \varphi(x) \in W^1_{p+1}(\Omega, \partial\Omega). \tag{9}$$

Here $W^1_{p+1}(\Omega,\partial\Omega)$ is a closing in the norm $W^1_{p+1}(\Omega)$ and the functions from $C^\infty(\Omega)$ vanishing near $\partial\,\Omega$. Let's formulate the auxiliary results from [10], [11]. Let

$$L_p u = div(|\nabla u|^{p-2}\nabla u), \quad p > 1$$

where p is a harmonic operator.

Lemma1. [10] There exists the positive eigen-value of spectral problem for the operator L_p, to which corresponds the positive in Ω eigen-function.

Lemma2. [11] Let $u, v \in W^1_p(\Omega)$ and $u \le v$ on $\partial\Omega$,

$$\int_\Omega L_p(u)\eta_{x_i} dx \le \int_\Omega L_p(\vartheta)\eta_{x_i} dx$$

for any $\eta \in \overset{0}{W}{}^1_p(\Omega)$ with $\eta \ge 0$. Then $u \le v$ on whole domain Ω.

We will call the generalized solution of problem (1)-(3) in $\Pi_{a,b}$ $u(x,t) \in W^1_p(\Pi_{a,b})$, $u(x,t) \in L_\infty(\Pi_{a,b})$ such that

$$-\int_{\Pi_{a,b}} \frac{\partial u}{\partial t}\psi dxdt + \sum_{i,j=1}^{u} \int_{\Pi_{a,b}} \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_j} dxdt = \int_{\Pi_{a,b}} F(x,t,u).\ \psi(x,t)dxdt \tag{10}$$

where $\psi(x,t)$ is an arbitrary function from $W^1_p(\Pi_{a,b})$, $\psi\big|_{\Gamma_{a,b}} = 0$, and $0 < a < b$ are any numbers. Let $u_0(x) > 0$ be an eigen-function of spectral problem for the operator L_p corresponding to $\lambda = \lambda_1 > 0, \ \int_\Omega u_0(x)dx = 1, \ \int_\Omega u_0^q \ dx = 1, \quad q \geq 1$. Let's assume that the condition

$$(L_p u_0, u) \geq (L_p u, u_0), \tag{A}$$

is fulfilled, i.e.,

$$\int_\Omega \left(\left| \frac{\partial u_0}{\partial x_i} \right|^{p-2} - \left| \frac{\partial u}{\partial x_i} \right|^{p-2} \right) \frac{\partial u}{\partial x_i} \frac{\partial u_0}{\partial x_i} dx \geq 0 .$$

Let's introduce the function which will be characterized by blow-up boundary regime at $t \to T$

$$G(t) = \lambda_1 \int_\Omega u_0(x) f(x,t) dx + \int_\Omega |f_t|^{\sigma+1} dx + \int_\Omega f^{\sigma+1} dx,$$

for $t < T$.

Theorem 1. Let $F(x,t,u) \geq \alpha_0 |u|^{\sigma-1} \cdot u$ at $(x,t) \in \Pi_0$ and $\sigma > 1$, $\alpha_0 = const > 0$. There exists $k = const$ such that, if $u(x,0) \geq 0, \ \int_\Omega u(x,0) u_0(x) dx \geq k$, conditions (4), (5) and $G(t) \to \infty$, at $t \to T$ are fulfilled, then $\lim_{t \to T-0} \max_\Omega u(x,t) = \infty$, $T = const > 0$.

Proof Assume the vice versa, then $u(x,t)$ is a solution of equation (1) in Π_0 and condition (2) is fulfilled on Γ_0. By virtue of Lemma 2, $u_0(x) > 0$ in Π_0. Substitute in (10) $\psi(x,t) = \varepsilon^{-1}(u_0(x) - f(x,t)), \ b = a + \varepsilon, a > 0, \ \varepsilon > 0,$ where $u_0(x) > 0$ in Ω is eigen- function of spectral problem for the operator L_p, corresponding to eigen-value $\lambda_1 > 0$. Such eigen-value and eigen-function exist by virtue of Lemma 1. Then we have

$$-\frac{\varepsilon^{-1}}{2}\left[\int_{\Omega} u_0(x)u(x,a+\varepsilon)dx-\int_{\Omega} u_0(x)u(x,a)dx\right]+\varepsilon^{-1}\int_{\Pi_{a,a+\varepsilon}}\frac{\partial u}{\partial t}f(x,t)dxdt+$$

$$+\varepsilon^{-1}\int_{\Pi a,a+\varepsilon}\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial \psi}{\partial x_i}dxdt-$$

$$\varepsilon^{-1}\int_{\Pi_a,a+\varepsilon} F(x,t,u)\psi(x,t)dxdt=0. \tag{11}$$

Using condition (A), as ε approaches to zero in equality (11), we have

$$-g^{/}(t)\le-\lambda_1\int_{\Omega} u_0(x)u(x,t)dx-\int_{\Omega} u\,\frac{\partial f(x,t)}{\partial t}dx+\lambda_1\int_{\Omega} u_0(x)f(x,t)dx-$$

$$-\int_{\Omega} u^{\sigma}f(x,t)dx+\int_{\Omega} u_0u^{\sigma}dx \tag{12}$$

where $g(t)=\int_{\Omega} u_0(x)u(x,t)dx$.

At first, let's estimate the second and the fourth integral on the right in (12). Apply the Cauchy inequality with $\varepsilon>0$ to get

$$\int_{\Omega} u\,\frac{\partial f(x,t)dx}{\partial t}=\frac{\varepsilon_1}{2}\int_{\Omega} u^2dx+c_1(\varepsilon_1)\int_{\Omega} f_t'^2dx\le\frac{\varepsilon_1}{2}\int_{\Omega} u^{\sigma+1}(x,t)dx+$$

$$c_1(\varepsilon_1)\int_{\Omega} f_t^{\prime\sigma+1}dx+c_3mes\Omega,$$

$$\int_{\Omega} u^{\sigma}f(x,t)dx\le\varepsilon_1\int_{\Omega} u^{\sigma+1}dx+c_3(\varepsilon_1)\int_{\Omega} f^{\sigma+1}(x,t)dx\ .$$

Now, let's estimate $\int_\Omega u^{\sigma+1}dx$ and $\int_\Omega u_0 u^\sigma dx$. By using the Holder inequality for these integrals under the normed conditions $u_0(x)$ we have

$$\int_\Omega u^{\sigma+1}dx \ge g^{\sigma+1}(t), \quad \int_\Omega u_0 u^\sigma dx \ge g^\sigma(t)$$

Substituting these estimates in (12) to get

$$g'(t) \ge -\lambda_1 g(t) + g^{\sigma+1}(t) + \lambda_1 \int_\Omega u_0(x) f(x,t)dx +.$$

$$\int_\Omega (f_t')^{\sigma+1}dx + \int_\Omega f^{\sigma+1}(x,t)dx \tag{13}$$

Then from (13) we obtain

$$g'(t) \ge -\lambda_1 g(t) + Cg^{\sigma+1}(t) + G(t). \tag{14}$$

Note that depending on a boundary regime there are various physical processes. At the power boundary regime with aggravation there is a heat localization. Such boundary regimes describe the solutions of equation stopped and thermal wave. Position of a point in front of a wave doesn't change in a current of all time of an aggravation $t \in (0,T)$ and thermal indications from the localization due to the fact that the temperature infinitely increase everywhere at $t \to T$. At this any boundary regime with an aggravation provides localization of thermal influence. For example, if $G(t) \le (T-t)^{-1/\sigma}, \sigma > 1, 0 < t < T$, there is a localization of heat, and at $G(t) \le (T-t)^n$, where $n < -1/\sigma$

there is no localization. If $g(0) > C_2 = \left(\frac{\lambda_1}{c}\right)^{1/\sigma+1}$, then from (14), under the condition on $G(t)$ we'll obtain $\lim_{t\to T-0} g(t) = +\infty$. Hence we have $\lim_{t\to T-0} \max_{\Omega} u(x,t) = \infty$. Theorem is proved.

Thus, if $u(x,0) \geq 0$ is not enough small, then the solution of equation (1) has not been in Π_0. Let's show that if $|u(x,0)|$ is sufficiently small then solution of problem (1), (2), (3) exists on whole domain Π_0. Here an important role is played by (9) and the condition which will coordinate the speeds of "attinity" of boundary and initial functions, i.e., for existence of solution at unboundedly tendency of boundary function to ∞, it is necessary sufficiently small initial function.

Theorem 2. Let $|F(x,t,u)| \leq (C_1 + C_2 t^m)|u|^\sigma, \sigma > 1$. Relative to the boundary function, the conditions of Theorem 1 is fulfilled and there exists $\delta > 0$ such that if $|\varphi(x)| \leq \delta$, then solution of problem (1)-(3) exists in Π_0, and $|u(x,t)| \leq ce^{-\alpha t}, \alpha = const > 0$.

Proof Let $\overline{\Omega} \subset B_R$ where $B_R = \{x \, |x| \leq R \}$, and $\vartheta > 0 \in B_R$ be an eigen-function corresponding to the eigen-value λ_1 of boundary-value problem

$$L_m u + \lambda u = 0, \quad x \in \Omega, u = 0, x \in \partial\Omega. \tag{15}$$

Consider the function $V(x,t) = \varepsilon e^{-\lambda_1 t/2}\vartheta(x)$. We have

$$V_t - L_m V - F(x,t,\vartheta) = \frac{1}{2}\varepsilon\lambda_1 e^{-\frac{\lambda_1 t}{2}}\vartheta(x) - (C_1 + C_2)\varepsilon^\sigma e^{-\frac{\lambda_1 t}{2}}\vartheta^\sigma, \tag{16}$$

$$(x,t) \in \Pi_0, \quad V > 0, \quad (x,t) \in \Gamma_0.$$

If $\varepsilon > 0$ is sufficiently small, inequality (16) is understood in weak sense. From (16) and Lemma 2 follows that $|u| \le V \le Ce^{-\lambda_1 t/2}$ if $|\varphi(x)| \le \delta = \varepsilon \min_{\Omega} \vartheta(x)$. Now let's determine the class of functions $\omega(x,t)$ consisting at the functions continuous in $\overline{\Pi}_{-\infty,+\infty}$, being equal to zero at $t \le T$, $|\omega| \le Ke^{-ht}, K$ and h such that $\overline{\Pi}_{-\infty,+\infty}$, $\|\omega\| = \sup_{\Pi_{-\infty,+\infty}} |\omega e^{ht}|$ is subset at Banach space continuous in $\theta(t) \subset C^{\infty}(R^1)$, $\theta(t) = 0$ functions with the norm $t \le \tau$. Let $\theta(t) = 1$ and $t > T + 1$, and let's determine the operator H on K substituting $H\omega = \theta(t)z,\ \omega \in K$. Thus, we solve the linear problem

$$\sum_{i,j=1}^{n} \frac{d}{dx_j}\left(\left| \frac{\partial z(x,t)}{\partial x_i} \right|^{m-2} \frac{\partial z}{\partial x_i} \right) \theta(t) \cdot \omega_{x_i x_j} - \frac{\partial z}{\partial t} + \theta(t) F(x,t,\vartheta) = 0$$

instead of the nonlinear problem. $z = 0$ at $(x,t) \in \Gamma_{-\infty,+\infty}$, consisting in the determination $\omega(x,t)$ in Ω by the known function $z(x,t)$. By means of semi-linear estimates above H terms K in K if T is sufficiently big. The operator H is quite continuous. This appears from the obtained estimation and theorem on Holderness of $\Pi_{-a,a}$ at solutions of parabolic equations for any a [12]. From Lere-Shouder theorem, the operator H has fixed point z. Theorem is proved.

REFERENCES

[1] Kaplan S. On the growth of solution of quasi-linear parabolic equations. Comm. Pure and Appl. Math., 1963, v. 16, pp.305-330

[2] Fujita H. On the blowing up solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J.Fac.Sci. Univ. Tokyo I, 1996, 13, pp.109-124

[3] Hayakawa K. On non existence of global solutions of some semi linear parabolic equations. Proc Japan Acad, Ser.A, Math. Sci., 1973, 49 503-505.

[4] Galaktionov V.A., Levine N. A general approach to critical Fujita exponents and systems. Nonlinear Anal., 1998, v.34, pp.1005-1027.

[5] Deng K., Levine N. The role of critical Fujita exponents in blow-up theorems. J.Math.Anal. Appl., 2000,243,pp.85-126.

[6] Samarskii A.A., Galaktinonov V.A., Kurdyumov, Mikahailov. Blow-up in quasi linear parabolic equations. Mockow, Nauka, 1987.

[7] Ebihara, Sh. Kawashima, N. Levine. On solution $u_{tt} - |x|^{\alpha} \Delta u = f(u)$, Funccial Ekvac , 38, (1995), No 3, p.539-544.

[8] H.Levine Instability and nonexistence of global solutions to nonlinear wave equations of the from $Pu_{tt} = -Au + F(u)$. Trans. Amer. Math. Soc.,1974, 192, p.1-21.

[9] Fujita H. On the blowing up of solutions to the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$ J. Fac. Sci. Univ. Tokyo. I, 1996, v.13, pp.109-124.

[10] Chanillo S. Whetden R. Weighted Poincare and Sobolev inequalities and estimates for weighted Peano maximal functions. Amer. J. Math. 1985. v.107, N 5, p.1191-1226.

[11] L.Tolksdorf. “On quasi linear boundary value problems in domains with corners. Nonlinear Anal. 1981,Vol.5, No 7. p.721-735.

[12] D. Gilbarg, N. Trudinger. Elliptic differential equation with partial derivatives of the second order. Moscow, Nauka, 1989.

[13] Ladyzhenskaya O.A., Solonnikov V.A., Uralceva N.N. Lineynie i kvazilineynie uravneniya parabolicheskogo tipa. Nauka, 1967 (ru)