

Positive solutions for nonlinear fractional differential equation with nonlocal boundary conditions

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Abstract

In this paper, we study the boundary value problem of a class of fractional differential equations involving the Riemann-Liouville fractional derivative with nonlocal integral boundary conditions. To establish the existence results for the given problems, we use the properties of the Green's function and the monotone iteration technique, one shows the existence of positive solutions and constructs two successively iterative sequences to approximate the solutions. The results are illustrated with an example.

1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equation

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

subject to the boundary conditions

$$u^{(i)}(0) = 0, \quad 0 \leq i \leq 2, \quad D_{0+}^{\beta} u(1) = \lambda I_{0+}^{\beta} u(\eta), \tag{1.2}$$

where D_{0+}^{α} , D_{0+}^{β} are the standard Riemann-Liouville fractional derivative of order $\alpha \in (3, 4]$, $\beta \in (0, 2]$, I_{0+}^{β} is the standard Riemann-Liouville fractional integral of order $\beta \in (0, 2]$ and $0 \leq \frac{\lambda \Gamma(\alpha - \beta) \eta^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)} < 1$.

The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, involves derivatives (or q-derivatives) of fractional order see for example [5, 6]. For more details we refer the reader to [2, 11] and the references cited therein.

Many mathematicians show strong interest in fractional differential equations and many wonderful results have been obtained. The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear fractional differential equations, plays an essential role in the research of this field, such as establishing the existence and the uniqueness or the multiplicity of solutions to nonlinear fractional differential equations boundary value problems, see [4, 7, 9, 10, 11, 12, 15, 16, 17, 18, 19] and the references therein.

In [3], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & t \in (0, 1), \\ u(0) = 0, D_{0+}^{\beta} u(1) = a D_{0+}^{\beta} u(\eta), \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \eta < 1$, $0 < \alpha, \beta < 1$, $f \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$ and D_{0+}^{α} , D_{0+}^{β} are the standard Riemann-Liouville fractional derivative of order α , β respectively. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

The monotone iteration scheme is an interesting and effective way to investigate the existence of solutions to nonlinear fractional problem (see for example [8, 13, 14]). Inspired and motivated by the works mentioned above, we focus on the existence of positive solutions for the nonlocal boundary value problem (1.1) – (1.2) by using the fixed point theorem for increasing operators on the order intervals, we also establish two iterative sequences to approximate the solutions. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, for more details; see [1]. The existence of the positive solutions to the problem (1.1) – (1.2), is proved and two successively iterative sequences to approximate the solutions are constructed and we give an example to illustrate our results in Section 3.

2. Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $K \subset E$ is said to be a cone provided that

- (i) $c_1 u + c_2 v \in K$ for all $c_1 \geq 0$, $c_2 \geq 0$, and
- (ii) $u \in K$, $-u \in K$ implies $u = 0$.

Every cone K induces an ordering in E given by $u \leq v$ if and only if $v - u \in K$.

Definition 2.2. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.3. [1]. The Riemann-Liouville fractional derivative order $\alpha > 0$ of a continuous function $u : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s) ds, \quad t > 0,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.4. [1] (i) If $u \in L^p(0, 1)$, $1 \leq p \leq +\infty$, $\beta > \alpha > 0$, then $D_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\beta-\alpha} u(t)$, $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$, $I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t)$.

(ii) If $\beta > \alpha > 0$, then $D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta)t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.

(iii) If $\alpha > 0$ and $\gamma \in (-1, +\infty)$, then $I_{0+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Lemma 2.5. [1] Let $\alpha > 0$ and for any $y \in L^1(0, 1)$. Then, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0 < t < 1$ is given by

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

where c_0, c_1, \dots, c_{n-1} are real constants and $n = [\alpha] + 1$.

Lemma 2.6. Let $y \in C[0, 1]$. Then the solution of the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \\ u^{(i)}(0) = 0, \quad 0 \leq i \leq 2, \\ D_{0+}^{\beta} u(1) = \lambda I_{0+}^{\beta} u(\eta), \end{cases} \quad (2.1)$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds, \quad (2.2)$$

where

$$G(t,s) = \begin{cases} \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}{\Delta}, & 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Lambda}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq t \leq s \leq 1, s \geq \eta, \end{cases} \quad (2.3)$$

where $\Delta = t^{\alpha-1} [\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} - \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}]$,

$\Lambda = \Gamma(\alpha+\beta)\Gamma(\alpha)(1-s)^{\alpha-\beta-1} t^{\alpha-1}$.

and $P = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)} \eta^{\alpha+\beta-1}$.

Proof. In view of Lemma 2.5, the general solution for the above equation is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + C_4 t^{\alpha-4},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

The boundary conditions $u(0) = u'(0) = u''(0) = 0$, implies that $c_2 = c_3 = c_4 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1}. \tag{2.4}$$

By (2.4) and Lemma 2.4, we get

$$D_{0+}^\beta u(t) = \frac{1}{\Gamma(\alpha-\beta)} \left[c_1 \Gamma(\alpha) t^{\alpha-\beta-1} - \int_0^t (t-s)^{\alpha-\beta-1} y(s) ds \right].$$

In view of boundary condition $D_{0+}^\beta u(1) = \lambda I_{0+}^\beta u(\eta)$, we conclude that

$$c_1 = \frac{1}{P} \left[\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds - \frac{\lambda}{\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} y(s) ds \right].$$

Therefore, the unique solution of the problem (2.1) is given by

$$u(t) = \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} y(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

For $t \leq \eta$, one has

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \left[\int_0^t (1-s)^{\alpha-\beta-1} y(s) ds + \int_t^\eta (1-s)^{\alpha-\beta-1} y(s) ds + \int_\eta^1 (1-s)^{\alpha-\beta-1} y(s) ds \right] \\ &- \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \left[\int_0^t (\eta-s)^{\alpha+\beta-1} y(s) ds + \int_t^\eta (\eta-s)^{\alpha+\beta-1} y(s) ds \right] - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \\ &= \int_0^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s) ds + \int_t^\eta \frac{\Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s) ds \\ &+ \int_\eta^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s) ds, \\ &= \int_0^1 G(t,s) y(s) ds. \end{aligned}$$

For $t \geq \eta$, one has

$$\begin{aligned} u(t) &= \int_0^\eta \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(\alpha-\beta)} y(s) ds \\ &+ \int_\eta^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s) ds \\ &+ \int_t^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s) ds, \dots \\ &= \int_0^1 G(t,s) y(s) ds. \end{aligned}$$

The proof is complete. □

We need some properties of function $G(t, s)$ to establish the existence of positive solutions.

Lemma 2.7. *The Green's function $G(t, s)$ has the following properties:*

- (i) *The function $G(t, s)$ is continuous and $G(t, s) > 0$ for all $t, s \in (0, 1)$.*
(ii) *For all $t, s \in (0, 1)$, we have*

$$t^{\alpha-1}w_2(s) \leq G(t, s) \leq t^{\alpha-1}w_1(s), \quad (2.5)$$

where

$$w_1(s) = \frac{\lambda \eta^{\alpha+\beta-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha+\beta-1} \right]}{P\Gamma(\alpha+\beta)},$$

and

$$w_2(s) = \frac{(1-s)^{\alpha-\beta-1}}{P\Gamma(\alpha-\beta)}.$$

Proof. It is easy to prove (i). Now, we prove (ii), assume that $0 \leq \frac{\lambda\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} < 1$, then for $0 \leq s \leq t \leq 1$, $s \leq \eta$, we get

$$\begin{aligned} P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t, s) &= -P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-\beta-1} \\ &\quad + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1} - \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}, \\ &= \lambda\Gamma(\alpha)\Gamma(\alpha-\beta) \left\{ \eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - (\eta-s)^{\alpha+\beta-1}t^{\alpha-1} \right\} \\ &\quad + \Gamma(\alpha)\Gamma(\alpha+\beta) \left\{ -(t-s)^{\alpha-1} + (1-s)^{\alpha-\beta-1}t^{\alpha-1} \right\}, \\ &\geq \lambda\eta^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\alpha-\beta)t^{\alpha-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha+\beta-1} \right], \end{aligned}$$

and

$$\begin{aligned} P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t, s) &= \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1} - \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}t^{\alpha-1}, \\ &\leq \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}. \end{aligned}$$

For $0 \leq \eta \leq s \leq t \leq 1$, we have

$$\begin{aligned} P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t, s) &= -P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &= -\Gamma(\alpha)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \\ &\quad + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &\geq \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} - \Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \\ &\quad + \Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &\geq \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}t^{\alpha-1} \\ &\geq \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}t^{\alpha-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha+\beta-1} \right], \end{aligned}$$

and

$$\begin{aligned} P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t, s) &= -P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &= -\Gamma(\alpha)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}t^{\alpha-1} + \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}(t-s)^{\alpha-1} \\ &\quad + \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &\leq \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}. \end{aligned}$$

For $0 \leq t \leq s \leq 1$, $s \geq \eta$, we get

$$\begin{aligned} P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)G(t, s) &= \Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, \\ &\geq \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}t^{\alpha-1} \left[(1-s)^{\alpha-\beta-1} - (1-s)^{\alpha+\beta-1} \right]. \end{aligned}$$

□

3. Existence results

We shall consider the Banach space $E = C[0, 1]$ equipped with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and let a closed cone $K \subset E$ by $K = \{u \in E : u \geq 0\}$ where 0 is the zero function. Then K is normal.

Set $K_a = \{u \in E : \|u\| \leq a\}$. Define the operator $T : K_a \rightarrow E$ as

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad t \in [0, 1], \tag{3.1}$$

where $G(t, s)$ is given by (2.3). It is not hard to see that fixed points of operator T coincide with the solutions to the problem (1.1) – (1.2).

Lemma 3.1. [9] *Let E be a Banach space ordered by a normal cone $K \subset E$. Assume that $T : [x_1, x_2] \rightarrow E$ is completely continuous operator such that $x_1 \leq Tx_2, x_2 \geq Tx_1$. Then T has a minimal fixed point x_* and a maximal fixed point x^* such that $x_1 \leq x_* \leq x^* \leq x_2$. Moreover, $x_* = \lim_{n \rightarrow \infty} T^n x_1, x^* = T^n x_2$, where $\{T^n x_1\}_{n=1}^\infty$ is an increasing sequence and $\{T^n x_2\}_{n=1}^\infty$ is a decreasing sequence.*

First, for the existence results of problem (1.1) – (1.2), we need the following assumptions.

(A₁) $f : [0, 1] \times [0, a] \rightarrow [0, \infty)$ is continuous and $f(t, 0) \neq 0$,

(A₂) There exists a nonnegative function $q \in C[0, 1] \subseteq L^1[0, 1]$ such that $|f(t, u)| \leq q(t), (t, u) \in [0, 1] \times [0, a]$,

(A₃) $f(t, \underline{u}) \leq f(t, \bar{u}), t \in [0, 1], 0 \leq \underline{u} \leq \bar{u} \leq a$.

Lemma 3.2. *Assume that (A₁) – (A₃) hold. Then the operator T defined in (3.1) is a completely continuous increasing operator.*

Proof. Firstly, the operator T is continuous in view of the continuity of functions $f(t, u(t))$ and $G(t, s)$. Secondly, we will show that $T(K_a)$ is bounded. Let

$$L = \int_0^1 q(t) dt < \infty.$$

Then, for any $u \in K_a$, we have

$$\|(Tu)(t)\| = \max_{t \in [0, 1]} \int_0^1 G(t, s) |f(s, u(s))| ds \leq \frac{L}{P\Gamma(\alpha - \beta)}, \quad t \in [0, 1].$$

For each $u \in K_a$, one can show that

$$\begin{aligned} |(Tu)'(t)| &= \left| \frac{(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha + \beta - 1} f(s, u(s)) ds \right. \\ &\quad \left. - \frac{\lambda(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} f(s, u(s)) ds - \frac{\alpha - 1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 2} f(s, u(s)) ds \right|, \\ &= \left| \frac{(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha + \beta - 1} f(s, u(s)) ds \right. \\ &\quad \left. - \frac{\lambda(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} f(s, u(s)) ds - \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha - 2} f(s, u(s)) ds \right|, \\ &\leq \frac{(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha - \beta)} \int_0^1 (1 - s)^{\alpha + \beta - 1} |f(s, u(s))| ds \\ &\quad + \frac{\lambda(\alpha - 1)t^{\alpha - 2}}{P\Gamma(\alpha + \beta)} \int_0^\eta (\eta - s)^{\alpha + \beta - 1} |f(s, u(s))| ds + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha - 2} |f(s, u(s))| ds, \\ &\leq \frac{(\alpha - 1)L}{P\Gamma(\alpha - \beta)} + \frac{\lambda(\alpha - 1)L}{P\Gamma(\alpha + \beta)} + \frac{L}{\Gamma(\alpha - 1)} = \bar{L}. \end{aligned}$$

Therefore, for any $t_1, t_2 \in [t_1, t_2] \subseteq \bar{L}(t_2 - t_1)$. with $t_1 < t_2$, we have

$$|(Tu)(t_1) - (Tu)(t_2)| \leq \int_{t_1}^{t_2} |(Tu)'(s)| ds \leq \bar{L}(t_2 - t_1) \rightarrow 0 \text{ as } t_2 \rightarrow t_1,$$

The Arzela-Ascoli theorem implies that the operator $T : K_a \rightarrow E$ is completely continuous. The assumption (A₃) provides that the operator $T : K_a \rightarrow E$ is an increasing operator. The proof is completed. □

Theorem 3.3. Assume that $(A_1) - (A_3)$ hold, and

$$\int_0^1 w_1(s) f(s, 0) ds \geq 0, \quad \int_0^1 w_2(s) f(a, as^{\alpha-1}) \leq a, \quad s \in [0, 1]. \quad (3.2)$$

Then the problem (1.1) – (1.2) has two positive solutions u^* , v^* satisfying $0 < u^* \leq v^* \leq a$. Moreover, there exist a non-decreasing iterative sequence $\{u_n\}_{n=0}^\infty$ with

$$\lim_{n \rightarrow \infty} u_n = u^*, \quad u_0 = 0, \quad u_{n+1} = Tu_n, \quad n = 0, 1, 2, \dots, \quad (3.3)$$

and a non-decreasing iterative sequence $\{v_n\}_{n=0}^\infty$ with

$$\lim_{n \rightarrow \infty} v_n = v^*, \quad v_0 = at^{\alpha-1}, \quad v_{n+1} = Tv_n, \quad n = 0, 1, 2, \dots \quad (3.4)$$

Proof. We only need to prove that $Tu_0 \geq u_0$ and $Tv_0 \leq u_0$.

$$\begin{aligned} (Tu_0)(t) &= \int_0^1 G(t, s) f(s, u_0) ds = \int_0^1 G(t, s) f(s, 0) ds, \\ &\geq t^{\alpha-1} \int_0^1 w_1(s) f(s, 0) ds \geq 0 = u_0, \quad t \in [0, 1], \end{aligned} \quad (3.5)$$

this implies $u_1 \geq u_0$, which combined with (A_3) gives

$$u_2 = (Tu_1)(t) = \int_0^1 G(t, s) f(s, u_1(t)) ds \geq u_1, \quad t \in [0, 1]$$

Similarly, we have

$$\begin{aligned} v_1 = Tv_0 &= \int_0^1 G(t, s) f(s, v_0) ds, \\ &\leq t^{\alpha-1} \int_0^1 w_2(s) f(s, at^{\alpha-1}) ds, \\ &\leq at^{\alpha-1} = v_0, \quad t \in [0, 1] \end{aligned} \quad (3.6)$$

Then, by (3.5) – (3.6) and induction, the iterative sequences $\{u_n\}$, $\{v_n\}$ satisfy

$$u_0(t) \leq u_1(t) \leq \dots \leq u_n(t) \leq \dots \leq v_n(t) \leq \dots \leq v_1(t) \leq v_0(t), \quad \forall t \in [0, 1].$$

By induction, one can prove that $u_{n+1} \geq u_n$ and $v_{n+1} \leq v_n$.

Lemma 3.1 shows that the operator T has a minimal fixed point u^* and a maximal fixed point v^* satisfying $0 \leq u^* \leq v^* \leq a$. From (A_1) we find that the zero function is not the solution to the problem (1.1) – (1.2). Thus $0 < u^* \leq v^* \leq a$. The proof is complete. \square

We construct an example to illustrate the applicability of the results presented.

Example 3.4. Consider the following boundary value problem

$$D_{0^+}^{\frac{7}{2}} u(t) + \frac{1}{\sqrt{t}} \left(t + u^{\frac{1}{3}}(t) \tanh(u(t)) + u^{\frac{1}{3}}(t) \right) = 0, \quad t \in (0, 1),$$

$$u(0) = u'(0) = u''(0) = 0, \quad D_{\frac{5}{2}} u(1) = \frac{1}{2} I_{0^+}^{\frac{5}{2}} u \left(\frac{1}{2} \right),$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$, $\lambda = \frac{1}{2}$, $\eta = \frac{1}{2}$ and $f(t, u(t)) = \frac{1}{\sqrt{t}} \left(t + u^{\frac{1}{3}}(t) \tanh(u(t)) + u^{\frac{1}{3}}(t) \right)$.

We take $a = 10$. By simple calculation we have

$$P = 3,3229182, \quad f(t, 0) = \frac{t}{\sqrt{t}} \text{ and } f\left(t, 10t^{\frac{5}{2}}\right) = \frac{1}{\sqrt{t}} \left(t + (10)^{\frac{1}{3}} t^{\frac{5}{6}} + t^{\frac{5}{6}} \right).$$

A simple calculation leads to

$$\int_0^1 w_1(s) f(s, 0) ds \simeq 0,0000239 \geq 0,$$

and

$$\int_0^1 w_2(s) f\left(s, 10s^{\frac{5}{2}}\right) ds \simeq 0,9124781 \leq 10.$$

Hence, all the assumptions of Theorem 3.3 are satisfied. Which implies that the boundary value (1.1) – (2.1) has two nontrivial solutions u^* , v^* with $0 \leq u^* \leq v^* \leq 10$, and the two monotone iterative sequences $\{u_n\}_{n=1}^\infty$ can be taken as

$$u_0 = 0, \quad u_{n+1} = Tu_n, \quad v_0 = 10t^{\alpha-1}, \quad v_{n+1} = Tv_n, \quad n = 0, 1, 2, \dots$$

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