Turk. J. Math. Comput. Sci.
17(1)(2025) 93–101
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DOI : 10.47000/tjmcs.1590392

On Dual Type Octonions and Their Properties

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Received: 24-11-2024 • Accepted: 14-04-2025

Abstract. In this study, we will define dual-type octonions by drawing inspiration from dual quaternions and Galilean geometry. Besides giving the basic properties of dual-type octonions and defining isotropic and non-isotropic dual-type octonions, we present Euler's and De Moivre's formulas for dual-type octonions. Finally, we give a matrix representation of dual-type octonions.

2020 AMS Classification: 11C20, 15B33

Keywords: Octonions, dual-type octonions, Euler's formula, De-Moivre formula, Galilean Geometry.

1. Introduction

Before starting, providing basic information about number systems that have been studied for a long time will help the reader understand which gap this article fills in the literature.

When the literature is examined, along with complex numbers ($\mathbb{C}$) present as two-dimensional, there are also dual numbers ($\mathbb{D}$) and hyperbolic numbers ($\mathbb{H}$) among the number systems that are well known to researchers. We can summarize these number systems and related references with the help of the table below.

Numbers	General Form	Property of Units	References
$\mathbb{C}$	$a + ib$	$i^2 = -1$	[1, 16, 38]
$\mathbb{D}$	$a + \varepsilon b$	$\varepsilon^2 = 0$	[22, 27, 37]
$\mathbb{H}$	$a + h\,b$	$h^2 = 1$	[4, 5, 18, 21, 28, 29, 33, 34, 36]

Table 1. Complex, Dual, and Hyperbolic numbers, and their general forms

In addition to the number systems given above, hybrid numbers, which are generalized versions of these numbers, are also defined as $z = a + ib + \epsilon c + hd$, here the units have same properties with the number system above. There are many different studies in the literature about these numbers, for details see [8, 14, 26].

Quaternions can be given as an example of a four-dimensional number system. They are obtained in two ways: the first way is obtained by changing the coefficients, the second way is obtained by changing the role of the quaternionic units. It can be summarized as below:

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i. Quaternions with different coefficients:
   a. $\boldsymbol{H}$ **− Real Quaternions**, $Q_1 = a_0 + ia_1 + ja_2 + ka_3, i^2 = j^2 = k^2 = -1$.
   b. $\boldsymbol{H_C}$ **− Complex Quaternions**, $Q = Q_1 + iQ_2, i^2 = -1, Q_{1,2} \in H$.
   c. $\boldsymbol{H_D}$ **− Dual Quaternions**, $Q = Q_1 + \varepsilon Q_2, \varepsilon^2 = 0, Q_{1,2} \in H$.
   d. $\boldsymbol{H_H}$ **− Hyperbolic Quaternions** , $Q = Q_1 + hQ_2, h^2 = 1, Q_{1,2} \in H$.

   For details, we refer to reader [2,17,25,30,31].

ii. Quaternions with different unit properties, for $Q_1 = a_0 + ia_1 + ja_2 + ka_3$ general form:
   a. $\boldsymbol{H}$ **− Real Quaternions** , $i^2 = j^2 = k^2 = -1, ijk = -1$.
   b. $\boldsymbol{H^S}$ **− Split Quaternions** , $i^2 = j^2 = -1, k^2 = 1, ijk = 1$.
   c. $\boldsymbol{H^D}$ **− Dual Type Quaternions** , $i^2 = j^2 = k^2 = 0, ijk = 0$.
   d. $\boldsymbol{H^H}$ **− Hyperbolic Type Quaternions** , $i^2 = j^2 = k^2 = 1, \quad ijk = 1$.

   For details, we refer to reader [8–13,20,23,32].

Some of the structures given above regarding quaternions are also given on octonions [6,7,15,19,35]. When the literature is examined, it is seen that dual-type octonions are not defined. The main purpose of this study is to fill this gap in the literature. For this purpose, firstly basic information about dual numbers will be given in order to provide preliminary information and understanding about the subject.

Dual numbers, $\mathbb{D} \equiv \mathbb{R}[\varepsilon]$, are an extension of real numbers with the dual unit ε. For the real numbers a and a^*, a dual number can be written as $z = a + \varepsilon a^*$. These numbers can be expressed as a combination of two real numbers with a dual unit. The dual unit ε is not equal to zero and $\varepsilon^2 = \varepsilon^3 = \cdots = 0, 0\varepsilon = \varepsilon 0 = 0$ and $1\varepsilon = \varepsilon 1 = \varepsilon$. The set of dual numbers can be defined as follows:

$$\mathbb{D} = \left\{ \mathrm{z} = a + \varepsilon a^* \mid a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}.$$

$1, \varepsilon$ are the base elements of dual numbers. The set of dual numbers is a 2 -dimensional vector space over $\mathbb{R}$. Moreover, dual numbers are described as $\mathbb{R}[x]/x^2$ in algebra. It is easy to see that $(\mathbb{D}, +, \cdot)$ is a commutative ring with unity. Dual numbers without real parts are called pure dual numbers, and they are zero-divisors such as $\varepsilon a, \varepsilon b$.

For dual numbers $\mathrm{z}_1 = a + \varepsilon a^*, \mathrm{z}_2 = b + \varepsilon b^*$ addition and multiplication operation are defined as follows:

$$z_1 + z_2 = (a + b) + \varepsilon (a^* + b^*),$$
$$z_1 \cdot z_2 = (a \cdot b) + \varepsilon (a^* b + ab^*).$$

It is easy to see that multiplication is associative and commutative. The conjugate of a dual number is denoted by $\overline{z}$ and defined by $\overline{z} = a - \varepsilon a^*$. Furthermore, the modulus of a dual number is denoted by $|\mathrm{z}|$ and is defined by $|\mathrm{z}| = \sqrt{z\overline{z}} = \sqrt{a^2} = |\mathrm{a}|$. This modulus corresponds to the distance in 2-dimensional Galilean plane. The Galilean plane is defined by the Galilean inner product, for the vectors $\vec{x} = (\mathrm{a}_1, \mathrm{b}_1), \vec{y} = (\mathrm{a}_2, \mathrm{b}_2) \in \mathbb{R}^2$, the Galilean inner product is defined by: $\langle \vec{x}, \vec{y} \rangle_G = \mathrm{a}_1 \mathrm{a}_2$. Dual numbers are isomorphic to the Galilean plane, $\mathbb{G}^2$, [22,27,37,38].

A dual number z is called a unit dual number if $|z| = 1$. The points that satisfy $|z| = 1$ are called Galilean unit circle on the dual plane. For $|\mathrm{z}| = a$, the circle can be demonstrated as in the following figure.

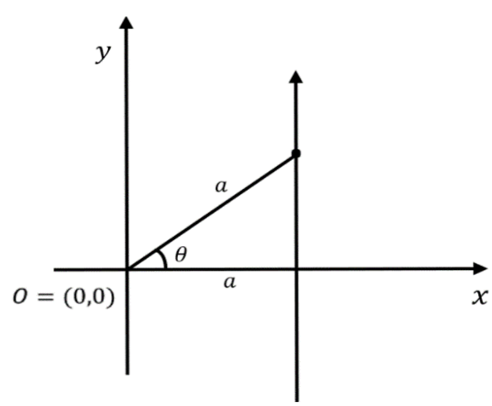

Figure 1. A circle with a radius a in the Galilean plane

In Galilean plane geometry, Galilean cosine and Galilean sine are shown as cosg and sing, respectively. The following statements can be given for Galilean plane geometry:

$$\begin{gathered}\cos g(\theta) := \frac{a}{a^*},\\ \sin g(\theta) := \frac{a^*}{a} = \theta,\\ \cos g(\alpha+\beta) = \cos g\alpha \cdot \cos g\beta - \varepsilon^2 \sin g\alpha \cdot \sin g\beta,\\ \sin g(\alpha+\beta) = \sin g\alpha \cdot \cos g\beta + \cos g\alpha \cdot \sin g\beta,\\ \cos^2\alpha + \varepsilon^2 \sin g^2\alpha = 1.\end{gathered}$$

For further details on dual numbers and Galilean geometry, see [10, 22, 23]. The octonions were first defined by Cayley. They are used in many areas such as quantum mechanics, electromagnetism, and sting theory [6, 7, 15, 19, 35]. The set of octonions can be given as follows:

$$\mathbb{O} = \left\{ O = \sum_{s=0}^{7} x_s e_s : x_s \in \mathbb{R}, e_0 = 1, e_0^2 = 1, e_i e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k \right\},$$

where e_0 is the identity element, e_s are basis elements of octonions, δ_{ij} is the Kronecker delta, and ε_{ijk} is a completely anti-symmetric tensor. It can be also said that the set of real numbers ($\mathbb{R}$), complex numbers ($\mathbb{C}$) and quaternions ($\mathbb{H}$) are subsets of octonions. Thus, a general octonion can be written as

$$O = x_0 e_0 + x_1 e_1 + x_2 e_2 + \cdots + x_7 e_7,$$

where the coefficients x_s are real numbers. When the coefficients are replaced by dual numbers, octonions with dual number coefficients are obtained. There are many works about octonions with dual number coefficients in the literature. In all of these studies, the name dual octonion is used. In fact, dual octonions in the literature should have been introduced as octonions with dual number coefficients. However, we are not working on these types of dual octonions in this study. Normally, the octonions that we will introduce in this study should be called dual octonions. In this study, in order to avoid confusion, we will use the name dual-type octonion. Changing the role of the octonion units e_s, we are going to define dual-type octonions, then we will define isotropic and non-isotropic dual octonions. Finally, we will present the properties of dual octonions.

2. Dual-Type Octonions

As we mentioned above, instead of the dual octonions known in the literature, dual-type octonions will be introduced with a new perspective, inspired by Galilean geometry. A general form of a dual-type octonion can be written as:

$$O = \sum_{s=0}^{7} x_s e_s = x_0 e_0 + x_1 e_1 + x_2 e_2 + \cdots + x_7 e_7,$$

where $x_0, x_1, \cdots, x_7 \in \mathbb{R}$, and $e_0, e_1, \cdots, e_7$ are the units of dual-type octonions. Additionally, $e_0 = 1$ and $e_0^2 = 1$. Any dual-type octonion can be written as $O = S_O + V_O$, where $S_O = x_0 e_0$ is the scalar part of and $V_O = x_1 e_1 + x_2 e_2 + \cdots + x_7 e_7$ is the vector part of O. If the scalar part equals zero, that is $S_O = 0$, then O is called an isotropic dual-type octonion. On the other hand, if $S_O \neq 0$, it is called a non-isotropic dual-type octonion.

Units of dual-type octonions satisfy the following rules:

- If O is a non-isotropic dual-type octonion,

$$e_1^2 = e_2^2 = \cdots = e_7^2 = 0 = e_i \times e_j,$$

  where $e_1^2 = e_1 \times e_1$, and the symbol " $\times$ " is used for multiplication of two non-isotropic dualtype octonions.
- If O is an isotropic dual-type octonion,

$$e_1^2 = e_2^2 = \cdots = e_7^2 = -1, \quad e_i \times_\delta e_j = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k,$$

  where $e_1^2 = e_1 \times_\delta e_1$, and the symbol " $\times_\delta$ " is used for multiplication of two isotropic dual-type octonions.

The product of two isotropic dual-type octonions is 0 . It can be observed by simple calculations. For dual-type octonions $O_1 = S_{O_1} + V_{O_1}, O_2 = S_{O_2} + V_{O_2}$, then

$$O_1 \times O_2 = (S_{O_1} + V_{O_1})(S_{O_2} + V_{O_2}) = S_{O_1}S_{O_2} + S_{O_1}V_{O_2} + S_{O_2}V_{O_1},$$
$$O_2 \times O_1 = (S_{O_2} + V_{O_2})(S_{O_1} + V_{O_1}) = S_{O_2}S_{O_1} + S_{O_2}V_{O_1} + S_{O_1}V_{O_2}.$$

From the equations above it can be seen that $O_1 \times O_2 = O_2 \times O_1$. $\mathbb{O}_{\mathbb{D}}$ and $\mathbb{O}_{\mathrm{D}}^p$ denote the set of non-isotropic dual-type octonions and isotropic dual octonions, respectively. Additionally, $\mathbb{O}_{\mathbb{D}}^p$ space is a subspace of $\mathbb{O}_{\mathbb{D}}$ and is isomorphic to $\mathbb{R}^7$. The sum of two dual-type octonions is also a dual-type octonion. The addition operation of dual-type octonions is defined as follows:

$$\oplus : \mathbb{O}_{\mathbb{D}} \times \mathbb{O}_{\mathrm{D}} \to \mathbb{O}_{\mathrm{D}}$$
$$(O_1, O_2) \to O_1 \oplus O_2 = (S_{O_1} + S_{O_2}) + (V_{O_1} + V_{O_2}).$$

$(\mathbb{Q}_{\mathrm{D}}, \oplus)$ is an Abelian group with the identity element. Moreover, the multiplication of a scalar and a dual-type octonion is defined by

$$\odot : \mathbb{R} \times \mathbb{O}_{\mathbb{D}} \to \mathbb{O}_{\mathbb{D}}$$
$$(\lambda, 0) \to \lambda \odot 0 = \lambda S_0 + \lambda V_0.$$

This operation implies the following statements:

1. $\lambda \odot (O_1 + O_2) = (\lambda \odot O_1) \oplus (\lambda \odot O_2)$,
2. $(\lambda_1 + \lambda_2) \odot 0 = (\lambda_1 \odot 0) \oplus (\lambda_2 \odot 0)$,
3. $(\lambda_1 \cdot \lambda_2) \odot 0 = \lambda_1 \odot (\lambda_2 \odot 0)$,
4. $1 \odot 0 = 0$.

Thus, it can be seen that $(\mathbb{O}_{\mathrm{D}}, \oplus, \odot)$ is a vector space over the real numbers.

Example 2.1. Let $O_1 = 3e_0 + 2e_1 + 5e_2 - 7e_3 + 2e_7$ and $O_2 = e_0 - 4e_2 - e_5 + 6e_7$ be two dual-type octonions. The addition of these dual-type octonions is as follows:

$$O_1 + O_2 = 4e_0 + 2e_1 + e_2 - 7e_3 - e_5 + 8e_7.$$

Definition 2.2. Let $O_1 = x_0e_0 + \sum_{s=1}^{7} x_se_s = S_{O_1} + V_{O_1}$ and $O_2 = y_0e_0 + \sum_{s=1}^{7} y_se_s = S_{O_2} + V_{O_2}$ be two dual-type octonions. Then, the multiplication of these dual-type octonions can be defined as follows:

- If O_1 and O_2 are non-isotropic dual-type octonions
$$O_1 \times O_2 = x_0y_0 + x_0 \sum_{s=1}^{7} y_se_s + y_0 \sum_{s=1}^{7} x_se_s = S_{O_1}S_{O_2} + S_{O_1}V_{O_2} + S_{O_2}V_{O_1}.$$
- If O_1 and O_2 are isotropic dual-type octonions
$$O_1 \times O_2 = - < V_{O_1}, V_{O_2} > + V_{O_1} \wedge V_{O_2},$$
where notation " $<,>$ " is denote the inner product, and " Λ " denote the vector product in Euclidean 7 -space.

Example 2.3. Let $O_1 = 2e_0 + e_1 - 3e_3 + 5e_7$ and $O_2 = e_0 - 2e_2 + 4e_5$ be two non-isotropic dual-type octonions. Multiplication of these non-isotropic dual-type octonions are as follows:

$$O_1 \times O_2 = 2e_0 + e_1 - 4e_2 - 3e_3 + 8e_5 + 5e_7.$$

Definition 2.4. Let $O = S_O + V_O$ be a dual-type octonion. The conjugate of O is denoted as $\overline{O}$ and defined as $\overline{O} = S_O - V_O$. The following statements are satisfied for dual-type octonions:

- If O be a non-isotropic dual-type octonion, then
$$O \times \overline{O} = \overline{O} \times O = S_O^2 - S_OV_O + S_OV_O = S_O^2 = x_0^2.$$
- If O an isotropic dual-type octonion, then
$$O \times_\delta \overline{O} = \overline{O} \times_\delta O = -\langle V_O, -V_O \rangle - V_O \wedge V_O = \langle V_O, V_O \rangle = \sum_{s=1}^{7} x_s^2.$$
- Let O_1 and O_2 be two dual-type octonions, then
$$\overline{O_1 + O_2} = \overline{O_1} + \overline{O_2}, \quad \overline{\left(\overline{O_1}\right)} = O_1, \quad \overline{\lambda O_1} = \lambda \overline{O_1}, \quad \lambda \in \mathbb{R}.$$

- Let $O = S_o + V_o$ be a dual-type octonion, then the following equations are hold:

$$S_O = \frac{O+\overline{O}}{2}, \quad V_O = \frac{O-\overline{O}}{2}.$$

Definition 2.5. Let $O = x_0e_0 + x_1e_1 + x_2e_2 + \cdots + x_7e_7$ be a dual-type octonion. Then the norm operation is defined as follows:

- If O is a non-isotropic dual-type octonion, then

$$\|O\| = \sqrt{\overline{O}\times O} = \sqrt{O\times\overline{O}} = \sqrt{x_0^2} = |x_0|\,.$$

- If O is an isotropic dual-type octonion, then

$$\|O\| = \sqrt{O\times_\delta\overline{O}} = \sqrt{\overline{O}\times_\delta O} = \sqrt{x_1^2+x_2^2+\cdots+x_7^2} = \sqrt{\sum_{s=1}^{7}x_s^2}.$$

Definition 2.6. A dual-type octonion O is called a unit dual-type octonion if $\|O\| = 1$.

Theorem 2.7. *For dual-type octonions O_1 and O_2, the following operations are hold:*

- $\|O_1\times O_2\| = \|O_1\|\times\|O_2\| = \|O_2\times O_1\|$,
- $\|O_1+O_2\| \leq \|O_1\|+\|O_2\|$,
- $\left\|O_1^2+O_2^2\right\| = \frac{1}{2}\|O_1+O_2\|^2+\|O_1-O_2\|^2$,
- $\|O_1\| = \left\|\overline{O_1}\right\|$,
- $\|O_1\| = 0 \Leftrightarrow O_1 = 0$.

Definition 2.8. Let $O = x_0e_0 + \sum_{s=1}^{7}x_se_s$ be a dual-type octonion. The inverse of is O is defined as follows:

- If O is a non-isotropic dual-type octonion, then

$$O^{-1} = \frac{\overline{O}}{\|O\|^2} = \frac{x_0-\sum_{s=1}^{7}x_se_s}{x_0^2}.$$

- If O is an isotropic dual-type octonion, then

$$O^{-1} = \frac{\overline{O}}{\|O\|^2} = \frac{-\sum_{s=1}^{7}x_se_s}{\sum_{s=1}^{7}x_s^2}.$$

For any dual-type octonion, the inverse operation satisfies the following properties:

- $\left\|O^{-1}\right\| = \|O\|^{-1}$,
- $O\times O^{-1} = O\times\left(\|O\|^{-2}\overline{O}\right) = \|O\|^{-2}(O\times\overline{O}) = \|O\|^{-2}\|O\|^2 = 1 = O^{-1}\times O$.

Definition 2.9. Let O_1 and O_2 be two dual-type octonions. The inner product of O_1 and O_2 is defined as follows:

- If O_1 and O_2 are non-isotropic dual-type octonions, then

$$\begin{aligned}
&<,>: O_{\mathrm{D}}\times O_{\mathrm{D}} \to \mathbb{R}\\
&< O_1, O_2 > = \frac{1}{2}\left(O_1\times\overline{O_2}+O_2\times\overline{O_1}\right)\\
&\qquad = \frac{1}{2}\left(S_{O_1}S_{O_2}-S_{O_1}V_{O_2}+S_{O_2}V_{O_1}+S_{O_2}S_{O_1}-S_{O_2}V_{O_1}+S_{O_1}V_{O_2}\right)\\
&\qquad = S_{O_1}S_{O_2}.
\end{aligned}$$

- If O_1 and O_2 are isotropic dual-type octonions, then

$$\begin{aligned}
&<,>_\delta: O_{\mathrm{D}}^p\times_\delta O_{\mathrm{D}}^p \to \mathbb{R}\\
&< O_1, O_2 >_\delta = \frac{1}{2}\left(O_1\times_\delta\overline{O_2}+O_2\times_\delta\overline{O_1}\right)\\
&\qquad = \frac{1}{2}\left(< V_{O_1}, V_{O_2} > -V_{O_1}\wedge V_{O_2}+ < V_{O_2}, V_{O_1} > -V_{O_2}\wedge V_{O_1}\right)\\
&\qquad =< V_{O_1}, V_{O_2} > .
\end{aligned}$$

2.1. **Polar Representation of Dual-type Octonions.** Let θ be the angle between the real axis and dual-type octonion $O = x_0 e_0 + \sum_{s=1}^{7} x_s e_s$. The polar representation of O is given as follows:

- If O is a non-isotropic dual-type octonion, then O can be written as follows:

$$O = \|O\| \frac{x_0 + \sum_{s=1}^{7} x_s e_s}{\|O\|} = \|O\| \left[\frac{x_0}{\|O\|} + \frac{\sum_{s=1}^{7} x_s e_s}{\|O\|} \right]$$

$$= \|O\| \left[\frac{x_0}{\|O\|} + \frac{\sqrt{\sum_{s=1}^{7} x_s^2}}{\|O\|} \frac{\sum_{s=1}^{7} x_s e_s}{\sqrt{\sum_{s=1}^{7} x_s^2}} \right],$$

where

$$n = \frac{\sum_{s=1}^{7} x_s e_s}{\sqrt{\sum_{s=1}^{7} x_s^2}}, \quad \cos g\theta = \frac{x_0}{\|O\|} = \frac{x_0}{x_0} = 1, \quad \sin g\theta = \frac{\sqrt{\sum_{s=1}^{7} x_s^2}}{\|O\|}.$$

Then, we obtain the polar form of a non-isotropic dual-type octonion as

$$O = \|O\|(\cos g\theta + n \operatorname{sing}\theta).$$

- If O is an isotropic dual-type octonion ($x_0 = 0$), then O can be written as follows:

$$O = \|O\| \frac{\sum_{s=1}^{7} x_s e_s}{\|O\|} = \|O\| \left[0 + \frac{\sqrt{\sum_{s=1}^{7} x_s^2}}{\|O\|} \frac{\sum_{s=1}^{7} x_s e_s}{\sqrt{\sum_{s=1}^{7} x_s^2}} \right],$$

where

$$n = \frac{\sum_{s=1}^{7} x_s e_s}{\sqrt{\sum_{s=1}^{7} x_s^2}}, \quad \cos\theta = \frac{0}{\|O\|} = 0, \quad \sin\theta = \frac{\sqrt{\sum_{s=1}^{7} x_s^2}}{\|O\|}.$$

Then, we obtain the polar form of a non-isotropic dual-type octonion as

$$O = \|O\|(\cos\theta + n \sin\theta).$$

2.2. **Euler's Formulas for Dual-type Octonions.** From the previous section, it is easy to see that a unit non-isotropic dual-type octonion can be expressed as $O = \operatorname{cosg}\theta + n \operatorname{sing}\theta$. Using the property $n^2 = n^3 = \cdots = n^a = 0\, (a \in \mathbb{Z}^+)$, for any angle θ the Euler formula of any non-isotropic dual-type octonion can be given as follows:

$$e^{n\theta} = 1 + n\theta + \frac{(n\theta)^2}{2!} + \frac{(n\theta)^3}{3!} + \cdots = 1 + n\theta = \cos g\theta + n \sin g\theta,$$

following equations hold for non-isotropic dual-type octonions:

i. $e^{n\theta_1} \times e^{n\theta_2} = e^{n(\theta_1+\theta_2)}$,

ii. $\frac{1}{e^{n\theta}} = e^{-n\theta}$,

iii. $\frac{e^{n\theta_1}}{e^{n\theta_2}} = e^{n(\theta_1-\theta_2)}$.

The Euler formula for isotropic dual-type octonions can be obtained as follows:

$$e^{n\theta} = \cos\theta + n \sin\theta,$$

where $n \wedge n = \overrightarrow{0}, < \text{n, n} >_\delta, \left\langle n, n >_{\mathbb{R}^7} = 1, n^2 = n \otimes_\delta n = -1 \right.$, and $n^3 = -n, n^4 = -1, \cdots$.

2.3. **De-Moivre's Formula and Inner Product for Dual-type Octonions.** Let $O_1 = \operatorname{cosg}\theta + n \operatorname{sing}\theta$ and $O_2 = \operatorname{cosg}\beta + n \operatorname{sing}\beta$ be two dual-type octonions. For $n = \frac{\sum_{s=1}^{7} x_s e_s}{\sqrt{\sum_{s=1}^{7} x_s^2}}$, it can be seen that

$$\cos g(\alpha + \beta) + n \sin g(\alpha + \beta) = (\cos g\alpha + n \sin g\alpha) \otimes (\cos g\beta + n \operatorname{sing}\beta).$$

Theorem 2.10. *Let O be a non-isotropic dual-type octonion. Then, O can be written as $O^k = \cos g(k\alpha) + n \sin g(k\alpha)$.*

Proof. The proof of this theorem can be easily obtained by induction. □

Definition 2.11. The inner product of a non-isotropic dual-type octonions is defined as follows:

$$<,>: O_{\mathrm{D}} \times O_{\mathrm{D}} \to \mathbb{R}$$
$$< O_1, O_2 > = \frac{1}{2}\left(O_1 \times \overline{O_2} + O_2 \times \overline{O_1}\right) = S_{O_1} S_{O_2} = x_0 y_0.$$

Example 2.12. Let $O_1 = 2e_0 - 9e_1 + 3e_5 - e_6$ and $O_2 = 5e_0 + e_4 - e_6 - 3e_7$ be two nonisotropic dual-type octonions. Inner product of these non-isotropic dual octonions are as follows:

$$< O_1, O_2 >= \frac{1}{2}\left(O_1 \times \overline{O_2} + O_2 \times \overline{O_1}\right) = 2 \cdot 5 = 10.$$

Definition 2.13. The inner product of an isotropic dual-type octonions is defined as follows:

$$<_1>_\delta : O^p_{\mathrm{B}} \times_\delta O^p_{\mathrm{D}} \to \mathbb{R}$$
$$< O_1, O_2 >_\delta = \frac{1}{2}\left(O_1 \times_\delta \overline{O_2} + O_2 \times_\delta \overline{O_1}\right) =< V_{O_1}, V_{O_2} >= \sum_{s=1}^{7} x_s y_s.$$

Example 2.14. Let $O_1 = -9e_1 + 2e_4 + 3e_5 - e_6 + 5e_7$ and $O_2 = e_1 + e_4 - e_6 - 3e_7$ be two isotropic dual-type octonions. Inner product of these isotropic dual octonions are as follows:

$$< O_1, O_2 >_\delta = \frac{1}{2}\left(O_1 \times_\delta \overline{O_2} + O_2 \times_\delta \overline{O_1}\right) = -9 + 2 + 1 - 15 = -21.$$

2.4. **Matrix Representation of Dual-type Octonions.** Let $O = x_0 e_0 + x_1 e_1 + x_2 e_2 + \cdots + x_7 e_7 = \sum_{s=0}^{7} x_s e_s$ be a dual-type octonion. For the linear map L_O which is defined as follows:

$$L_O : O_{\mathbb{D}} \to O_{\mathrm{D}}$$
$$L_O(O_1) = O \times O_1,$$

using the basis elements of dual-type vector space $\{e_0 = 1, e_1, e_2, \ldots, e_7\}$ and the operator above, we can write

$$\begin{aligned}
L_O(1) &= O \times 1 = x_0 \cdot 1 + x_1 \cdot e_1 + x_2 \cdot e_2 + x_3 \cdot e_3 + x_4 \cdot e_4 + x_5 \cdot e_5 + x_6 \cdot e_6 + x_7 \cdot e_7\\
L_O(e_1) &= O \times e_1 = 0 \cdot 1 + x_0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4 + 0 \cdot e_5 + 0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_2) &= O \times e_2 = 0 \cdot 1 + 0 \cdot e_1 + x_0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4 + 0 \cdot e_5 + 0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_3) &= O \times e_3 = 0 \cdot 1 + 0 \cdot e_1 + 0 \cdot e_2 + x_0 \cdot e_3 + 0 \cdot e_4 + 0 \cdot e_5 + 0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_4) &= O \times e_4 = 0 \cdot 1 + 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + x_0 \cdot e_4 + 0 \cdot e_5 + 0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_5) &= O \times e_5 = 0 \cdot 1 + 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4 + x_0 \cdot e_5 + 0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_6) &= O \times e_6 = 0 \cdot 1 + 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4 + 0 \cdot e_5 + x_0 \cdot e_6 + 0 \cdot e_7\\
L_O(e_7) &= O \times e_7 = 0 \cdot 1 + 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3 + 0 \cdot e_4 + 0 \cdot e_5 + 0 \cdot e_6 + x_0 \cdot e_7.
\end{aligned}$$

Then, we can obtain the real matrix representation of the dual-type octonion as follows:

$$L_O = \begin{pmatrix}
x_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
x_1 & x_0 & 0 & 0 & 0 & 0 & 0 & 0\\
x_2 & 0 & x_0 & 0 & 0 & 0 & 0 & 0\\
x_3 & 0 & 0 & x_0 & 0 & 0 & 0 & 0\\
x_4 & 0 & 0 & 0 & x_0 & 0 & 0 & 0\\
x_5 & 0 & 0 & 0 & 0 & x_0 & 0 & 0\\
x_6 & 0 & 0 & 0 & 0 & 0 & x_0 & 0\\
x_7 & 0 & 0 & 0 & 0 & 0 & 0 & x_0
\end{pmatrix}.$$

Example 2.15. Let $O_1 = 5e_0 + 2e_1 - 3e_3 + 4e_5 - 9e_7$ be a dual type octonion. Real matrix representation of O_1 is as follows:

$$L_O = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ -9 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

3. Conclusion and Future Remarks

In this paper, a new number system has been defined, and it is added to the types of octonion number systems. This new system is obtained by changing the role of octonionic units and is introduced as the dual-type octonion system. This study is inspired by Galilean space and the article [9, 22, 32]. Dual-type octonions have been studied as isotropic and non-isotropic dualtype octonions. In addition to giving the basic properties of dual octonions, polar representation, Euler's formula, De-Moivre's formula, and matrix representation of dual-type octonions are also given. As a future direction and study, we will examine the application of this new number system to Fibonacci and the other sequences. The results studied and presented in [3, 24] can be examined with this new number system.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors have read and agreed to the published version of the manuscript.

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