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Solution of Shortest Paths in Non-Euclidean Farey Graph with Floyd-Warshall Algorithm

İbrahim Gökcan[1]

[1]Faculty of Arts and Sciences, Artvin Çoruh University, 08000, Artvin, TURKEY
https://orcid.org/0000-0002-6933-8494

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Abstract: Algorithm applications on graphs are intensively researched. Graph theory systematizes complex and difficult problems and algorithms provide fast and clear solutions, which increases interest in the discipline. The Floyd-Warshall algorithm determines the shortest paths between all the vertices in a graph. In this paper, we consider the Floyd-Warshall algorithm on the Farey graph defined in a non-Euclidean hyperbolic space. A Farey graph with 15 edges and 9 vertices is constructed and the shortest paths from all vertices to other vertices are detected. By defining the weight between consecutive vertices, the shortest paths between the vertices are measured in terms of the number of steps.

Keywords: Farey graph, Floyd-Warshall algorithm, non-Euclidean hyperbolic space

1. Introduction

Algorithms are at the core of the advance of computer and internet technology. Algorithms basically separate a problem into steps and systematically produce results in the context of inputs and outputs. The design and selection of the algorithm differs according to the type of problem. Algorithms provide solutions to a specific type of problem under certain conditions. In the scientific progress, many operations that were previously performed with the pen and the power of thought are now achieved much faster and with clearer results via algorithms. In this paper, the Floyd-Warshall algorithm, one of the shortest path algorithms, is used to obtain the shortest paths between vertices on a Farey graph defined in a non-Euclidean space.

Several algorithms exist which aim to find the shortest path between two vertices or vertices in a graph. Dijkstra's algorithm, which has been intensely investigated in the literature and is one of the most reliable shortest path algorithms, produces the shortest path between two selected vertices in a graph. Dijkstra's algorithm does not apply if the edges are negative weighted. Also, the Bellman-Ford algorithm is used to identify the shortest path between two selected vertices in the graph. We can refer to [1], [2], [3], [4], [5], and [6] for shortest path algorithms. Unlike Dijkstra and Bellman-Ford algorithms, the Floyd-Warshall algorithm is used to determine the shortest paths between all vertices in the graph and runs in the presence of a negatively weighted edge, but not in the presence of a negative cycle. For more detailed information on the Floyd-Warshall algorithm, see [7].

Researches on graph theory have been increasing in recent years and have been associated with many disciplines. For the literature on graph theory, see [8] and [9]. In [10], new concepts for suborbital graph act were presented. Numerous studies have been conducted on the relevant concepts. In [11], in particular, the Farey graph $\boldsymbol{F}$, suborbit graphs $G_{u,N}$ ve $F_{u,N}$ on the rational projective line $\widehat{\mathbb{Q}}$ with $(u,N)=1$ in hyperbolic geometry were realized. Moreover, the relationships of the vertices of suborbit graphs with continued fractions by means of the Modular group Γ were achieved in [12]. In addition, in [13], continued fractions were associating with special number sequences and generalizations were studied for vertex values. Also, the minimum length condition between vertices in suborbital graphs was analyzed in [14]. The minimum length condition between vertices introduced in [14] was shown as an algorithm in [15]. Besides, in [16], a Farey sequence between the vertices of $F_{u,N}$ was constructed and Dijkstra's algorithm was used to generate a tree with minimum lengths from the initial vertex to the other vertices.

2. Material and Method

2.1. Graph Theory

Graph theory is extensively researched in mathematics and is associated with different sub-disciplines. While there are different assumptions about the origin of graph theory, there are two prominent views. The first is based on Plato's uniform objects. The unfolding of these uniform objects results in vertices and edges between vertices, which are the basis of graph theory. The second is based on Euler's article “The Seven Bridges of Konigsberg”, published in 1736. Seven bridges over the Pregel River, which runs through the town of Kaliningrad in Belarus, were used by people attempting to cross the bridges according to a certain rule. Starting from a certain bridge, the rule was trying if you could reach the starting point after passing through each bridge without crossing the same bridge and path again. Euler carried the problem to the literature with “The Seven Bridges of Konigsberg” article and proved that starting from a bridge and without crossing the bridges and paths again; the starting point cannot be reached by passing through each bridge. In the context of developments in graph theory, in 1822, J.J. Sylvester used the term graph for the first time in his work. Moreover, Gustav Kirchhoff published Kirchhoff circuit theories based on Graph theory in 1945 and Francis Guthrie published the four-color problem in 1852. Furthermore, in 1936, D. König published the first book on graph theory.
Here are some definitions related to graph theory that will be used in the study.

A graph is roughly a mathematical object consisting of vertices and edges connecting the vertices. An element at both ends of an edge is called a vertex. The element between two vertices is called an edge. A directed graph is a graph whose edges contain directional information. A sequence of the edges to be traced from one vertex to another is called a path. The length of a path is equal to the number of edges traversed. An edge with the same initial and ending vertices is called a loop. A path that begins at a vertex and returns to the same vertex and does not pass through a vertex twice is called a cycle. A graph has at least one cycle if the number of edges equals or exceeds the number of vertices. A connected graph without cycles is called a tree. Adding an edge to a tree creates a loop and the number of edges in a tree is one less than the number of vertices. A graph that has no cycles is called a forest. A forest is formed by trees and a tree alone is a forest. In a graph, edges can take values and these values are included in the structure of the graph. A graph where all edges have values is called a weighted graph. The sum of all values in a weighted graph provides the total cost of the graph. Once the least-weighted path is calculated for any two vertices in a graph, the sum of the weights of edges is the total weight of this path. The more central of the two vertices is the one with the lower weight. The least-weight vertex in a graph is called a central vertex of the graph. A spanning tree is a tree that covers all vertices in a graph. Given all the spanning trees in a graph, a spanning tree with the lowest weight is a least weighted spanning tree of that graph. For a graph, if there is at least one path between two vertices x and y, the weight of a shortest of these paths is called the distance between x and y.

2.2. Non-Euclidean Geometries

The existence of cases where 5th postulate in Euclid's Elements is not satisfied has been studied for centuries in the scientific world. Especially in the 18th and 19th centuries, Girolamo Saccheri (1667-1733), Carl Friedrich Gauss (1777-1855), Nikolai Ivanovich Lobachevsky (1792-1856), János Bolyai (1802-1860), Bernhard Riemann (1826-1866), Henry Poincaré (1854-1912) and Eugenio Beltrami (1835-1900) discussed the problem comprehensively and non-Euclidean geometries were defined. Non-Euclidean geometries can be analyzed under two headings: Hyperbolic and Elliptic geometries. In a non-planar region, Hyperbolic geometries are constructed in case the sum of the interior angles of the triangle is less than 180° and Elliptic geometries are constructed in case it is greater than 180°. Obviously, in non-Euclidean geometries, the edges are curves, not lines. In this paper, in particular, Farey graphs constructed from hyperbolic triangles will be focused on.

2.3. Farey Graph

The Farey graph is formed basically by an infinite number of curved triangles that get smaller as they approach to the circle line. Given the two vertices $\frac{a}{b}$ and $\frac{c}{d}$ on the Farey graph, the middle vertex is determined by mediant rule. Let assume that vertices $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive vertices on Farey graph satisfying $\frac{a}{b} < \frac{c}{d}$ and $|bc - ad| = 1$. Then, the mediant of these consecutive vertices is presented by $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$. For example, if vertices $\frac{0}{1}$ and $\frac{1}{1}$ are consecutive vertices on Farey graph, the mediant of these consecutive vertices is $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$. In other words, the middle vertex of vertices $\frac{0}{1}$ and $\frac{1}{1}$ is $\frac{1}{2}$. The Farey graph is constructed between all consecutive pairs of numbers arranged around a circle. An example of Farey graph between $\frac{a}{b}$ and $\frac{c}{d}$ only is illustrated below:

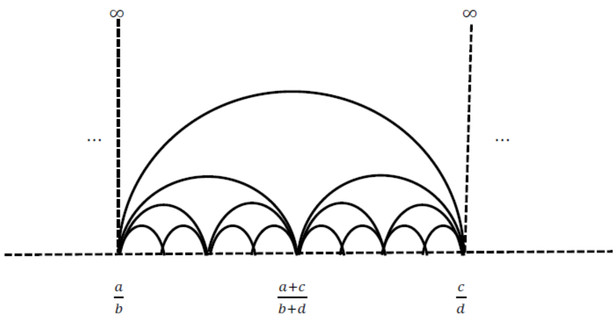

Figure 1. Farey graph constructed between $\frac{a}{b}$ and $\frac{c}{d}$

In special, taking the vertices of the graph as $\frac{0}{1}$ and $\frac{1}{1}$, by using the mediant, the vertices of the graph can be represented as elements of the Farey sequence, some elements of which are presented as follows.

$$\frac{0}{1}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

2.4. Suborbital Graphs and Farey Graph

In [10], the motion of G on the set Ω is described, where G is a group and Ω is the set whose elements are the vertices of the graph. [17] and [18] investigated the idea introduced in [10] on finite groups. In [11], however, $G = \Gamma$ Modular group and $\Omega = \widehat{\mathbb{Q}}$

were scrutinised. Additionally, in [11] the next lemma for group action between vertices in a Farey graph was presented.

2.4.1. Lemma Assume that $\frac{a}{b}, \frac{c}{d} \in \widehat{\mathbb{Q}}$ are reduced rationals, $\boldsymbol{F}$ is the Farey graph and F_n is the sequence generated from the Farey graph. In this case, the conditions below are equivalent.

$i)$ $\frac{a}{b}$ and $\frac{c}{d}$ are the neighbor vertices in the Farey graph $\boldsymbol{F}$.

$ii)$ $ad - bc = \pm 1$.

$iii)$ $\frac{a}{b}$ and $\frac{c}{d}$ are the neighbor vertices in the Farey sequence F_n for $n \in \mathbb{N}$.

2.5. Floyd-Warshall Algorithm

The Floyd-Warshall algorithm is a dynamic programming algorithm that determines shortest paths between all pairs of vertices in a graph. The Floyd-Warshall algorithm is a shortest path algorithm similar to the Dijkstra and Bellman-Ford algorithms. The Dijkstra and Bellman-Ford algorithms have a source vertex in a graph and produce shortest path from this source vertex to any other vertex. In the contrary, the Floyd-Warshall algorithm provides shortest paths between all pairs of vertices in a graph. Obviously, in Floyd-Warshall algorithm, every vertex can be considered as a source vertex. It is applicable to directed and undirected graphs. The Floyd-Warshall algorithm is also applicable to pseudo graphs consisting of loops and parallel edges. However, in this case, all 0's are substituted by ∞. The algorithm fails on graphs with negative cycles; a negative cycle means that the sum of the edges in a cycle is negative. The Floyd-Warshall algorithm is a dynamic program. In other words, it recalls the shortest distances between vertices in the previous step and uses them in subsequent steps. In addition, the time complexity of the Floyd - Warshall algorithm is $O(n^3)$.

The w_{ij} value of the Floyd-Warshall algorithm is defined as a piecewise function as follows: Assume that i, k and j are vertices and $w(ij)$ is the weight of the edge if exists between i and j in a graph G. Then;

$$w_{ij} = \begin{cases} 0, & \text{if } i = j \text{ ;} \\ w(ij), & \text{if } i \neq j \text{ and there is an edge between } i \text{ and } j\text{;} \\ +\infty & \text{if } i \neq j \text{ and there is an edge between } i \text{ and } j. \end{cases}$$

Thus, the matrix $W = (w_{ij})$ is generated by using the values $w(ij)$. Now, suppose that the graph is composed of vertices $v_1, v_2, \cdots, v_n$. Assume that L_0 is the first generated matrix based on the connectivity between vertices, that is, $L_0 = W$. Let $d_{v_i v_j}^{(v_k)}$ path from be length of a shortest path v_i to v_j via an intermediate vertex. There are two cases depending on the state of reaching vertex v_j from vertex v_i with intermediate vertex v_k.

$i.$ The situation where the paths from vertex v_i to vertex v_j do not pass via intermediate vertex v_k. In this case, the length of the shortest path is defined as $d_{v_i v_j}^{(v_{k-1})}$.

$ii.$ The situation where the paths from vertex v_i to vertex v_j pass via intermediate vertex v_k. To obtain a shortest path, the path should not pass two times via the same vertex. Thus, get a path with the shortest length from vertex v_i to vertex v_j, the paths from vertex v_i to vertex v_k and from vertex v_k to vertex v_j should be selected as short as possible. The length of the shortest path is thus equal to $d_{v_i v_k}^{(v_{k-1})} + d_{v_k v_j}^{(v_{k-1})}$ in $\{v_1, v_2, \cdots, v_k\}$. Therefore, it is obvious that

$$d_{v_i, v_j}^{(v_k)} = \min\left\{d_{v_i, v_j}^{(v_{k-1})}, d_{v_i, v_k}^{(v_{k-1})} + d_{v_k, v_j}^{(v_{k-1})}\right\}$$

As the algorithm updates the matrix at each step, the distance matrix L_{v_k} is updated after each intermediate vertex v_k.
Pseudocode of the Floyd-Warshall algorithm is presented below:

```
floyd − warshall (n, w)  {
        d = array[1..n, 1..n]                    // distance matrix
        for (i = 1 to n) {                       // initialize
        for (j = 1 to n) {
          d[i, j] = w[i, j]
          helper[i, j] = null
        }
    }
    for (k = 1 to n) {                           //use intermediates {1..k}
       for (i = 1 to n) {                        // ... from i
          for (j = 1 to n) {                     // ... from j
             if (d[i, k] + d[k, j] < d[i, j])  {
                d[i, j] = d[i, k] + d[k, j] // new shorter path length
                helper[i, j] = k        // new path is through k
             }
          }
       }
   }
   return d                          //d[i, j] holds the distance from i to j
```

3. Results

3. Determination of Shortest Paths in Farey Graph with Floyd-Warshall Algorithm

Consider the vertices $\frac{0}{1}$ and $\frac{1}{1}$ of the graph given in Figure 1 and analyze the shortest paths between the vertices with the Floyd-Warshall Algorithm. In particular, consider the edges of the graph as right-oriented from $\frac{0}{1}$ to $\frac{1}{1}$. We study with the graph shown in Figure 2.

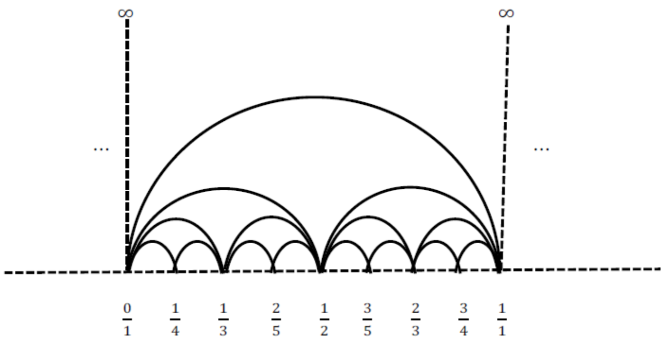

Figure 2. Farey graph between $\frac{0}{1}$ and $\frac{1}{1}$

Let the weight between vertices in the graph be identified as $|ad - bc|$ by using Lemma 2.4.1, where $\frac{a}{b}$ and $\frac{c}{d}$ are neighboring vertices. Hence, the weighted matrix $W = L_0$ constructed based on the weights of edges between the vertices of the graph is obtained by means of the piecewise function w_{ij} as follows. Let $\frac{a}{b},\frac{c}{d}$ be two vertices in the above Farey graph $\boldsymbol{F}$**.** Then

$$w_{ij}=\begin{cases} 0, & \text{if } \dfrac{a}{b}=\dfrac{c}{d};\\ |ad-bc| & \text{if } \dfrac{a}{b}\rightarrow\dfrac{c}{d};\\ +\infty, & \text{if } \dfrac{a}{b}\nrightarrow\dfrac{c}{d}.\end{cases}$$

Then L_0 is the following matrix:

$$\begin{array}{cc} w_{ij} & \begin{array}{ccccccccc} v_1=\frac{0}{1} & v_2=\frac{1}{4} & v_3=\frac{1}{3} & v_4=\frac{2}{5} & v_5=\frac{1}{2} & v_6=\frac{3}{5} & v_7=\frac{2}{3} & v_8=\frac{3}{4} & v_9=\frac{1}{1}\end{array} \\ \begin{array}{c} v_1=\frac{0}{1}\\ v_2=\frac{1}{4}\\ v_3=\frac{1}{3}\\ v_4=\frac{2}{5}\\ v_5=\frac{1}{2}\\ v_6=\frac{3}{5}\\ v_7=\frac{2}{3}\\ v_8=\frac{3}{4}\\ v_9=\frac{1}{1}\end{array} & \begin{pmatrix} 0 & 1 & 1 & \infty & 1 & \infty & \infty & \infty & 1\\ 1 & 0 & 1 & \infty & \infty & \infty & \infty & \infty & \infty\\ 1 & 1 & 0 & 1 & 1 & \infty & \infty & \infty & \infty\\ \infty & \infty & 1 & 0 & 1 & \infty & \infty & \infty & \infty\\ 1 & \infty & 1 & 1 & 0 & 1 & 1 & \infty & 1\\ \infty & \infty & \infty & \infty & 1 & 0 & 1 & \infty & \infty\\ \infty & \infty & \infty & \infty & 1 & 1 & 0 & 1 & 1\\ \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 & 1\\ 1 & \infty & \infty & \infty & 1 & \infty & 1 & 1 & 0\end{pmatrix}\end{array}$$

Let us scrutinise the shortest paths through all intermediate vertices.

Using the intermediate vertex $v_2=\frac{1}{4}$, the length $d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{1}{3}}$ is calculated as follows:

$$d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{1}{3}}=\min\left\{d^{\left(\frac{0}{1}\right)}_{\frac{0}{1},\frac{1}{3}},d^{\left(\frac{0}{1}\right)}_{\frac{0}{1},\frac{1}{4}}+d^{\left(\frac{0}{1}\right)}_{\frac{1}{4},\frac{1}{3}}\right\}=\min\{1,1+1\}=1$$

Therefore, $L_{\frac{1}{4}}=L_0$.

For the intermediate vertex $v_3=\frac{1}{3}$, lengths $d^{\left(\frac{1}{3}\right)}_{\frac{0}{1},\frac{2}{5}},d^{\left(\frac{1}{3}\right)}_{\frac{0}{1},\frac{1}{2}},d^{\left(\frac{1}{3}\right)}_{\frac{1}{4},\frac{2}{5}}$ and $d^{\left(\frac{1}{3}\right)}_{\frac{1}{4},\frac{1}{2}}$ are examined.

$$d^{\left(\frac{1}{3}\right)}_{\frac{0}{1},\frac{2}{5}}=\min\left\{d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{2}{5}},d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{1}{3}}+d^{\left(\frac{1}{4}\right)}_{\frac{1}{3},\frac{2}{5}}\right\}=\min\{\infty,1+1\}=2,$$

$$d^{\left(\frac{1}{3}\right)}_{\frac{0}{1},\frac{1}{2}}\min\left\{d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{1}{2}},d^{\left(\frac{1}{4}\right)}_{\frac{0}{1},\frac{1}{3}}+d^{\left(\frac{1}{4}\right)}_{\frac{1}{3},\frac{1}{2}}\right\}=\min\{1,1+1\}=1,$$

$$d^{\left(\frac{1}{3}\right)}_{\frac{1}{4},\frac{2}{5}}\min\left\{d^{\left(\frac{1}{4}\right)}_{\frac{1}{4},\frac{2}{5}},d^{\left(\frac{1}{4}\right)}_{\frac{1}{4},\frac{1}{3}}+d^{\left(\frac{1}{4}\right)}_{\frac{1}{3},\frac{2}{5}}\right\}=\min\{\infty,1+1\}=2,$$

$$d_{\frac{1}{4},\frac{1}{2}}^{\left(\frac{1}{3}\right)} \min\left\{d_{\frac{1}{4},\frac{2}{5}}^{\left(\frac{1}{4}\right)}, d_{\frac{1}{4},\frac{1}{3}}^{\left(\frac{1}{4}\right)} + d_{\frac{1}{3},\frac{2}{5}}^{\left(\frac{1}{4}\right)}\right\} = \min\{\infty, 1+1\} = 2.$$

The distance matrix $L_{\frac{1}{3}}$ is presented below, showing the changes in red.

$$\left(\begin{array}{cccccccccc}
w_{ij} & v_1=\frac{0}{1} & v_2=\frac{1}{4} & v_3=\frac{1}{3} & v_4=\frac{2}{5} & v_5=\frac{1}{2} & v_6=\frac{3}{5} & v_7=\frac{2}{3} & v_8=\frac{3}{4} & v_9=\frac{1}{1} \\
v_1=\frac{0}{1} & 0 & 1 & 1 & 2 & 1 & \infty & \infty & \infty & 1 \\
v_2=\frac{1}{4} & 1 & 0 & 1 & 2 & 2 & \infty & \infty & \infty & \infty \\
v_3=\frac{1}{3} & 1 & 1 & 0 & 1 & 1 & \infty & \infty & \infty & \infty \\
v_4=\frac{2}{5} & 2 & 2 & 1 & 0 & 1 & \infty & \infty & \infty & \infty \\
v_5=\frac{1}{2} & 1 & 2 & 1 & 1 & 0 & 1 & 1 & \infty & 1 \\
v_6=\frac{3}{5} & \infty & \infty & \infty & \infty & 1 & 0 & 1 & \infty & \infty \\
v_7=\frac{2}{3} & \infty & \infty & \infty & \infty & 1 & 1 & 0 & 1 & 1 \\
v_8=\frac{3}{4} & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 & 1 \\
v_9=\frac{1}{1} & 1 & \infty & \infty & \infty & 1 & \infty & 1 & 1 & 0
\end{array}\right)$$

The only length that is analyzable for the intermediate vertex $v_4 = \frac{2}{5}$ is $d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{2}{5}\right)}$. This results in

$$d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{2}{5}\right)} = \min\left\{d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{1}{3}\right)}, d_{\frac{1}{3},\frac{2}{5}}^{\left(\frac{1}{3}\right)} + d_{\frac{2}{5},\frac{1}{2}}^{\left(\frac{1}{3}\right)}\right\} = \min\{1,1+1\} = 1,$$

and thus no update to the distance matrix is required. Therefore the distance matrix $L_{\frac{2}{5}}$ is the same as the distance matrix $L_{\frac{1}{3}}$.

The lengths $d_{\frac{0}{1},\frac{1}{1}}^{\left(\frac{1}{2}\right)}, d_{\frac{0}{1},\frac{2}{3}}^{\left(\frac{1}{2}\right)}, d_{\frac{0}{1},\frac{3}{5}}^{\left(\frac{1}{2}\right)}, d_{\frac{1}{3},\frac{1}{1}}^{\left(\frac{1}{2}\right)}, d_{\frac{1}{3},\frac{2}{3}}^{\left(\frac{1}{2}\right)}, d_{\frac{1}{3},\frac{3}{5}}^{\left(\frac{1}{2}\right)}, d_{\frac{2}{5},\frac{1}{1}}^{\left(\frac{1}{2}\right)}, d_{\frac{2}{5},\frac{2}{3}}^{\left(\frac{1}{2}\right)}, d_{\frac{2}{5},\frac{3}{5}}^{\left(\frac{1}{2}\right)}$ are obtainable via the intermediate vertex $v_5 = \frac{1}{2}$.

$$d_{\frac{0}{1},\frac{1}{1}}^{\left(\frac{1}{2}\right)} = \min\left\{d_{\frac{0}{1},\frac{1}{1}}^{\left(\frac{2}{5}\right)}, d_{\frac{0}{1},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{1}{1}}^{\left(\frac{2}{5}\right)}\right\} = \min\{1,1+1\} = 1,$$

$$d_{\frac{0}{1},\frac{2}{3}}^{\left(\frac{1}{2}\right)} = \min\left\{d_{\frac{0}{1},\frac{2}{3}}^{\left(\frac{2}{5}\right)}, d_{\frac{0}{1},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{0}{1},\frac{3}{5}}^{\left(\frac{1}{2}\right)} = \min\left\{d_{\frac{0}{1},\frac{3}{5}}^{\left(\frac{2}{5}\right)}, d_{\frac{0}{1},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{3}{5}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{1}{3},\frac{1}{1}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{1}{3},\frac{1}{1}}^{\left(\frac{2}{5}\right)}, d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{1}{1}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{1}{3},\frac{2}{3}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{1}{3},\frac{2}{3}}^{\left(\frac{2}{5}\right)}, d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{1}{3},\frac{3}{5}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{1}{3},\frac{3}{5}}^{\left(\frac{2}{5}\right)}, d_{\frac{1}{3},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{3}{5}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{2}{5},\frac{1}{1}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{2}{5},\frac{1}{1}}^{\left(\frac{2}{5}\right)}, d_{\frac{2}{5},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{1}{1}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{2}{5},\frac{2}{3}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{2}{5},\frac{2}{3}}^{\left(\frac{2}{5}\right)}, d_{\frac{2}{5},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d_{\frac{2}{5},\frac{3}{5}}^{\left(\frac{1}{2}\right)} \min\left\{d_{\frac{2}{5},\frac{3}{5}}^{\left(\frac{2}{5}\right)}, d_{\frac{2}{5},\frac{1}{2}}^{\left(\frac{2}{5}\right)} + d_{\frac{1}{2},\frac{3}{5}}^{\left(\frac{2}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2.$$

Hence, the distance matrix $L_{\frac{1}{2}}$ is given as follows.

$$\begin{array}{cc}
w_{ij} & \begin{array}{ccccccccc} v_1=\frac{0}{1} & v_2=\frac{1}{4} & v_3=\frac{1}{3} & v_4=\frac{2}{5} & v_5=\frac{1}{2} & v_6=\frac{3}{5} & v_7=\frac{2}{3} & v_8=\frac{3}{4} & v_9=\frac{1}{1} \end{array} \\
\begin{array}{c} v_1=\frac{0}{1} \\ v_2=\frac{1}{4} \\ v_3=\frac{1}{3} \\ v_4=\frac{2}{5} \\ v_5=\frac{1}{2} \\ v_6=\frac{3}{5} \\ v_7=\frac{2}{3} \\ v_8=\frac{3}{4} \\ v_9=\frac{1}{1} \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 2 & 1 & 2 & 2 & \infty & 1 \\
1 & 0 & 1 & 2 & 2 & \infty & \infty & \infty & \infty \\
1 & 1 & 0 & 1 & 1 & 2 & 2 & \infty & 2 \\
2 & 2 & 1 & 0 & 1 & 2 & 2 & \infty & 2 \\
1 & 2 & 1 & 1 & 0 & 1 & 1 & \infty & 1 \\
2 & \infty & 2 & 2 & 1 & 0 & 1 & \infty & \infty \\
2 & \infty & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\
\infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 & 1 \\
1 & \infty & 2 & 2 & 1 & \infty & 1 & 1 & 0
\end{pmatrix}
\end{array}$$

The only length that is obtained by the intermediate vertex $v_6 = \frac{3}{5}$ is $d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{3}{5}\right)}$:

$$d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{3}{5}\right)} = \min\left\{d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{1}{2}\right)}, d_{\frac{1}{2},\frac{3}{5}}^{\left(\frac{1}{2}\right)} + d_{\frac{3}{5},\frac{2}{3}}^{\left(\frac{1}{2}\right)}\right\} = \min\{1, 1+1\} = 1.$$

The distance matrix $L_{\frac{3}{5}}$ is then the same as the distance matrix $L_{\frac{1}{2}}$.

The lengths $d_{\frac{1}{2},\frac{3}{4}}^{\left(\frac{2}{3}\right)}$, $d_{\frac{1}{2},\frac{1}{1}}^{\left(\frac{2}{3}\right)}$, $d_{\frac{3}{5},\frac{3}{4}}^{\left(\frac{2}{3}\right)}$ and $d_{\frac{3}{5},\frac{1}{1}}^{\left(\frac{2}{3}\right)}$ are achieved via the intermediate vertex $v_7 = \frac{2}{3}$.

$$d_{\frac{1}{2},\frac{3}{4}}^{\left(\frac{2}{3}\right)} = \min\left\{d_{\frac{1}{2},\frac{3}{4}}^{\left(\frac{3}{5}\right)}, d_{\frac{1}{2},\frac{2}{3}}^{\left(\frac{3}{5}\right)} + d_{\frac{2}{3},\frac{3}{4}}^{\left(\frac{3}{5}\right)}\right\} = \min\{\infty, 1+1\} = 2,$$

$$d^{\left(\frac{2}{3}\right)}_{\frac{1}{2},\frac{1}{1}} = \min\left\{d^{\left(\frac{3}{5}\right)}_{\frac{1}{2},\frac{1}{1}}, d^{\left(\frac{3}{5}\right)}_{\frac{1}{2},\frac{2}{3}} + d^{\left(\frac{3}{5}\right)}_{\frac{2}{3},\frac{1}{1}}\right\} = \min\{1,1+1\} = 1,$$

$$d^{\left(\frac{2}{3}\right)}_{\frac{3}{5},\frac{3}{4}} = \min\left\{d^{\left(\frac{3}{5}\right)}_{\frac{3}{5},\frac{3}{4}}, d^{\left(\frac{3}{5}\right)}_{\frac{3}{5},\frac{2}{3}} + d^{\left(\frac{3}{5}\right)}_{\frac{2}{3},\frac{3}{4}}\right\} = \min\{\infty,1+1\} = 2,$$

$$d^{\left(\frac{2}{3}\right)}_{\frac{3}{5},\frac{1}{1}} = \min\left\{d^{\left(\frac{3}{5}\right)}_{\frac{3}{5},\frac{1}{1}}, d^{\left(\frac{3}{5}\right)}_{\frac{3}{5},\frac{2}{3}} + d^{\left(\frac{3}{5}\right)}_{\frac{2}{3},\frac{1}{1}}\right\} = \min\{\infty,1+1\} = 2.$$

So, the distance matrix $L_{\frac{2}{3}}$ is presented as follows:

$$\begin{pmatrix}
w_{ij} & v_1=\frac{0}{1} & v_2=\frac{1}{4} & v_3=\frac{1}{3} & v_4=\frac{2}{5} & v_5=\frac{1}{2} & v_6=\frac{3}{5} & v_7=\frac{2}{3} & v_8=\frac{3}{4} & v_9=\frac{1}{1} \\
v_1=\frac{0}{1} & 0 & 1 & 1 & 2 & 1 & 2 & 2 & \infty & 1 \\
v_2=\frac{1}{4} & 1 & 0 & 1 & 2 & 2 & \infty & \infty & \infty & \infty \\
v_3=\frac{1}{3} & 1 & 1 & 0 & 1 & 1 & 2 & 2 & \infty & 2 \\
v_4=\frac{2}{5} & 2 & 2 & 1 & 0 & 1 & 2 & 2 & \infty & 2 \\
v_5=\frac{1}{2} & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 1 \\
v_6=\frac{3}{5} & 2 & \infty & 2 & 2 & 1 & 0 & 1 & 2 & 2 \\
v_7=\frac{2}{3} & 2 & \infty & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\
v_8=\frac{3}{4} & \infty & \infty & \infty & \infty & 2 & 2 & 1 & 0 & 1 \\
v_9=\frac{1}{1} & 1 & \infty & 2 & 2 & 1 & 2 & 1 & 1 & 0
\end{pmatrix}$$

The only length that is achievable for the intermediate vertex $v_8 = \frac{3}{4}$ is $d^{\left(\frac{3}{4}\right)}_{\frac{2}{3},\frac{1}{1}}$.

$$d^{\left(\frac{3}{4}\right)}_{\frac{2}{3},\frac{1}{1}} = \min\left\{d^{\left(\frac{2}{3}\right)}_{\frac{2}{3},\frac{1}{1}}, d^{\left(\frac{2}{3}\right)}_{\frac{2}{3},\frac{3}{4}} + d^{\left(\frac{2}{3}\right)}_{\frac{3}{4},\frac{1}{1}}\right\} = \min\{1,1+1\} = 1.$$

This means that there is no need to update the distance matrix $L_{\frac{2}{3}}$.

The values of ∞ remaining in the distance matrix are found as the following:

$$d^{\left(\frac{2}{3}\right)}_{\frac{0}{1},\frac{3}{4}} = \min\left\{d^{\left(\frac{3}{5}\right)}_{\frac{0}{1},\frac{3}{4}}, d^{\left(\frac{3}{5}\right)}_{\frac{0}{1},\frac{2}{3}} + d^{\left(\frac{3}{5}\right)}_{\frac{2}{3},\frac{3}{4}}\right\} = \min\{\infty,2+1\} = 3,$$

$$d^{\left(\frac{1}{2}\right)}_{\frac{1}{4},\frac{3}{5}} = \min\left\{d^{\left(\frac{2}{5}\right)}_{\frac{1}{4},\frac{3}{5}}, d^{\left(\frac{2}{5}\right)}_{\frac{1}{4},\frac{1}{2}} + d^{\left(\frac{2}{5}\right)}_{\frac{1}{2},\frac{3}{5}}\right\} = \min\{\infty,2+1\} = 3,$$

$$d^{\left(\frac{3}{5}\right)}_{\frac{1}{4},\frac{2}{3}} = \min\left\{d^{\left(\frac{1}{2}\right)}_{\frac{1}{4},\frac{2}{3}}, d^{\left(\frac{1}{2}\right)}_{\frac{1}{4},\frac{1}{2}} + d^{\left(\frac{1}{2}\right)}_{\frac{1}{2},\frac{2}{3}}\right\} = \min\{3,3+1\} = 3,$$

$$d_{\frac{1}{4},\frac{3}{4}}^{\left(\frac{2}{3}\right)} = \min\left\{d_{\frac{1}{4},\frac{3}{4}}^{\left(\frac{3}{5}\right)}, d_{\frac{1}{4},\frac{2}{3}}^{\left(\frac{3}{5}\right)} + d_{\frac{2}{3},\frac{3}{4}}^{\left(\frac{3}{5}\right)}\right\} = \min\{\infty, 3+1\} = 4,$$

$$d_{\frac{1}{4},\frac{1}{1}}^{\left(\frac{3}{4}\right)} = \min\left\{d_{\frac{1}{4},\frac{1}{1}}^{\left(\frac{2}{3}\right)}, d_{\frac{1}{4},\frac{2}{3}}^{\left(\frac{2}{3}\right)} + d_{\frac{2}{3},\frac{1}{1}}^{\left(\frac{2}{3}\right)}\right\} = \min\{3,4+1\} = 3,$$

$$d_{\frac{1}{3},\frac{3}{4}}^{\left(\frac{2}{3}\right)} = \min\left\{d_{\frac{1}{3},\frac{3}{4}}^{\left(\frac{3}{5}\right)}, d_{\frac{1}{3},\frac{2}{3}}^{\left(\frac{3}{5}\right)} + d_{\frac{2}{3},\frac{3}{4}}^{\left(\frac{3}{5}\right)}\right\} = \min\{\infty, 2+1\} = 3,$$

$$d_{\frac{2}{5},\frac{3}{4}}^{\left(\frac{2}{3}\right)} = \min\left\{d_{\frac{2}{5},\frac{3}{4}}^{\left(\frac{3}{5}\right)}, d_{\frac{2}{5},\frac{2}{3}}^{\left(\frac{3}{5}\right)} + d_{\frac{2}{3},\frac{3}{4}}^{\left(\frac{3}{5}\right)}\right\} = \min\{\infty, 2+1\} = 3.$$

Consequently, the distance matrix formed is the following:

$$\begin{array}{cc}
w_{ij} & \begin{array}{ccccccccc} v_1=\frac{0}{1} & v_2=\frac{1}{4} & v_3=\frac{1}{3} & v_4=\frac{2}{5} & v_5=\frac{1}{2} & v_6=\frac{3}{5} & v_7=\frac{2}{3} & v_8=\frac{3}{4} & v_9=\frac{1}{1} \end{array} \\
\begin{array}{c} v_1=\frac{0}{1} \\ v_2=\frac{1}{4} \\ v_3=\frac{1}{3} \\ v_4=\frac{2}{5} \\ v_5=\frac{1}{2} \\ v_6=\frac{3}{5} \\ v_7=\frac{2}{3} \\ v_8=\frac{3}{4} \\ v_9=\frac{1}{1} \end{array} &
\begin{pmatrix}
0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1 \\
1 & 0 & 1 & 2 & 2 & 3 & 3 & 4 & 3 \\
1 & 1 & 0 & 1 & 1 & 2 & 2 & 3 & 2 \\
2 & 2 & 1 & 0 & 1 & 2 & 2 & 3 & 2 \\
1 & 2 & 1 & 1 & 0 & 1 & 1 & 2 & 1 \\
2 & 3 & 2 & 2 & 1 & 0 & 1 & 2 & 2 \\
2 & 3 & 2 & 2 & 1 & 1 & 0 & 1 & 1 \\
3 & 4 & 3 & 3 & 2 & 2 & 1 & 0 & 1 \\
1 & 3 & 2 & 2 & 1 & 2 & 1 & 1 & 0
\end{pmatrix}
\end{array}$$

4. Conclusion

The shortest lengths between all vertices in a Farey graph were determined by analyzing the Farey graph defined in hyperbolic geometry, which has been intensely discussed in the relevant literature, with the Floyd-Warshall algorithm. The lengths between vertices in the Farey graph are associated by the number of steps, and the step condition is given by * as below:

* The number of steps between neighboring vertices is 1 regardless of the distance, based on Lemma 2.4.1. The step number is 2 if the two edges have a common vertex, provided that the middle vertex is not obtained with the mediant. The number of steps increases based on this condition. Therefore, the length of a shortest path between two vertices is defined as follows:

$$d_{v_i,v_j}^{(v_k)} = \text{Number of steps between } \mathrm{v_i} \text{ and } \mathrm{v_j} \text{ depending on the condition } *$$

by the Floyd-Warshall algorithm. As an example, the shortest path between $\frac{1}{4}$ and $\frac{3}{4}$ is 4 as the number of steps is 4 with the condition * as in Figure 3.

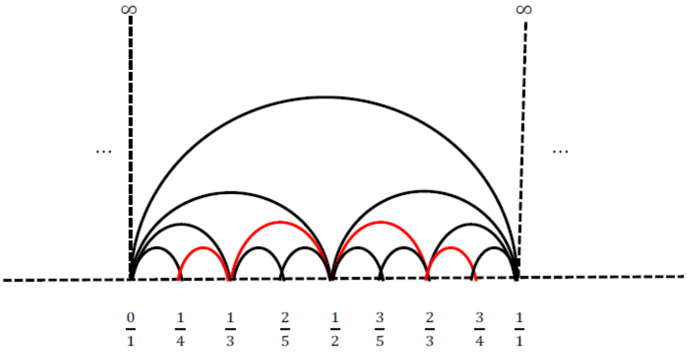

Figure 3. 4 step path from $\frac{1}{4}$ to $\frac{3}{4}$

In addition, the results obtained with the Floyd-Warshall algorithm are very useful because Floyd-Warshall algorithm give the results in a single table, but they are obtained separately with the Dijkstra and Bellman-Ford algorithms for each vertex.

Authorship Contribution Statement

İ. Gökcan: Conceptualization, Methodology, Validation, Formal Analysis, Investigation, Resource, Data Curation, Original Draft Writing, Review and Editing, Visualization

Declaration of competing interest

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Ethics Committee Approval and/or Informed Consent Information

As the author of this study, I declare that I do not have any ethics committee approval and/or informed consent statement.

References

[1] E. W. Dijkstra, “A note on two problems in connexion with graphs”, *Numerische Mathematik*, 1(1), 269–271, 1959.

[2] R. Bellman, “On a routing problem”, *Quarterly Applied Mathematics*, 16, 87– 90, 1958.

[3] P. D. Whiting and J.A. Hillier, “A method for finding the shortest route trough a road network”, *Operations Research Quarterly*, 11, 37-40, 1960.

[4] P. E. Hart, N. J. Nilsson and B. Raphael, “A formal basis for the heuristic determination of minimum cost paths”, *IEEE Transactions on Systems Science and Cybernetics*, 4(2), 100–107, 1968.

[5] P. E. Hart, N. J. Nilsson and B. Raphael, “Correction to "A formal basis for the heuristic determination of minimum cost paths"", *SIGART Newsletter*, 28–29, 1972.

[6] S. Dreyfus, “An appraisal of some shortest-path algorithms”, *Operations Research*, 17, 395-412, 1969.

[7] T. Veerarajan, Discrete Mathematics, with Graph Theory and Combinatorics, Mc Graw Hill India, 288 pages, 2006.

[8] R. Diestel, Graph Theory, New York, Springer-Verlag Heidelberg, 322 pages, 2005.

[9] K. Ruohonen, Graph Theory (Translation by Janne Tamminen, Kung-Chung Lee and Robert Piché), 114 pages, 2006.

[10] C. C. Sims, “Graphs and finite permutation groups”, *Mathematische Zeitschrift*, 95,76-86, 1967.

[11] G. A. Jones, D. Singerman and K. Wicks, “The modular group and generalized farey graphs”, *London Mathematical Society Lecture Note Series*, 160, 316-338, 1991.

[12] A. H. Deger, M. Besenk ve B. O. Guler, “On suborbital graphs and related continued fractions”, *Applied Mathematics and Computation*, 218(3), 746-750, 2011.

[13] A. H. Deger, “Relationships with the Fibonacci numbers and the special vertices of the suborbital graphs”, *Gümüşhane University Journal of Science and Technology*, 7,168−180, 2017.
[14] A. H. Deger, “Vertices of paths of minimal lengths on suborbital graphs”, *Filomat*, 31(4), 913-923, 2017.
[15] Ü. Akbaba, A. H. Değer and T. Berber, “An Algorıthm For The Vertıces On The Paths Of Mınımal Length In The Suborbıtal Graphs”, Proceedings of the 6th International Conference on Computational Mathematics and Engineering Sciences (CMES-2022) , 1(1), 250-255, 2022.
[16] İ. Gökcan, “Application of Dijkstra Algorithm with Sequence F_m in $F_{u,N}$”, in Research&Reviews in Science and Mathematics-1, Gece Publishing, 2021, pp. 1-19.
[17] P. M. Neumann, Finite Permutation Groups, Edge Coloured Graphs and Matrices, Topics in Group Theory and Computation, Curran M.P.J. (eds.), London, Academic Press, , 118 pages, 1977.
[18] T. Tsukuzu, Finite Groups and Finite Geometries, Cambridge, Cambridge University Press, , 328 pages, 1982.