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Point estimation of the process capability index S'_{pmk} for the generalized KM exponential model with applications

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Abstract. In this paper, we conduct statistical inferences on the process capability index S'_{pmk} for a lifetime distribution called the generalized KM exponential distribution. Some statistical properties of the distribution are reported, accompanied by figures that illustrate its shape characteristics based on the probability density function and the hazard rate function. The process capability index S'_{pmk} is used to evaluate processes that deviate from a non-normal distribution. The maximum likelihood estimate for S'_{pmk} is obtained using the invariance property of the maximum likelihood estimator. Furthermore, point estimates for S'_{pmk} are derived based on least squares, weighted least squares, Anderson-Darling, and Cramér-von Mises estimators. Furthermore, two real data analyses are performed to assess the applicability of the S'_{pmk} process capability index to the generalized KM exponential distribution.

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1. Introduction

In statistics, various distributions are developed, particularly lifetime distributions. These distributions play a crucial role in the analysis of survival rates, event durations, and reliability, enabling the modeling and interpretation of time-to-event data. Lifetime distributions find extensive application in fields such as engineering, medicine, and environmental science, where assessing the probability of survival or failure over time is vital. By reflecting how various data types behave across time, these distributions offer valuable insights into factors influencing lifetime and reliability, making them essential for reliability analysis, risk assessment, and predictive modeling. The literature provides numerous lifetime distributions for modeling survival data, including exponential, Weibull, and gamma distributions, alongside many newly developed alternative distributions. For insights into some lifetime distributions, refer to [3,10,12]. When constructing lifetime distributions, the aim is to propose more flexible distributions with enhanced data modeling capabilities, providing a superior fit for survival data.

Quality control charts determine the presence of variation in a quality control process. The process capability index (PCI) evaluates the extent to which products obtained from the production process meet customer requirements and aim to measure the overall capability of the process to fulfill specific criteria. While numerous indices, such as C_p [5], C_{pk} [6], C_{pm} [14], and C_{pmk} [2], are utilized when measurements follow a normal distribution, it is important to note that the distribution of measurements does not always conform to normality. Several PCI are developed for such cases. These indices may include S_{pmk} [1] and C_{pyk} [11]. Aside from these two indices, an index referred to as S'_{pmk} is proposed by [4] based on S_{pmk}. The S_{pmk} index, introduced by [1], was one of the first attempts to extend the traditional process capability indices to handle non-normally distributed data. The S_{pmk} index uses the

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empirical distribution of data to adapt the calculation of process capability, offering a more flexible and accurate assessment when data deviate from normality. The S_{pmk} index evaluates the performance of the process concerning both the upper and lower specification limits and takes into account the asymmetry or skewness of the data. One of the key developments in this line of research is the work of [4], who proposed a new version of the S_{pmk} index, denoted as S'_{pmk}. This updated index is derived based on a refined methodology that improves the robustness of the original S_{pmk} index, especially in terms of estimating process capability in the presence of extreme values or outliers in the data. The S'_{pmk} index is shown to provide more stable and accurate estimates compared to its predecessors, particularly when applied to non-normal data.

Since measurement data obtained in practice often do not follow a normal distribution, it is important to recognize the significant value of new indices developed for non-normally distributed measurements. In this study, the S'_{pmk} index is adapted to the Generalized KM Exponential (GKME) distribution introduced by [7].

The remainder of the article is organized as follows: Section 2 provides the probability density function (pdf), cumulative distribution function (cdf), hazard rate function (hrf), and quantile function (qf) related to the GKME distribution. The basic information on index S_{pmk} and S'_{pmk} are presented in general form and the S'_{pmk} is examined based on the GKME distribution in Section 3. In Section 4, point estimation is performed S'_{pmk} based on the GKME distribution. The performances of the estimators are evaluated with a Monte Carlo simulation in Section 5. In Section 6, two real data analyses are conducted to demonstrate the applicability of the S'_{pmk} index adapted to the GKME distribution. The article concludes with the conclusions in Section 7.

2. Generalized KM Exponential Distribution

In this section, the GKME distribution introduced by [7], is presented. The pdf, cdf, hrf, and qf are given, respectively, by

$$f(x) = \frac{e}{e-1}\alpha\theta e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\left(1-e^{-\theta x}\right)^{\alpha}}, x>0, \alpha, \theta>0,$$

$$F(x) = \frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x}\right)^{\alpha}}\right],$$

$$h(x) = \frac{e\alpha\theta}{\left(e^{1-\left(1-e^{-\theta x}\right)^{\alpha}}-1\right)}e^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1}e^{-\left(1-e^{-\theta x}\right)^{\alpha}},$$

and

$$Q(p) = \frac{-1}{\theta}\log\left[1-\left(-\log\left(1-p(e-1)\right)\right)^{\frac{1}{\alpha}}\right].$$

Moreover, the survival function (sf) of the GKME distribution can also be easily obtained as $S(x) = 1-F(x)$, where F is cdf in Equation (2). The plots of pdf and hrf for various parameter choices are presented in Figures 1 and 2. It can be observed from Figure 1 that the pdf of the GKME distribution exhibits shapes characterized by decreasing, increasing, and increasing-decreasing patterns. Figure 2 illustrates that the hrf of the GKME distribution also follows decreasing, increasing, and increasing-decreasing trends, demonstrating its flexibility in modeling different failure rate behaviors.

3. The S'_{pmk} Based on Generalized KM Exponential Distribution

In this section, the PCIs S_{pmk} and S'_{pmk} are given and then the S'_{pmk} are introduced based on the GKME distribution. Firstly, the S_{pmk} index suggested by [1], is given as

$$S_{pmk} = \frac{\Phi^{-1}\left(\frac{1+F(USL)-F(LSL)}{2}\right)}{3\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}} = \frac{\Phi^{-1}\left(1-\frac{P}{2}\right)}{3\sqrt{1+\left(\frac{\mu-T}{\sigma}\right)^2}},$$

where Φ^{-1} is the inverse cdf of the standard normal distribution, $F(x)$ is the cdf of the process and P is the percentage of nonconforming items, μ is process mean, σ is the standard deviation of the process. Furthermore, LSL, USL, and T are the lower and upper specification limits and target values of the

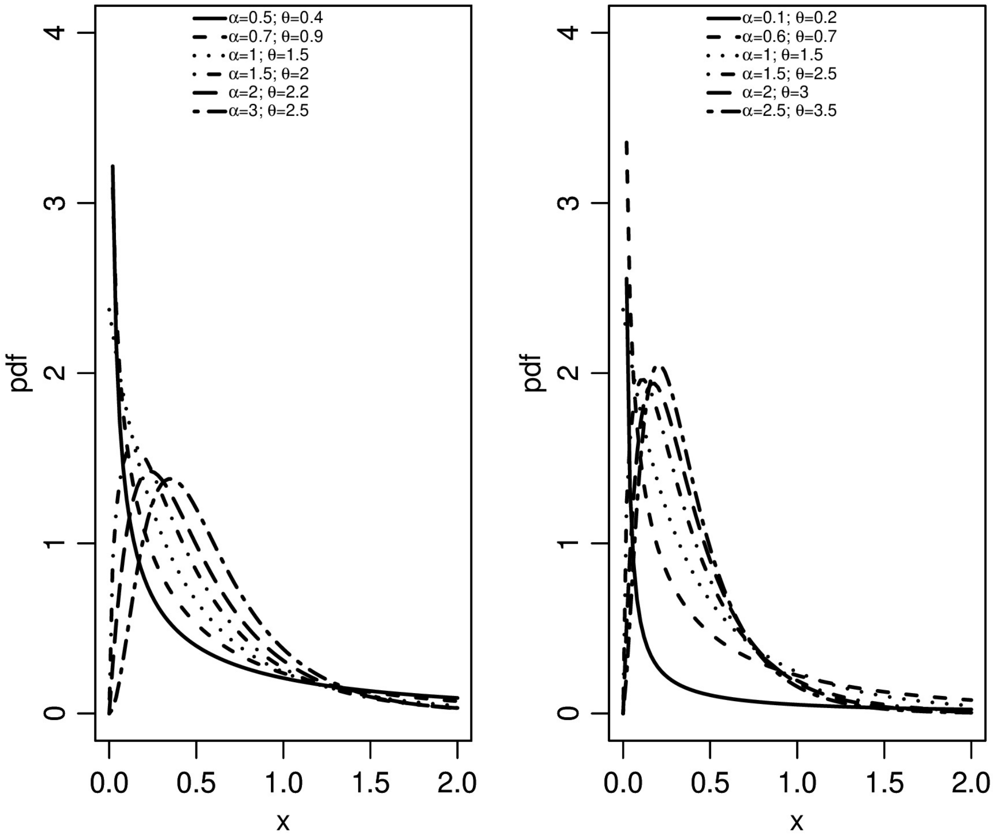

FIGURE 1. The pdf plots of GKME for selected parameter values

process, respectively. S_{pmk} is used for non-normal distributions and considers departures from the target value using the quadratic loss function. The PCI S'_{pmk} was introduced by [4] and defined as

$$S^{'}_{pmk} = \frac{\Phi^{-1}\left(\frac{1+F(USL)-F(LSL)}{2}\right)}{3\sqrt{1+\frac{2}{\sigma^2}\frac{e^{\gamma(\mu-T)}-\gamma(\mu-T)-1}{\gamma^2}}},$$

where γ is a constant that determines the degree of asymmetry.

Now the PCI S'_{pmk} is introduced based on GKME as follow

$$S^{'}_{pmk} = \frac{\Phi^{-1}\left(1-\frac{P}{2}\right)}{3\sqrt{1+\frac{2}{\sigma^2}\frac{e^{\gamma(\mu-T)}-\gamma(\mu-T)-1}{\gamma^2}}},$$

where $P = 1-\left[\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta(USL)}\right)^{\alpha}}\right]-\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta(LSL)}\right)^{\alpha}}\right]\right]$ is the exact proportion of non-conformity. In the following section, several point estimators for S'_{pmk} based on GKME distribution are introduced.

4. POINT ESTIMATION

In this section, five estimators, such as maximum likelihood estimate (MLE), least squares estimate (LSE), weighted least squares estimate (WLSE), Anderson-Darling estimate (ADE), and Cramer-von Mises estimate (CvME) are studied to estimate the unknown parameters α and θ of GKME distribution and the corresponding estimator of $S^{'}_{pmk}$. Let $X_1, X_2, ..., X_n$ be a random sample from the GKME(α, θ)

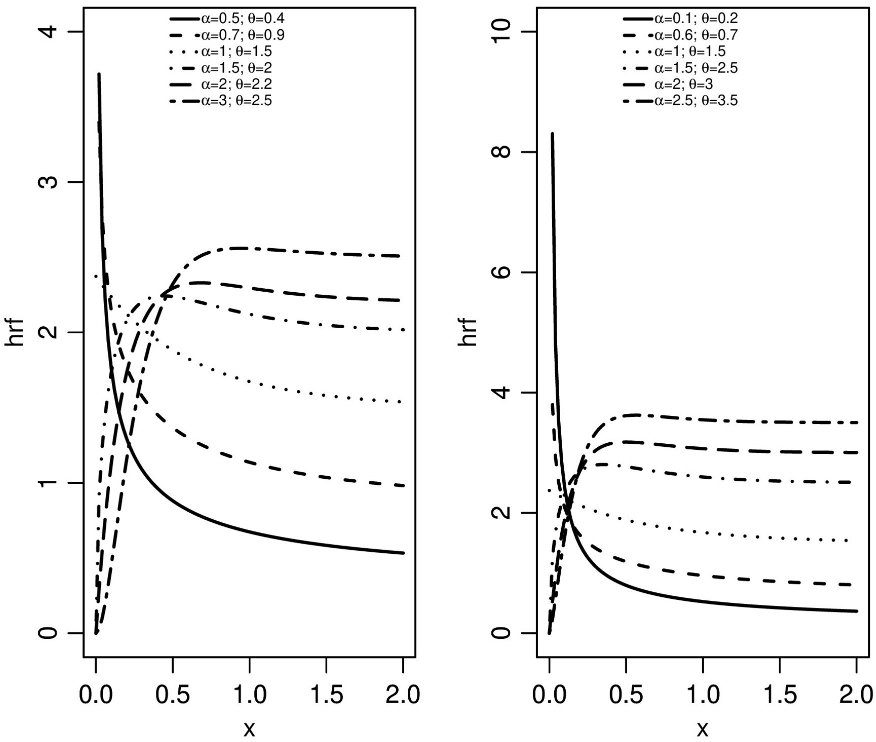

Figure 2. The hrf plots of GKME for selected parameter values

distribution and $x_1, x_2, ..., x_n$ is observed values of the sample. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics based on sample $X_1, X_2, ..., X_n$ with the realization $x_{(1)}, x_{(2)}, ..., x_{(n)}$. Then, the likelihood and log-likelihood function can be written by

$$L(\Xi) = \prod_{i=1}^{n} \frac{e}{e-1} \alpha\theta e^{-\theta x_i} \left(1 - e^{-\theta x_i}\right)^{\alpha-1} e^{-\left(1-e^{-\theta x_i}\right)^{\alpha}}$$

and

$$\begin{aligned} \ell(\Xi) \quad \approx \quad & n\log(\alpha) + n\log(\theta) - \theta\sum_{i=1}^{n} x_i \\ & +(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\theta x_i}\right) - \sum_{i=1}^{n}\left(1-e^{-\theta x_i}\right)^{\alpha}, \end{aligned}$$

where $\Xi = (\alpha, \theta)$. The MLE $\widehat{\Xi} = \left(\widehat{\alpha}, \widehat{\theta}\right)$ for Ξ is obtained by

$$\widehat{\Xi} = \underset{(\alpha,\theta)\in(0,\infty)\times(0,\infty)}{\arg\max}\, \ell(\Xi) \quad .$$

Then $\widehat{\Xi}$ is obtained by solving the simultaneously the following two likelihood equations

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\theta x_i}\right) - \sum_{i=1}^{n}\log\left(1-e^{-\theta x_i}\right)\left(1-e^{-\theta x_i}\right)^{\alpha}$$

and

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{n}\frac{x_i e^{-\theta x_i}}{1-e^{-\theta x_i}} - \alpha\sum_{i=1}^{n} x_i e^{-\theta x_i}\left(1-e^{-\theta x_i}\right)^{\alpha-1}.$$

The $\widehat{\Xi}$ are not closed forms. However, some numerical methods can be used to determine them with a high level of numerical precision. Let us tackle the following four functions to obtain other estimators for Ξ as

$$\Psi_{LSE}\left(\Xi\right)=\sum_{i=1}^{n}\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x_{(i)}}\right)^{\alpha}}\right]-\frac{i}{n+1}\right)^{2}, \tag{1}$$

$$\Psi_{WLSE}\left(\Xi\right)=\sum_{i=1}^{n}\frac{\left(n+2\right)\left(n+1\right)^{2}}{i\left(n-i+1\right)}\left(\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x_{(i)}}\right)^{\alpha}}\right]-\frac{i}{n+1}\right)^{2}, \tag{2}$$

$$\begin{aligned}\Psi_{ADE}\left(\Xi\right) &= -n-\sum_{i=1}^{n}\frac{2i-1}{n}\left[\log\left\{\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x_{(i)}}\right)^{\alpha}}\right]\right\}\right.\\ &\quad\left.+\log\left\{1-\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x_{(n+i-1)}}\right)^{\alpha}}\right]\right\}\right],\end{aligned} \tag{3}$$

and

$$\Psi_{CvME}\left(\Xi\right)=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\theta x_{(i)}}\right)^{\alpha}}\right]-\frac{2i-1}{2n}\right]^{2}, \tag{4}$$

Then the LSE, WLSE, ADE, and CvME are obtained by minimizing Equations (1)-(4), respectively, as

$$\begin{aligned}\widehat{\Xi}_2 &= \arg\min\Psi_{LSE}\left(\Xi\right),\\ (\alpha,\theta) &\in (0,\infty)\times(0,\infty)\end{aligned}$$

$$\begin{aligned}\widehat{\Xi}_3 &= \arg\min\Psi_{WLSE}\left(\Xi\right),\\ (\alpha,\theta) &\in (0,\infty)\times(0,\infty)\end{aligned}$$

$$\begin{aligned}\widehat{\Xi}_4 &= \arg\min\Psi_{ADE}\left(\Xi\right),\\ (\alpha,\theta) &\in (0,\infty)\times(0,\infty)\end{aligned}$$

and

$$\begin{aligned}\widehat{\Xi}_5 &= \arg\min\Psi_{CvME}\left(\Xi\right).\\ (\alpha,\theta) &\in (0,\infty)\times(0,\infty)\end{aligned}$$

Some numerical methods, such as Nelder-Mead or BFGS, can solve all maximizing and minimization problems. The MLE, LSE, WLSE, ADE, and CvME of $S_{pmk}^{'}$ can be expressed respectively, by

$$S_{pmk}^{'MLE}=\frac{\Phi^{-1}\left(1-\frac{\widehat{p}_{MLE}}{2}\right)}{3\sqrt{1+\frac{2}{\widehat{\sigma}^{2}}\frac{e^{\gamma(\widehat{\mu}-T)}-\gamma(\widehat{\mu}-T)-1}{\gamma^{2}}}},$$

where

$$\begin{aligned}\widehat{P}_{MLE} &= 1-\left[\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\widehat{\theta}_{MLE}(USL)}\right)^{\widehat{\alpha}_{MLE}}}\right]\right.\\ &\quad\left.-\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\widehat{\theta}_{MLE}(LSL)}\right)^{\widehat{\alpha}_{MLE}}}\right]\right],\end{aligned}$$

$$S_{pmk}^{'LSE}=\frac{\Phi^{-1}\left(1-\frac{\widehat{p}_{LSE}}{2}\right)}{3\sqrt{1+\frac{2}{\widehat{\sigma}^{2}}\frac{e^{\gamma(\widehat{\mu}-T)}-\gamma(\widehat{\mu}-T)-1}{\gamma^{2}}}},$$

where

$$\widehat{P}_{LSE} = 1-\left[\frac{e}{e-1}\left[1-e^{-\left(1-e^{-\widehat{\theta}_{LSE}(USL)}\right)^{\widehat{\alpha}_{LSE}}}\right]\right.$$

$$- \frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{LSE}(LSL)}\right)^{\widehat{\alpha}_{LSE}}}\right]\Bigg],$$

$$S_{pmk}^{'WLSE} = \frac{\Phi^{-1}\left(1 - \frac{\widehat{p}_{WLSE}}{2}\right)}{3\sqrt{1 + \frac{2}{\widehat{\sigma}^2}\frac{e^{\gamma(\widehat{\mu}-T)} - \gamma(\widehat{\mu}-T) - 1}{\gamma^2}}},$$

where

$$\widehat{P}_{WLSE} = 1 - \left[\frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{WLSE}(USL)}\right)^{\widehat{\alpha}_{WLSE}}}\right]\right.$$
$$\left. - \frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{WLSE}(LSL)}\right)^{\widehat{\alpha}_{WLSE}}}\right]\right],$$

$$S_{pmk}^{'ADE} = \frac{\Phi^{-1}\left(1 - \frac{\widehat{p}_{ADE}}{2}\right)}{3\sqrt{1 + \frac{2}{\widehat{\sigma}^2}\frac{e^{\gamma(\widehat{\mu}-T)} - \gamma(\widehat{\mu}-T) - 1}{\gamma^2}}},$$

where

$$\widehat{P}_{ADE} = 1 - \left[\frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{ADE}(USL)}\right)^{\widehat{\alpha}_{ADE}}}\right]\right.$$
$$\left. - \frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{ADE}(LSL)}\right)^{\widehat{\alpha}_{ADE}}}\right]\right],$$

and

$$S_{pmk}^{'CvME} = \frac{\Phi^{-1}\left(1 - \frac{\widehat{p}_{CvME}}{2}\right)}{3\sqrt{1 + \frac{2}{\widehat{\sigma}^2}\frac{e^{\gamma(\widehat{\mu}-T)} - \gamma(\widehat{\mu}-T) - 1}{\gamma^2}}},$$

where

$$\widehat{P}_{CvME} = 1 - \left[\frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{CvME}(USL)}\right)^{\widehat{\alpha}_{CvME}}}\right]\right.$$
$$\left. - \frac{e}{e-1}\left[1 - e^{-\left(1-e^{-\widehat{\theta}_{CvME}(LSL)}\right)^{\widehat{\alpha}_{CvME}}}\right]\right].$$

In all estimators of S'_{pmk}, μ and σ are estimated by the sample mean and sample standard deviation, respectively. In the following, a simulation study is conducted to evaluate the performance of these estimators for S'_{pmk}.

5. Monte Carlo Simulation Study

In this section, the bias, mean squared error (MSE), average absolute bias (ABB), and mean relative error (MRE) of MLE, LSE, WLSE, ADE, and CvME values of S'_{pmk} based on GKME distribution are estimated based on 5000 trials. The sample sizes are selected as $n = 15, 25, 50, 100, 250$ and 500. The performance of the point estimators for S'_{pmk} is assessed in both on-target and off-target scenarios using bias, ABB, MSE, and MRE criteria. While the on-target scenario target values correspond to the true process mean, the off-target scenarios represent cases where varying levels of deviation from the true process mean are introduced ($1 \times \sigma$, $0.5 \times \sigma$, and $0.1 \times \sigma$). For the parameter values (2, 0.5), the MLE method demonstrates the best performance across all criteria under the on-target scenario. In the off-target scenario with $1 \times \sigma$, the best estimation methods according to the bias, MSE, ABB, and MRE criteria are MLE and CvME. In the off-target scenario with $0.5 \times \sigma$, MLE and LSE emerge as the best estimation methods based on all criteria. For the off-target scenario with $0.1 \times \sigma$, MLE again shows the best performance according to the bias, MSE, ABB, and MRE criteria. Similarly, for the parameter values (3, 1.5), MLE, and ADE consistently demonstrate optimal performance across the bias, MSE, ABB, and MRE criteria for the on-target scenario as well as for the off-target scenarios with $1 \times \sigma$, $0.5 \times \sigma$, and $0.1 \times \sigma$. Overall, the results suggest that MLE possesses broad applicability for estimating S'_{pmk} under the GKME distribution and generally outperforms the other estimation methods in various scenarios.

TABLE 1. Simulation results for LSL=0.3950 and USL=6.3356 and $\Xi = (2, 0.5)$

		$T = 2.5044$ (on target)					$T = 4.6530$ (off target with $1 \times \sigma$)				
Bias	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0064	-0.0426	-0.0374	-0.0236	0.0066	0.0076	-0.0187	-0.0164	-0.0091	0.0092
	25	0.0459	0.0154	0.0209	0.0276	0.0455	-0.0487	-0.0661	-0.0630	-0.0594	-0.0487
$\gamma = -0.5$	50	-0.0005	-0.0155	-0.0112	-0.0094	0.0000	-0.0546	-0.0633	-0.0607	-0.0596	-0.0543
	100	0.0065	-0.0007	0.0019	0.0021	0.0072	-0.0278	-0.0323	-0.0306	-0.0304	-0.0277
	250	0.0001	-0.0030	-0.0016	-0.0017	0.0001	-0.0592	-0.0609	-0.0601	-0.0602	-0.0590
	500	0.0010	-0.0008	0.0002	0.0000	0.0008	-0.0491	-0.0500	-0.0495	-0.0496	-0.0490
	15	0.0023	-0.0470	-0.0410	-0.0279	0.0021	-0.1207	-0.1547	-0.1511	-0.1413	-0.1207
	25	0.0159	-0.0127	-0.0077	-0.0018	0.0180	-0.1090	-0.1299	-0.1263	-0.1217	-0.1084
$\gamma = 0.5$	50	0.0019	-0.0126	-0.0082	-0.0068	0.0030	0.0679	0.0566	0.0598	0.0612	0.0677
	100	0.0117	0.0040	0.0071	0.0072	0.0119	0.0449	0.0392	0.0412	0.0415	0.0448
	250	0.0004	-0.0026	-0.0012	-0.0014	0.0006	0.0068	0.0048	0.0058	0.0056	0.0071
	500	-0.0001	-0.0017	-0.0008	-0.0010	-0.0001	-0.0146	-0.0157	-0.0151	-0.0153	-0.0145
	15	0.0348	-0.0122	-0.0069	0.0049	0.0373	-0.0221	-0.0610	-0.0569	-0.0464	-0.0213
	25	0.0108	-0.0208	-0.0151	-0.0085	0.0095	-0.0467	-0.0702	-0.0662	-0.0612	-0.0453
$\gamma = 2$	50	0.0037	-0.0104	-0.0067	-0.0050	0.0052	-0.0128	-0.0254	-0.0217	-0.0204	-0.0126
	100	0.0041	-0.0036	-0.0007	-0.0005	0.0043	0.0349	0.0284	0.0309	0.0310	0.0349
	250	0.0038	0.0003	0.0020	0.0018	0.0035	0.0075	0.0049	0.0063	0.0060	0.0075
	500	0.0004	-0.0010	-0.0003	-0.0005	0.0006	0.0046	0.0035	0.0041	0.0039	0.0049
MSE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0110	0.0155	0.0140	0.0111	0.0170	0.0072	0.0084	0.0077	0.0069	0.0100
	25	0.0080	0.0082	0.0073	0.0069	0.0112	0.0072	0.0101	0.0092	0.0085	0.0089
$\gamma = -0.5$	50	0.0029	0.0044	0.0036	0.0033	0.0045	0.0056	0.0072	0.0066	0.0064	0.0063
	100	0.0014	0.0021	0.0017	0.0016	0.0022	0.0022	0.0028	0.0025	0.0025	0.0026
	250	0.0006	0.0009	0.0007	0.0007	0.0009	0.0041	0.0044	0.0043	0.0043	0.0042
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0027	0.0029	0.0028	0.0028	0.0028
	15	0.0111	0.0161	0.0147	0.0116	0.0172	0.0199	0.0307	0.0289	0.0252	0.0229
	25	0.0064	0.0086	0.0074	0.0063	0.0100	0.0152	0.0218	0.0202	0.0184	0.0174
$\gamma = 0.5$	50	0.0028	0.0043	0.0035	0.0032	0.0044	0.0064	0.0058	0.0058	0.0058	0.0074
	100	0.0016	0.0021	0.0018	0.0017	0.0023	0.0029	0.0029	0.0028	0.0028	0.0034
	250	0.0006	0.0008	0.0007	0.0007	0.0008	0.0004	0.0006	0.0005	0.0005	0.0006
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0004	0.0005	0.0005	0.0005	0.0005
	15	0.0127	0.0145	0.0133	0.0108	0.0190	0.0052	0.0112	0.0097	0.0069	0.0097
	25	0.0069	0.0095	0.0081	0.0070	0.0104	0.0050	0.0097	0.0082	0.0069	0.0075
$\gamma = 2$	50	0.0030	0.0044	0.0036	0.0033	0.0046	0.0016	0.0032	0.0024	0.0021	0.0029
	100	0.0015	0.0021	0.0017	0.0016	0.0022	0.0019	0.0020	0.0018	0.0018	0.0025
	250	0.0006	0.0009	0.0007	0.0007	0.0009	0.0003	0.0005	0.0004	0.0004	0.0006
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0002	0.0003	0.0002	0.0002	0.0003
ABB	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0823	0.1018	0.0956	0.0849	0.1003	0.0706	0.0728	0.0698	0.0680	0.0776
	25	0.0691	0.0697	0.0653	0.0639	0.0800	0.0680	0.0844	0.0797	0.0755	0.0777
$\gamma = -0.5$	50	0.0428	0.0539	0.0486	0.0463	0.0532	0.0615	0.0720	0.0686	0.0669	0.0666
	100	0.0300	0.0364	0.0326	0.0319	0.0372	0.0377	0.0438	0.0412	0.0407	0.0418
	250	0.0190	0.0234	0.0206	0.0205	0.0233	0.0592	0.0612	0.0603	0.0604	0.0594
	500	0.0136	0.0164	0.0145	0.0144	0.0164	0.0491	0.0501	0.0496	0.0496	0.0491
	15	0.0830	0.1047	0.0991	0.0878	0.1017	0.1221	0.1600	0.1555	0.1432	0.1331
	25	0.0622	0.0735	0.0680	0.0631	0.0762	0.1097	0.1344	0.1300	0.1231	0.1168
$\gamma = 0.5$	50	0.0422	0.0527	0.0473	0.0450	0.0524	0.0707	0.0624	0.0639	0.0650	0.0717
	100	0.0314	0.0363	0.0330	0.0323	0.0376	0.0473	0.0442	0.0446	0.0449	0.0487
	250	0.0189	0.0230	0.0204	0.0203	0.0231	0.0163	0.0195	0.0176	0.0175	0.0201
	500	0.0136	0.0166	0.0147	0.0147	0.0166	0.0165	0.0191	0.0177	0.0177	0.0184
	15	0.0862	0.0946	0.0895	0.0816	0.1034	0.0565	0.0885	0.0815	0.0673	0.0768
	25	0.0645	0.0781	0.0716	0.0665	0.0787	0.0572	0.0845	0.0779	0.0701	0.0715
$\gamma = 2$	50	0.0434	0.0533	0.0482	0.0460	0.0535	0.0313	0.0462	0.0401	0.0374	0.0429
	100	0.0300	0.0370	0.0330	0.0323	0.0373	0.0370	0.0355	0.0347	0.0347	0.0399
	250	0.0195	0.0235	0.0209	0.0207	0.0237	0.0146	0.0183	0.0160	0.0159	0.0189
	500	0.0135	0.0165	0.0146	0.0146	0.0166	0.0102	0.0132	0.0114	0.0113	0.0134
MRE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.1520	0.1879	0.1764	0.1566	0.1852	0.2338	0.2410	0.2311	0.2249	0.2568
	25	0.1381	0.1393	0.1305	0.1277	0.1599	0.1880	0.2331	0.2203	0.2087	0.2146
$\gamma = -0.5$	50	0.0781	0.0984	0.0886	0.0844	0.0970	0.1650	0.1933	0.1840	0.1795	0.1786
	100	0.0555	0.0672	0.0603	0.0590	0.0688	0.1083	0.1258	0.1183	0.1168	0.1200
	250	0.0346	0.0427	0.0377	0.0374	0.0426	0.1547	0.1599	0.1576	0.1577	0.1552
	500	0.0248	0.0300	0.0265	0.0264	0.0301	0.1317	0.1342	0.1328	0.1331	0.1317
	15	0.1514	0.1910	0.1809	0.1602	0.1856	0.2434	0.3189	0.3100	0.2855	0.2653
	25	0.1164	0.1376	0.1274	0.1181	0.1428	0.2206	0.2703	0.2615	0.2475	0.2350
$\gamma = 0.5$	50	0.0771	0.0964	0.0864	0.0823	0.0958	0.2182	0.1925	0.1972	0.2007	0.2212
	100	0.0585	0.0676	0.0614	0.0602	0.0700	0.1351	0.1262	0.1274	0.1283	0.1390
	250	0.0345	0.0421	0.0373	0.0370	0.0421	0.0416	0.0499	0.0449	0.0446	0.0513
	500	0.0248	0.0303	0.0269	0.0267	0.0302	0.0399	0.0463	0.0429	0.0429	0.0447
	15	0.1675	0.1837	0.1740	0.1585	0.2009	0.1210	0.1897	0.1746	0.1443	0.1645
	25	0.1194	0.1445	0.1325	0.1231	0.1457	0.1155	0.1708	0.1574	0.1416	0.1445
$\gamma = 2$	50	0.0795	0.0976	0.0885	0.0843	0.0981	0.0671	0.0992	0.0860	0.0803	0.0922
	100	0.0552	0.0680	0.0606	0.0594	0.0686	0.0880	0.0844	0.0826	0.0827	0.0949
	250	0.0358	0.0430	0.0383	0.0380	0.0434	0.0326	0.0408	0.0357	0.0353	0.0422
	500	0.0246	0.0302	0.0267	0.0266	0.0302	0.0224	0.0291	0.0252	0.0250	0.0297

Table 2. Simulation results for LSL=0.3950 and USL=6.3356 and $\Xi = (2, 0.5)$

		$T = 3.5787$ (off target with $0.5 \times \sigma$)					$T = 2.7193$ (off target with $0.1 \times \sigma$)				
Bias	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
$\gamma = -0.5$	15	0.2092	0.1703	0.1745	0.1851	0.2093	0.0739	0.0280	0.0331	0.0447	0.0755
	25	0.0391	0.0147	0.0189	0.0242	0.0393	0.0468	0.0169	0.0223	0.0285	0.0468
	50	0.0908	0.0780	0.0815	0.0831	0.0908	-0.0074	-0.0231	-0.0187	-0.0165	-0.0078
	100	0.0068	0.0001	0.0026	0.0029	0.0066	-0.0021	-0.0095	-0.0068	-0.0067	-0.0017
	250	-0.0103	-0.0128	-0.0116	-0.0118	-0.0102	-0.0051	-0.0081	-0.0067	-0.0068	-0.0050
	500	0.0206	0.0191	0.0200	0.0198	0.0205	-0.0039	-0.0060	-0.0049	-0.0051	-0.0044
$\gamma = 0.5$	15	0.0906	0.0487	0.0531	0.0646	0.0908	0.0002	-0.0465	-0.0416	-0.0291	0.0021
	25	0.0104	-0.0146	-0.0099	-0.0050	0.0117	-0.0028	-0.0336	-0.0285	-0.0212	-0.0035
	50	-0.0124	-0.0248	-0.0214	-0.0200	-0.0111	-0.0015	-0.0170	-0.0125	-0.0109	-0.0017
	100	-0.0046	-0.0105	-0.0082	-0.0081	-0.0036	0.0074	-0.0005	0.0026	0.0029	0.0073
	250	-0.0160	-0.0189	-0.0175	-0.0176	-0.0161	0.0094	0.0059	0.0076	0.0074	0.0091
	500	0.0189	0.0175	0.0183	0.0181	0.0189	0.0009	-0.0006	0.0002	0.0000	0.0010
$\gamma = 2$	15	0.0786	0.0337	0.0390	0.0508	0.0784	0.1076	0.0611	0.0661	0.0783	0.1103
	25	0.0870	0.0597	0.0645	0.0702	0.0875	0.0087	-0.0210	-0.0157	-0.0095	0.0092
	50	0.0039	-0.0101	-0.0060	-0.0044	0.0042	0.0328	0.0174	0.0219	0.0237	0.0328
	100	-0.0055	-0.0128	-0.0101	-0.0098	-0.0056	0.0134	0.0056	0.0087	0.0088	0.0135
	250	0.0137	0.0107	0.0122	0.0120	0.0136	-0.0019	-0.0049	-0.0034	-0.0036	-0.0018
	500	-0.0031	-0.0046	-0.0038	-0.0040	-0.0032	-0.0032	-0.0048	-0.0040	-0.0042	-0.0032
MSE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
$\gamma = -0.5$	15	0.0524	0.0395	0.0402	0.0426	0.0565	0.0158	0.0141	0.0134	0.0119	0.0220
	25	0.0068	0.0070	0.0064	0.0060	0.0093	0.0078	0.0079	0.0072	0.0066	0.0110
	50	0.0110	0.0098	0.0098	0.0099	0.0122	0.0027	0.0043	0.0035	0.0032	0.0041
	100	0.0014	0.0019	0.0016	0.0015	0.0020	0.0012	0.0020	0.0016	0.0015	0.0020
	250	0.0006	0.0009	0.0007	0.0007	0.0009	0.0005	0.0008	0.0006	0.0006	0.0008
	500	0.0007	0.0007	0.0007	0.0007	0.0008	0.0003	0.0004	0.0003	0.0003	0.0004
$\gamma = 0.5$	15	0.0148	0.0115	0.0111	0.0107	0.0196	0.0099	0.0148	0.0133	0.0104	0.0156
	25	0.0042	0.0062	0.0052	0.0044	0.0070	0.0056	0.0088	0.0075	0.0063	0.0088
	50	0.0021	0.0038	0.0030	0.0027	0.0035	0.0026	0.0042	0.0034	0.0030	0.0042
	100	0.0010	0.0017	0.0013	0.0012	0.0017	0.0014	0.0020	0.0016	0.0015	0.0021
	250	0.0006	0.0010	0.0008	0.0008	0.0009	0.0006	0.0008	0.0007	0.0006	0.0009
	500	0.0005	0.0006	0.0006	0.0006	0.0007	0.0002	0.0004	0.0003	0.0003	0.0004
$\gamma = 2$	15	0.0127	0.0104	0.0101	0.0090	0.0176	0.0221	0.0175	0.0169	0.0163	0.0290
	25	0.0113	0.0095	0.0090	0.0090	0.0144	0.0057	0.0082	0.0070	0.0059	0.0089
	50	0.0017	0.0031	0.0024	0.0021	0.0032	0.0037	0.0041	0.0037	0.0035	0.0052
	100	0.0009	0.0017	0.0013	0.0012	0.0016	0.0015	0.0020	0.0016	0.0016	0.0022
	250	0.0005	0.0007	0.0006	0.0006	0.0008	0.0005	0.0008	0.0006	0.0006	0.0008
	500	0.0002	0.0003	0.0002	0.0002	0.0003	0.0003	0.0004	0.0003	0.0003	0.0004
ABB	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
$\gamma = -0.5$	15	0.2110	0.1731	0.1771	0.1873	0.2113	0.0981	0.0879	0.0851	0.0837	0.1097
	25	0.0681	0.0648	0.0620	0.0620	0.0750	0.0684	0.0677	0.0639	0.0622	0.0786
	50	0.0937	0.0832	0.0855	0.0868	0.0946	0.0413	0.0534	0.0476	0.0455	0.0510
	100	0.0299	0.0346	0.0314	0.0311	0.0355	0.0281	0.0362	0.0316	0.0308	0.0356
	250	0.0197	0.0243	0.0217	0.0216	0.0235	0.0183	0.0234	0.0204	0.0202	0.0229
	500	0.0224	0.0224	0.0222	0.0221	0.0234	0.0130	0.0164	0.0143	0.0143	0.0161
$\gamma = 0.5$	15	0.1026	0.0798	0.0785	0.0832	0.1076	0.0788	0.1001	0.0938	0.0824	0.0971
	25	0.0513	0.0626	0.0566	0.0525	0.0643	0.0595	0.0769	0.0704	0.0644	0.0737
	50	0.0362	0.0498	0.0438	0.0415	0.0472	0.0407	0.0524	0.0466	0.0441	0.0512
	100	0.0253	0.0332	0.0289	0.0282	0.0325	0.0289	0.0354	0.0312	0.0307	0.0361
	250	0.0203	0.0257	0.0229	0.0228	0.0243	0.0192	0.0225	0.0202	0.0200	0.0232
	500	0.0199	0.0202	0.0198	0.0197	0.0212	0.0124	0.0156	0.0136	0.0135	0.0156
$\gamma = 2$	15	0.0910	0.0745	0.0724	0.0737	0.0989	0.1189	0.0969	0.0954	0.0987	0.1289
	25	0.0903	0.0740	0.0738	0.0768	0.0946	0.0592	0.0732	0.0668	0.0615	0.0735
	50	0.0331	0.0442	0.0384	0.0364	0.0443	0.0474	0.0499	0.0466	0.0457	0.0553
	100	0.0243	0.0333	0.0285	0.0277	0.0322	0.0300	0.0353	0.0317	0.0311	0.0368
	250	0.0186	0.0209	0.0191	0.0190	0.0221	0.0181	0.0226	0.0198	0.0196	0.0224
	500	0.0108	0.0149	0.0125	0.0125	0.0146	0.0131	0.0166	0.0144	0.0144	0.016
MRE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
$\gamma = -0.5$	15	0.9261	0.7601	0.7774	0.8223	0.9274	0.2147	0.1924	0.1862	0.1832	0.2400
	25	0.1677	0.1595	0.1527	0.1528	0.1848	0.1390	0.1375	0.1299	0.1264	0.1597
	50	0.2583	0.2295	0.2358	0.2395	0.2609	0.0753	0.0974	0.0869	0.0830	0.0930
	100	0.0661	0.0765	0.0696	0.0687	0.0785	0.0516	0.0665	0.0580	0.0565	0.0654
	250	0.0418	0.0516	0.0461	0.0459	0.0499	0.0334	0.0427	0.0372	0.0369	0.0417
	500	0.0507	0.0507	0.0502	0.0499	0.0529	0.0238	0.0300	0.0261	0.0261	0.0294
$\gamma = 0.5$	15	0.2712	0.2108	0.2074	0.2197	0.2843	0.1455	0.1849	0.1734	0.1523	0.1794
	25	0.1106	0.1350	0.1221	0.1133	0.1387	0.1092	0.1410	0.1293	0.1181	0.1352
	50	0.0735	0.1011	0.0889	0.0841	0.0958	0.0750	0.0965	0.0858	0.0812	0.0943
	100	0.0520	0.0683	0.0594	0.0580	0.0669	0.0539	0.0662	0.0583	0.0572	0.0674
	250	0.0406	0.0513	0.0458	0.0455	0.0486	0.0360	0.0422	0.0377	0.0375	0.0435
	500	0.0426	0.0433	0.0425	0.0422	0.0456	0.0229	0.0287	0.0250	0.0248	0.0288
$\gamma = 2$	15	0.2154	0.1764	0.1713	0.1743	0.2342	0.2712	0.2209	0.2175	0.2250	0.2939
	25	0.2175	0.1782	0.1777	0.1851	0.2279	0.1102	0.1363	0.1245	0.1146	0.1368
	50	0.0661	0.0883	0.0768	0.0727	0.0885	0.0926	0.0974	0.0910	0.0893	0.1080
	100	0.0477	0.0654	0.0559	0.0544	0.0631	0.0565	0.0664	0.0597	0.0585	0.0693
	250	0.0377	0.0425	0.0389	0.0385	0.0450	0.0331	0.0415	0.0362	0.0360	0.0410
	500	0.0211	0.0292	0.0245	0.0245	0.0287	0.0239	0.0304	0.0263	0.0263	0.0299

TABLE 3. Simulation results for LSL=0.2550 and USL=2.3774 and $\Xi = (3, 1.5)$

		$T = 1.0510$ (on target)					$T = 1.7257$ (off target with $1 \times \sigma$)				
Bias	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0689	0.0195	0.0260	0.0395	0.0676	0.0717	0.0396	0.0435	0.0523	0.0711
	25	0.0061	-0.0232	-0.0178	-0.0117	0.0071	-0.1196	-0.1404	-0.1365	-0.1322	-0.1204
$\gamma = -0.5$	50	0.0084	-0.0064	-0.0024	-0.0006	0.0091	0.0296	0.0193	0.0222	0.0234	0.0297
	100	0.0035	-0.0038	-0.0010	-0.0010	0.0041	0.0475	0.0421	0.0442	0.0444	0.0474
	250	0.0041	0.0009	0.0025	0.0022	0.0041	0.0406	0.0387	0.0397	0.0395	0.0409
	500	-0.0002	-0.0018	-0.0010	-0.0012	-0.0003	-0.0092	-0.0099	-0.0095	-0.0096	-0.0088
	15	0.0426	-0.0085	-0.0015	0.0118	0.0396	-0.0084	-0.0418	-0.0378	-0.0290	-0.0081
	25	0.0026	-0.0273	-0.0219	-0.0159	0.0030	-0.0715	-0.0926	-0.0889	-0.0843	-0.0713
$\gamma = 0.5$	50	0.0222	0.0061	0.0106	0.0125	0.0216	0.0360	0.0257	0.0285	0.0296	0.0369
	100	0.0001	-0.0073	-0.0046	-0.0043	0.0006	-0.0134	-0.0194	-0.0171	-0.0169	-0.0138
	250	0.0004	-0.0025	-0.0010	-0.0013	0.0006	0.0051	0.0028	0.0039	0.0037	0.0051
	500	0.0009	-0.0007	0.0001	-0.0001	0.0009	-0.0152	-0.0165	-0.0158	-0.0160	-0.0153
	15	0.0266	-0.0230	-0.0165	-0.0032	0.0250	-0.0913	-0.1282	-0.1236	-0.1137	-0.0922
	25	0.0076	-0.0227	-0.0173	-0.0107	0.0076	-0.0683	-0.0911	-0.0866	-0.0822	-0.0684
$\gamma = 2$	50	0.0013	-0.0132	-0.0089	-0.0075	0.0024	-0.0556	-0.0672	-0.0638	-0.0626	-0.0554
	100	0.0034	-0.0046	-0.0016	-0.0013	0.0033	0.0085	0.0028	0.0050	0.0051	0.0088
	250	0.0020	-0.0014	0.0003	0.0000	0.0017	-0.0091	-0.0115	-0.0103	-0.0105	-0.0090
	500	-0.0005	-0.0022	-0.0013	-0.0015	-0.0006	-0.0056	-0.0070	-0.0063	-0.0064	-0.0057
MSE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0158	0.0131	0.0133	0.0120	0.0197	0.0116	0.0092	0.0091	0.0090	0.0143
	25	0.0063	0.0089	0.0078	0.0066	0.0095	0.0184	0.0249	0.0233	0.0217	0.0203
$\gamma = -0.5$	50	0.0030	0.0043	0.0035	0.0033	0.0046	0.0032	0.0034	0.0031	0.0030	0.0041
	100	0.0014	0.0022	0.0017	0.0016	0.0022	0.0035	0.0034	0.0034	0.0033	0.0040
	250	0.0006	0.0008	0.0007	0.0006	0.0009	0.0022	0.0022	0.0022	0.0021	0.0024
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0003	0.0004	0.0004	0.0004	0.0004
	15	0.0130	0.0132	0.0128	0.0109	0.0174	0.0060	0.0091	0.0084	0.0066	0.0089
	25	0.0062	0.0093	0.0080	0.0068	0.0097	0.0086	0.0134	0.0121	0.0108	0.0105
$\gamma = 0.5$	50	0.0034	0.0041	0.0035	0.0033	0.0049	0.0032	0.0036	0.0032	0.0031	0.0045
	100	0.0015	0.0022	0.0018	0.0017	0.0022	0.0012	0.0018	0.0015	0.0014	0.0017
	250	0.0006	0.0009	0.0007	0.0007	0.0009	0.0004	0.0006	0.0005	0.0005	0.0006
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0004	0.0006	0.0005	0.0005	0.0005
	15	0.0115	0.0134	0.0128	0.0104	0.0160	0.0134	0.0232	0.0214	0.0179	0.0167
	25	0.0061	0.0086	0.0074	0.0064	0.0092	0.0078	0.0130	0.0115	0.0101	0.0100
$\gamma = 2$	50	0.0029	0.0044	0.0036	0.0033	0.0045	0.0047	0.0071	0.0061	0.0058	0.0058
	100	0.0015	0.0021	0.0017	0.0016	0.0022	0.0009	0.0013	0.0011	0.0010	0.0015
	250	0.0006	0.0009	0.0007	0.0007	0.0009	0.0004	0.0007	0.0005	0.0005	0.0006
	500	0.0003	0.0004	0.0003	0.0003	0.0004	0.0002	0.0003	0.0003	0.0003	0.0003
ABB	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0960	0.0872	0.0851	0.0831	0.1056	0.0907	0.0724	0.0726	0.0765	0.0923
	25	0.0616	0.0766	0.0705	0.0647	0.0766	0.1204	0.1438	0.1394	0.1334	0.1265
$\gamma = -0.5$	50	0.0429	0.0520	0.0469	0.0450	0.0527	0.0463	0.0454	0.0439	0.0440	0.0499
	100	0.0298	0.0373	0.0329	0.0321	0.0376	0.0511	0.0479	0.0485	0.0487	0.0521
	250	0.0189	0.0229	0.0204	0.0202	0.0232	0.0413	0.0401	0.0406	0.0404	0.0420
	500	0.0134	0.0165	0.0145	0.0145	0.0164	0.0146	0.0169	0.0156	0.0156	0.0166
	15	0.0874	0.0914	0.0875	0.0815	0.1006	0.0615	0.0769	0.0722	0.0639	0.0732
	25	0.0622	0.0784	0.0717	0.0663	0.0772	0.0766	0.1012	0.0960	0.0891	0.0869
$\gamma = 0.5$	50	0.0456	0.0505	0.0466	0.0454	0.0539	0.0472	0.0463	0.0446	0.0447	0.0522
	100	0.0306	0.0376	0.0334	0.0327	0.0374	0.0277	0.0346	0.0312	0.0307	0.0331
	250	0.0192	0.0236	0.0208	0.0206	0.0236	0.0167	0.0195	0.0178	0.0177	0.0199
	500	0.0135	0.0163	0.0144	0.0144	0.0163	0.0173	0.0198	0.0185	0.0185	0.0191
	15	0.0834	0.0933	0.0892	0.0810	0.0979	0.0966	0.1365	0.1316	0.1174	0.1107
	25	0.0613	0.0749	0.0691	0.0638	0.0748	0.0735	0.1003	0.0943	0.0871	0.0852
$\gamma = 2$	50	0.0431	0.0533	0.0480	0.0459	0.0531	0.0584	0.0733	0.0684	0.0661	0.0649
	100	0.0302	0.0366	0.0327	0.0319	0.0367	0.0241	0.0290	0.0257	0.0253	0.0300
	250	0.0194	0.0235	0.0209	0.0207	0.0236	0.0160	0.0208	0.0183	0.0181	0.0201
	500	0.0135	0.0167	0.0146	0.0146	0.0166	0.0113	0.0146	0.0128	0.0127	0.0143
MRE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.1998	0.1816	0.1772	0.1730	0.2198	0.3152	0.2516	0.2523	0.2661	0.3207
	25	0.1134	0.1409	0.1297	0.1190	0.1408	0.2482	0.2965	0.2874	0.2751	0.2609
$\gamma = -0.5$	50	0.0796	0.0963	0.0869	0.0834	0.0976	0.1363	0.1335	0.1293	0.1294	0.1469
	100	0.0548	0.0684	0.0604	0.0589	0.0690	0.1572	0.1475	0.1495	0.1501	0.1604
	250	0.0347	0.0421	0.0374	0.0371	0.0426	0.1241	0.1204	0.1219	0.1214	0.1262
	500	0.0245	0.0301	0.0265	0.0264	0.0300	0.0381	0.0441	0.0407	0.0407	0.0431
	15	0.1724	0.1804	0.1727	0.1609	0.1985	0.1575	0.1970	0.1849	0.1636	0.1875
	25	0.1137	0.1432	0.1310	0.1212	0.1410	0.1665	0.2202	0.2088	0.1937	0.1889
$\gamma = 0.5$	50	0.0868	0.0960	0.0887	0.0863	0.1026	0.1321	0.1297	0.1249	0.1251	0.1461
	100	0.0558	0.0687	0.0610	0.0597	0.0682	0.0680	0.0848	0.0765	0.0753	0.0812
	250	0.0350	0.0431	0.0379	0.0376	0.0430	0.0427	0.0499	0.0455	0.0452	0.0509
	500	0.0246	0.0298	0.0264	0.0263	0.0299	0.0420	0.0481	0.0449	0.0449	0.0464
	15	0.1593	0.1784	0.1706	0.1547	0.1871	0.1927	0.2724	0.2627	0.2344	0.2208
	25	0.1130	0.1381	0.1275	0.1176	0.1379	0.1523	0.2078	0.1953	0.1805	0.1765
$\gamma = 2$	50	0.0787	0.0973	0.0876	0.0838	0.0970	0.1230	0.1543	0.1440	0.1392	0.1368
	100	0.0553	0.0670	0.0600	0.0585	0.0672	0.0583	0.0701	0.0623	0.0613	0.0725
	250	0.0356	0.0431	0.0382	0.0379	0.0432	0.0370	0.0482	0.0423	0.0419	0.0464
	500	0.0246	0.0304	0.0267	0.0266	0.0303	0.0262	0.0340	0.0297	0.0296	0.0332

Table 4. Simulation results for LSL=0.2550 and USL=2.3774 and $\Xi = (3, 1.5)$

		$T = 1.3884$ (off target with $0.5 \times \sigma$)					$T = 1.1185$ (off target with $0.1 \times \sigma$)				
Bias	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	-0.0348	-0.0772	-0.0719	-0.0605	-0.0364	-0.0109	-0.0595	-0.0531	-0.0406	-0.0125
	25	-0.0197	-0.0451	-0.0402	-0.0354	-0.0194	0.0313	0.0013	0.0068	0.0133	0.0309
$\gamma = -0.5$	50	-0.0595	-0.0725	-0.0689	-0.0674	-0.0590	0.0003	-0.0148	-0.0107	-0.0090	0.0006
	100	-0.0189	-0.0256	-0.0232	-0.0228	-0.0187	0.0035	-0.0038	-0.0011	-0.0008	0.0040
	250	0.0154	0.0126	0.0139	0.0137	0.0154	-0.0052	-0.0085	-0.0070	-0.0071	-0.0053
	500	0.0103	0.0089	0.0097	0.0095	0.0103	0.0032	0.0014	0.0023	0.0021	0.0029
	15	0.0208	-0.0221	-0.0168	-0.0054	0.0197	0.0766	0.0282	0.0349	0.0471	0.0755
	25	-0.0405	-0.0666	-0.0623	-0.0563	-0.0401	0.0087	-0.0215	-0.0158	-0.0094	0.0083
$\gamma = 0.5$	50	-0.0453	-0.0595	-0.0555	-0.0539	-0.0458	-0.0018	-0.0168	-0.0127	-0.0108	-0.0015
	100	0.0055	-0.0008	0.0016	0.0017	0.0062	0.0035	-0.0039	-0.0011	-0.0010	0.0039
	250	0.0060	0.0033	0.0046	0.0044	0.0061	-0.0050	-0.0081	-0.0067	-0.0069	-0.0049
	500	0.0036	0.0023	0.0030	0.0028	0.0038	0.0013	-0.0003	0.0006	0.0004	0.0013
	15	-0.0318	-0.0764	-0.0708	-0.0586	-0.0340	-0.0061	-0.0543	-0.0485	-0.0349	-0.0068
	25	-0.0523	-0.0793	-0.0742	-0.0687	-0.0523	-0.0047	-0.0354	-0.0297	-0.0234	-0.0056
$\gamma = 2$	50	0.0245	0.0114	0.0149	0.0166	0.0254	0.0214	0.0066	0.0107	0.0124	0.0219
	100	0.0037	-0.0027	-0.0003	-0.0002	0.0045	-0.0036	-0.0110	-0.0082	-0.0081	-0.0032
	250	-0.0116	-0.0146	-0.0131	-0.0133	-0.0117	0.0020	-0.0010	0.0005	0.0002	0.0022
	500	0.0016	0.0001	0.0009	0.0007	0.0016	-0.0008	-0.0026	-0.0017	-0.0018	-0.0011
MSE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0088	0.0152	0.0138	0.0110	0.0124	0.0100	0.0156	0.0143	0.0111	0.0145
	25	0.0053	0.0086	0.0074	0.0063	0.0077	0.0066	0.0077	0.0069	0.0061	0.0096
$\gamma = -0.5$	50	0.0060	0.0087	0.0077	0.0072	0.0072	0.0028	0.0043	0.0034	0.0031	0.0043
	100	0.0015	0.0024	0.0019	0.0018	0.0021	0.0013	0.0020	0.0016	0.0015	0.0021
	250	0.0007	0.0008	0.0007	0.0007	0.0009	0.0005	0.0008	0.0006	0.0006	0.0008
	500	0.0003	0.0004	0.0004	0.0003	0.0004	0.0003	0.0004	0.0003	0.0003	0.0004
	15	0.0077	0.0097	0.0089	0.0072	0.0115	0.0164	0.0135	0.0135	0.0124	0.0207
	25	0.0061	0.0105	0.0091	0.0078	0.0084	0.0058	0.0082	0.0071	0.0060	0.0088
$\gamma = 0.5$	50	0.0042	0.0066	0.0056	0.0052	0.0054	0.0027	0.0043	0.0034	0.0031	0.0043
	100	0.0011	0.0017	0.0013	0.0013	0.0018	0.0013	0.0020	0.0015	0.0015	0.0020
	250	0.0004	0.0007	0.0005	0.0005	0.0007	0.0005	0.0009	0.0007	0.0007	0.0008
	500	0.0002	0.0003	0.0003	0.0003	0.0003	0.0003	0.0004	0.0003	0.0003	0.0004
	15	0.0078	0.0148	0.0134	0.0102	0.0118	0.0101	0.0150	0.0137	0.0110	0.0145
	25	0.0069	0.0122	0.0106	0.0091	0.0094	0.0056	0.0090	0.0077	0.0064	0.0087
$\gamma = 2$	50	0.0026	0.0033	0.0027	0.0025	0.0040	0.0031	0.0039	0.0033	0.0031	0.0046
	100	0.0010	0.0016	0.0012	0.0011	0.0017	0.0013	0.0021	0.0017	0.0016	0.0021
	250	0.0005	0.0008	0.0007	0.0006	0.0008	0.0005	0.0008	0.0006	0.0006	0.0008
	500	0.0002	0.0003	0.0002	0.0002	0.0003	0.0003	0.0004	0.0003	0.0003	0.0004
ABB	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.0741	0.1035	0.0979	0.0849	0.0898	0.0797	0.1049	0.0990	0.0867	0.0971
	25	0.0569	0.0757	0.0692	0.0632	0.0700	0.0625	0.0691	0.0644	0.0609	0.0748
$\gamma = -0.5$	50	0.0642	0.0802	0.0749	0.0721	0.0711	0.0416	0.0521	0.0466	0.0446	0.0515
	100	0.0308	0.0397	0.0353	0.0345	0.0372	0.0287	0.0358	0.0316	0.0309	0.0360
	250	0.0212	0.0229	0.0215	0.0213	0.0241	0.0186	0.0235	0.0205	0.0204	0.0230
	500	0.0147	0.0163	0.0151	0.0150	0.0169	0.0128	0.0156	0.0137	0.0137	0.0158
	15	0.0701	0.0783	0.0735	0.0669	0.0826	0.0991	0.0876	0.0856	0.0845	0.1084
	25	0.0623	0.0862	0.0798	0.0727	0.0751	0.0601	0.0734	0.0676	0.0621	0.0734
$\gamma = 0.5$	50	0.0532	0.0691	0.0632	0.0608	0.0610	0.0410	0.0528	0.0470	0.0447	0.0518
	100	0.0261	0.0324	0.0286	0.0280	0.0330	0.0283	0.0352	0.0310	0.0303	0.0354
	250	0.0168	0.0201	0.0178	0.0177	0.0206	0.0188	0.0236	0.0207	0.0205	0.0231
	500	0.0116	0.0144	0.0126	0.0126	0.0146	0.0128	0.0160	0.0140	0.0139	0.0161
	15	0.0708	0.1029	0.0968	0.0831	0.0881	0.0798	0.1026	0.0967	0.0860	0.0965
	25	0.0678	0.0950	0.0881	0.0803	0.0809	0.0590	0.0780	0.0712	0.0647	0.0740
$\gamma = 2$	50	0.0406	0.0440	0.0403	0.0393	0.0482	0.0431	0.0486	0.0447	0.0433	0.0520
	100	0.0248	0.0318	0.0275	0.0269	0.0322	0.0292	0.0373	0.0329	0.0319	0.0365
	250	0.0184	0.0236	0.0207	0.0206	0.0226	0.0180	0.0221	0.0195	0.0192	0.0222
	500	0.0111	0.0142	0.0123	0.0123	0.0143	0.0129	0.0161	0.0141	0.0140	0.0160
MRE	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
	15	0.1480	0.2066	0.1953	0.1694	0.1792	0.1454	0.1914	0.1806	0.1581	0.1771
	25	0.1164	0.1549	0.1416	0.1292	0.1431	0.1227	0.1355	0.1262	0.1194	0.1467
$\gamma = -0.5$	50	0.1200	0.1500	0.1400	0.1348	0.1329	0.0767	0.0961	0.0859	0.0823	0.0950
	100	0.0616	0.0794	0.0707	0.0691	0.0745	0.0531	0.0662	0.0585	0.0572	0.0666
	250	0.0456	0.0491	0.0461	0.0458	0.0518	0.0340	0.0428	0.0374	0.0371	0.0419
	500	0.0311	0.0344	0.0319	0.0317	0.0357	0.0237	0.0289	0.0254	0.0253	0.0292
	15	0.1540	0.1721	0.1615	0.1470	0.1814	0.2124	0.1877	0.1836	0.1812	0.2324
	25	0.1197	0.1656	0.1533	0.1398	0.1444	0.1125	0.1373	0.1265	0.1163	0.1373
$\gamma = 0.5$	50	0.1005	0.1306	0.1194	0.1148	0.1153	0.0754	0.0972	0.0865	0.0822	0.0953
	100	0.0543	0.0673	0.0594	0.0581	0.0686	0.0524	0.0652	0.0574	0.0562	0.0656
	250	0.0348	0.0416	0.0369	0.0366	0.0427	0.0343	0.0431	0.0377	0.0374	0.0422
	500	0.0240	0.0297	0.0260	0.0259	0.0301	0.0236	0.0296	0.0258	0.0256	0.0297
	15	0.1363	0.1983	0.1864	0.1600	0.1696	0.1456	0.1872	0.1765	0.1569	0.1760
	25	0.1248	0.1749	0.1622	0.1478	0.1488	0.1077	0.1423	0.1299	0.1181	0.1350
$\gamma = 2$	50	0.0864	0.0936	0.0858	0.0837	0.1027	0.0828	0.0934	0.0858	0.0831	0.0998
	100	0.0504	0.0646	0.0561	0.0546	0.0655	0.0533	0.0680	0.0600	0.0583	0.0666
	250	0.0363	0.0465	0.0408	0.0405	0.0444	0.0332	0.0408	0.0359	0.0355	0.0410
	500	0.0224	0.0287	0.0249	0.0248	0.0289	0.0236	0.0296	0.0258	0.0258	0.0294

6. Real Data Examples

In this section, two real data analyses are carried out. This analysis demonstrates the applicability of the GKME distribution with the PCI S'_{pmk}. The parameter estimates based on all estimators, the MLE of S'_{pmk}, and estimates of S'_{pmk} based on other estimators are obtained for both on-target and off-target processes. Furthermore, the Kolmogorov-Smirnov (KS) test statistics, and the associated p-value are reported for both datasets.

6.1. **Real Data Example I.** The first data set represents the ball size of wire bonding for an electronic connection from the integrated circuit apparatus to the lead frame. The first data set is taken from Leiva et al [9]. The first data set is used by [8] and [13]. The data consists of 100 samples. The data are 2.891, 4.035, 4.495, 2.890, 2.312, 3.158, 5.228, 3.334, 5.896, 5.639, 3.842, 1.590, 1.954, 1.842, 0.680, 2.752, 1.301, 2.260, 0.889, 2.381, 0.619, 2.788, 1.050, 3.750, 3.508, 6.123, 6.549, 5.954, 2.207, 4.417, 4.805, 1.516, 2.227, 2.797, 1.636, 1.066, 0.940, 4.101, 4.542, 1.295, 1.770, 3.492, 5.706, 3.722, 6.644, 2.472, 1.383, 4.494, 1.694, 2.892, 2.111, 3.591, 2.093, 3.222, 2.891, 2.582, 0.665, 3.234, 1.102, 1.083, 1.508, 1.811, 2.803, 6.659, 0.923, 6.229, 3.177, 2.333, 1.311, 4.419, 2.495, 0.921, 4.061, 9.725, 1.600, 4.281, 3.360, 1.131, 1.618, 4.489, 3.696, 1.982, 2.413, 5.480, 1.992, 2.573, 1.845, 4.620, 6.221, 1.694, 4.882, 1.380, 3.982, 2.260, 2.366, 2.899, 3.782, 2.336, 1.175, 3.055. The quality characteristic to be monitored is the ball size, with LSL = 0.5 mil and USL = 8 mil (1 mil = 1/1000 in = 0.0254 mm). The parameters are obtained as follows: $\widehat{\alpha} = 3.9246$, $\widehat{\theta} = 0.5793$, KS = 0.0443, and KS p-value = 0.9895. The S'_{pmk} based on the ball size data set are reported in Table (5) for different values of γ. Figure 3 presents the pdf, cdf, sf, and probability–probability (P–P) plots. From Figure 3, it can be observed that the GKME distribution provides a good fit to the data. Although the S'_{pmk} values vary depending on the estimation method, the differences are not substantial. This indicates that the MLE, LSE, WLSE, ADE, and CvME methods provide similarly reliable estimates of the S'_{pmk} index. However, the MLE method generally tends to yield the highest values, whereas the LSE method tends to produce the lowest. Moreover, as the value of γ increases, the S'_{pmk} values show an increasing trend. This suggests that the γ parameter has a positive effect on the index, indicating that higher γ values are associated with better process capability.

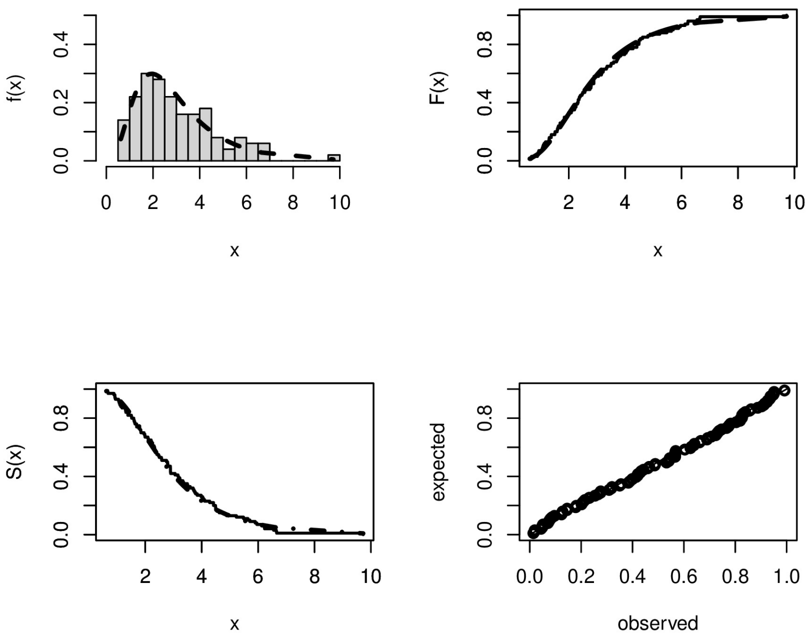

Figure 3. The fitted pdf, cdf, sf, and P–P plots for GKME distribution for real data I

TABLE 5. Results for real data analysis I with on-target and off-target

T	γ	**Estimate**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
On-target	-0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.4288	0.3999	0.4116	0.4122	0.4051
	0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.5177	0.4828	0.4969	0.4976	0.4890
	2	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.5930	0.5530	0.5692	0.5699	0.5602
Off-target ($1 \times \sigma$)	-0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.2253	0.2102	0.2163	0.2166	0.2129
	0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.3769	0.3515	0.3618	0.3623	0.3561
	2	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.5037	0.4697	0.4835	0.4841	0.4758
Off-target ($0.5 \times \sigma$)	-0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.3072	0.2865	0.2949	0.2953	0.2902
	0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.4357	0.4063	0.4182	0.4188	0.4116
	2	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.5431	0.5064	0.5213	0.5220	0.5130
Off-target ($0.1 \times \sigma$)	-0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.4007	0.3737	0.3846	0.3851	0.3785
	0.5	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.4991	0.4654	0.4791	0.4797	0.4714
	2	$\widehat{\alpha}$	3.9246	3.3609	3.5735	3.5949	3.4681
		$\widehat{\theta}$	0.5793	0.5187	0.5433	0.5440	0.5290
		S'_{pmk}	0.5820	0.5428	0.5587	0.5594	0.5498

6.2. **Real Data Example II.** The second data set represents the characteristic monitored protein amount (in g) in the restricted diet for adult patients in the Hospital Carlos Van Buren of the city of Valparaiso, Chile. The second data set is taken from [9]. The second data set is used by [8]. The data consists of 100 samples. The data are: 66.09, 146.48, 97.99, 56.77, 74.23, 76.20, 108.90, 89.51, 62.40, 141.15, 68.22, 80.44, 89.81, 201.05, 210.32, 64.25, 39.43, 43.87, 59.56, 49.97, 34.35, 51.28, 100.43, 42.37, 164.05, 59.63, 49.21, 51.41, 93.32, 102.91, 124.91, 56.16, 70.94, 77.26, 68.62, 41.69, 59.01, 59.78, 37.67, 88.27, 35.60, 42.54, 169.82, 105.08, 102.16, 67.55, 17.76, 83.73, 44.41, 25.55, 41.12, 163.50, 123.85, 100.43, 88.14, 54.69, 85.32, 34.07, 61.95, 60.03, 139.39. The LSL is taken as 30 g, and the USL is taken as 96 g. The parameters are obtained as follows: $\widehat{\alpha} = 5.8655$, $\widehat{\theta} = 0.0265$, KS $= 0.0538$, and KS p-value $= 0.9944$. The S'_{pmk} based on the ball size data set are reported in Table (6) for different values of γ. Figure 4 presents the pdf, cdf, sf, and P–P plots. From Figure 4, it can be observed that the GKME distribution provides a good fit to the data. The results of the real data analysis reveal that the S'_{pmk} index values vary depending on both whether the process is on-target and the value of the γ parameter. In particular, a noticeable increase in S'_{pmk} values is observed as the γ parameter increases. This indicates that the γ parameter has a positive effect on process capability in both on-target and off-target scenarios.

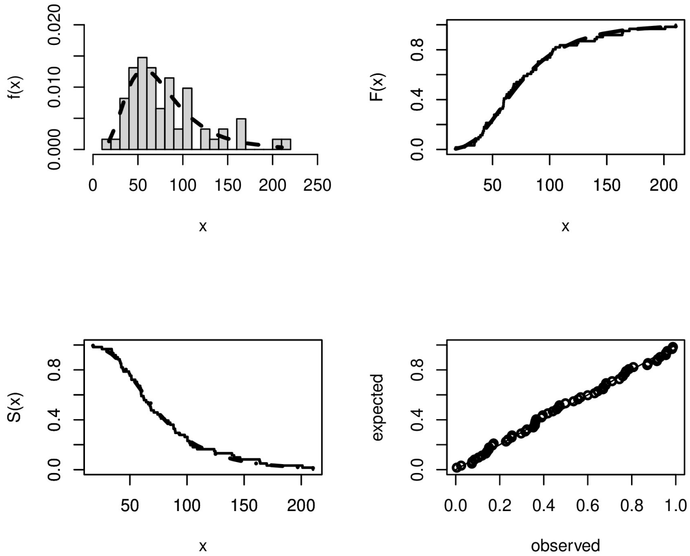

FIGURE 4. The fitted pdf, cdf, sf, and P–P plots for GKME distribution for real data II

TABLE 6. Results for real data analysis II with on-target and off-target

T	γ	**Estimate**	**MLE**	**LSE**	**WLSE**	**ADE**	**CvME**
On-target	-0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.0687	0.0668	0.0668	0.0679	0.0683
	0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6037	0.5865	0.5870	0.5963	0.5999
	2	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6122	0.5948	0.5953	0.6046	0.6084
Off-target ($1 \times \sigma$)	-0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.0000	0.0000	0.0000	0.0000	0.0000
	0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.5780	0.5615	0.5620	0.5708	0.5744
	2	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6051	0.5880	0.5885	0.5977	0.6014
Off-target ($0.5 \times \sigma$)	-0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.0003	0.0003	0.0003	0.0003	0.0003
	0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.5904	0.5736	0.5741	0.5831	0.5868
	2	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6086	0.5913	0.5918	0.6011	0.6049
Off-target ($0.1 \times \sigma$)	-0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.0240	0.0233	0.0233	0.0237	0.0239
	0.5	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6009	0.5839	0.5844	0.5936	0.5973
	2	$\widehat{\alpha}$	5.8679	5.4894	5.4733	5.6966	5.8208
		$\widehat{\theta}$	0.0265	0.0259	0.0257	0.0262	0.0266
		S'_{pmk}	0.6114	0.5941	0.5946	0.6039	0.6077

7. Conclusion

In this study, a process capability index based on a lifetime distribution is examined, and various estimators are proposed for estimating the S'_{pmk} index. The performance of these estimators is assessed using Monte Carlo simulations in terms of point estimation. The simulation results indicate that all estimators perform well, particularly for large sample sizes. Furthermore, the GKME distribution is analyzed using the S'_{pmk} index within the context of process capability analysis, and its applicability to engineering is demonstrated through two real data analyses. This study provides a new perspective on process capability analysis for non-normal distributions, contributing to the development of appropriate process evaluation methods, particularly for asymmetric distributions. The findings of this study can be applied in areas such as quality control, reliability analysis, and process improvement in manufacturing. Additionally, the scalability of the proposed modeling strategies and their applicability across various engineering disciplines present a significant starting point for future research. In this context, the obtained results are expected to contribute to the existing literature and serve as a guide for further studies.

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