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GOVERNING EQUATIONS FOR ISOTHERMAL FLOW THROUGH WOVEN FIBER MATS BY EMPLOYING LOCAL VOLUME AVERAGING TECHNIQUE

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ABSTRACT

Accurate mathematical modeling of resin flow in liquid composite molding (LCM) processes is important for effective simulations of the mold-filling process. Recent experiments indicate that the physics of resin flow in woven fiber mats is very different from the flow in random fiber mats. In this study, the mathematically rigorous volume averaging method is adapted to derive the averaged form of mass and momentum balance equations for unsaturated flow in LCM. The two phases used in the volume averaging method are the dense bundle of fibers called tows, and the surrounding gap present in the woven fiber mats. Averaging the mass balance equation yields a macroscopic equation of continuity which is similar to the conventional continuity equation for a single-phase flow. Similar averaging of the momentum balance equation is accomplished for the dual-scale porous medium.

Keywords: *Porous media flow, Mathematical modelling; Resin transfer moulding (RTM); Local volume averaging.*

FİBERLİ GÖZENEKLİ BİR ORTAMDAKİ İZOTERMAL BİR AKIŞKANIN YEREL HACİMSEL ORTALAMA METODU KULLANILARAK HAREKET DENKLEMLERİNİN ELDE EDİLMESİ

ÖZET

Sıvı kompozit kalıp işlemi sırasındaki resin akışının doğru bir matematiksel modelinin oluşturulması, simülasyon ve kalıp doldurma işlemi için çok önemlidir. Bu çalışmada, yerel hacimsel ortalama metodu detaylı olarak kullanılmak sureti ile akış alanına ait kütle ve momentum denklemleri elde edildi. Oluşan matematiksel model akışkan resine ait ortalama hız bileşenleri ile basıncı verir. Benzer ortalama teknikleri ile, çift skalalı gözenekli ortama ait momentum balans deklemleri elde edilebilir.

Anahtar Kelimeler: *Gözenekli ortam akışı, matematiksel modelleme, RTM.*

1. Introduction

Liquid composite molding (LCM) technologies such as the resin transfer molding are very important in the manufacture of polymeric composites [1]. These composites consist of polymeric matrix, which is interspersed with reinforcements such as carbon and glass fibers. In LCM, the composites are created by impregnating a mold cavity packed with fibers with liquid polymer or resin by injecting it through the inlet gates of the mold. Numerical simulation of such mold-filling process in LCM is becoming indispensable for optimizing the mold design [2] and[3]. The flow of resin through the fiber packed mold is modeled as a single-phase flow of a viscous liquid through a porous medium behind a progressing resin front, as the medium behind the front is assumed to be *fully saturated.* Since the resin is very viscous and resin speed never exceeds a few centimeters per second, the local Reynolds number is many orders smaller than unity. Such a flow is modeled using the macroscopic equation of continuity [4], [5] and [6] with the Darcy's law for momentum balance. Since the resin is undergoing an exothermic chemical reaction during the flow, and the release of heat affects the flow through the temperature dependent viscosity, both energy and chemical species transport equation are solved simultaneously.

Of the various material parameters needed for the LCM mold flow simulations, the permeability S is most important as it is very sensitive to the packing arrangement of fibers. Various techniques for the measurement of this quantity generate a simple radial or rectilinear flow in a fibrous medium using a non-reacting test liquid. Permeability is computed by substituting the measured pressure gradient and flow rates for such flows[7] and[8]. One such permeability measurement setup was used by Parseval et al. [9] to create transient mold-filling experiments with flat rectangular molds. Here, the change in inlet pressure with time is recorded as the mold is filled at a constant flow rate. The theoretical inlet pressure increases linearly with time. Literature indicates that the inlet pressure histories for woven mats are non-linear and could not be explained with the help of the simple Darcy' s equations. A two-layer model, where the tow and gap regions alternate in the flow domain and are aligned with the flow direction, was proposed by the author [10] and[12] to explain the deviant behavior of flow in woven fiber mats. As soon as the macroscopic flow-front passed a certain point, the fluid pressure in the gap forced the liquid into the tows. Such a movement of the liquid and micro-front (i.e. liquid front inside the tows) was modeled using the Darcy's law, and was aided by the capillary suction pressure [13]. Unlike mass balance, the momentum balance equation was not derived rigorously and was borrowed from the single-scale medium in the form of Darcy's law. Application of this equation of continuity and Darcy's law to rectilinear flow in a rectangular mold filled with woven mats successfully replicated the drooping inlet pressure history. It could also explain the difference in the inlet pressure histories for random and woven mats in the radial injection case [10] and [7]. The partial saturation behind the flow-front could now be explained in terms of the degree of impregnation of tows. A similar models based on the concept of dual-scale porous media [15] and [16] explained anomalous pressure drops and predicted void distribution in woven fiber mats.

In this paper, the two governing equations for isothermal flow in dual-scale porous media will be derived rigorously using the well-established volume averaging method as adapted to such media. In the past, these methods have been successfully used in deriving transport equations for single-phase flow in the single-scale porous media [4], [5], [6], [17], [18], [19] and [20] and were later adapted for the dual-scale fractured porous media [6], [21] and [22]. Here, we will be applying the volume averaging method to establish the mass and momentum transport equations for flow in the gaps between the fiber tows and show that a workable system of equations can be thus created to model unsaturated flow in woven fiber mats.

2. Volume averaging method adapted for the dual-scale porous medium

2.1 Nomenclature for porous media flow

The manufacturing process employing RTM contains two phases, the resin (i.e. the fluid phase) and the fiber reinforcement (i.e. the solid phase), and it will be required to distinguish between their respective properties. The two phases under consideration will be indicated by subscripts "f" for the fluid phase and "s" for the solid phase. Terms associated with either phase will be denoted with the corresponding subscript, while the terms associated with both phases will have no subscript. For example, ρ is the density of both phases, while ρ_f is the density of the fluid and $\boldsymbol{\sigma}_s$ is the total stress tensor in the solid.

The control volume for which we derive the mathematical model to describe the flow of the resin through fibers and to which we apply the local volume averaging technique is a microscopic control volume. When this control volume is examined, the individual fibers and the resin in the pores between the fibers can be seen, but the molecular structure of the fibers and resin cannot be seen. On this scale we may use a continuum description of the fibers and the resin, and identify pressure, velocity, stress, etc., as field variables that have values at each point in space. We assume that these variables vary continuously with position $\boldsymbol{x}$, despite the fact that they may have jump discontinuities on the surface of the solid where no slip boundary condition is present.
In order to develop a mathematical development for flow through porous media, it is useful to give a formal description of the microscopic geometry of the porous medium. This is done by defining phase functions, as follows:

$$X_f(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} \text{ lies in the fluid} \\ 0 & \text{if } \boldsymbol{x} \text{ lies in any other phase} \end{cases} \tag{1}$$

The solid phase function can also be defined in a similar fashion. If the solid media is not stationary, the phase functions will be time-dependent as well. When solving real flow problems, the character of the porous medium is never considered in detail, but the phase functions are necessary functions to obtain the governing conservation and constitutive equations of the flow field.

2.2 Local volume averaging technique

The basic concepts in porous media theory are the local volume averaging technique and the use of averaged variables instead of instantaneous ones in field equations. Rather than modeling the microscopic resin flow around each fiber, porous media theory predicts averaged velocity, pressure and stress fields for the flow phase. This approach is similar to the averaging procedure used in continuum mechanics for homogenous materials. Instead of computing the position and velocity of every atom in a gas (i.e. the Lagrangian viewpoint), one develops the continuum theories to calculate the velocity, pressure and temperature for small regions that contain many atoms (i.e. the Eularian viewpoint). Within the mathematical aspects of continuum mechanics, it is possible to consider the velocity at a point. However, the results of continuum mechanics loose their meaning when the size of the point approaches the mean free path-line (streak-lines) of an atom. Similarly, porous media theory gives the averaged velocity, pressure and stress at each point, but these results lose their meanings when we examine the material on the scale of the individual pores and fibers.

To understand the governing conservation and constitutive equations, and to derive them for flow through porous media, we must first give the definitions of these averaged variables. First, let us consider a representative control volume V with each point $\boldsymbol{x}$ in the medium. The size of this volume has to be large enough to include representative amounts of both fluid and solid phases, but small enough so that long-range variations do not affect the average. As an illustrative example, let us consider V as a sphere with its center on $\boldsymbol{x}$, as shown in Figure.1. The surface S bounds V.

Let B represent any variable that has a value at each point. There are three different ways to represent and calculate an average value for B. The spatial average is defined as the average value of B within V. The spatial average will be given by angle brackets as:

$$\langle B \rangle = \frac{1}{V} \int_V B \mathrm{dV} \tag{2}$$

Since B has a value at each point inside the volume V, the spatial average gives the combined properties of both phases. For instance, $\langle \rho \rangle$ is the total mass of both fluid and solid per unit total volume.

The phase average contains only the points that are associated with a single phase, but still averages over the entire volume V. Phase averages will also be given by an angle bracket, but the variable inside the bracket will show a subscript denoting the phase over which the average is performed. The phase average of B over the fluid phase is given as

$$\langle B_\mathrm{f} \rangle = \frac{1}{V} \int_V B \;\; X_\mathrm{f} \, dV \;\; = \frac{1}{V} \int_{V_\mathrm{f}} B_\mathrm{f} \, \mathrm{d}V \tag{3}$$

The multiplication of X_f with variable B in the first integral causes only the points lying within the fluid to be counted; in the second integral V_f means the portion of V occupied by the fluid. Therefore, we can say that the phase average describes the properties of a single phase. The terms “volume average” and “local volume average” indicate a phase average.

The third average takes into account only points associated with a single phase and averages their values occupied by that phase. This is termed as an intrinsic phase average. It is denoted by angle brackets with a superscript showing the relevant phase:

$$\left\langle B_{\mathrm{f}}\right\rangle^{\mathrm{f}}=\frac{1}{V_{\mathrm{f}}}\int_{V_{\mathrm{f}}}B_{\mathrm{f}}\,\mathrm{d}V=\frac{\int_{V}B_{\mathrm{f}}X_{\mathrm{f}}\mathrm{d}V}{\int_{V}X_{\mathrm{f}}\mathrm{d}V} \tag{4}$$

All three types of averages given above are used, and can appear in governing conservation and constitutive equations for porous media. The intrinsic phase average is a mean to calculate and report the averaged pressure; as an example, if all the fluid is at one pressure, then that pressure will be equal to $\left\langle p_{\mathrm{f}}\right\rangle^{\mathrm{f}}$. The phase average is a way to calculate and report averaged velocity; for example, the volume flow rate of fluid crossing a surface dS that has unit normal vector $\boldsymbol{n}$ is given by $\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle \cdot \boldsymbol{n}\mathrm{d}S$.

The various averages are related to one another in a simple fashion. First, the spatial average is the sum of the solid and fluid phase averages:

$$\left\langle B\right\rangle=\left\langle B_{\mathrm{f}}\right\rangle+\left\langle B_{\mathrm{s}}\right\rangle \tag{5}$$

To obtain a relationship between the phase average and the intrinsic phase average, first we define the volume fractions of the phases as

$$\varepsilon_{\mathrm{f}}=\frac{V_{\mathrm{f}}}{V}=\frac{1}{V}\int_{V}X_{\mathrm{f}}\,\mathrm{d}V \qquad \text{and} \qquad \varepsilon_{\mathrm{s}}=\frac{V_{\mathrm{s}}}{V}=\frac{1}{V}\int_{V}V_{\mathrm{s}}\,dV \tag{6}$$

where the fluid portion ε_{f} is often called the porosity and the solid fraction is ε_{s}. From these definitions, we see that the phase and intrinsic averages are now related by

$$\varepsilon_{\mathrm{f}}\left\langle B_{\mathrm{f}}\right\rangle^{\mathrm{f}}=\left\langle B_{\mathrm{f}}\right\rangle \tag{7}$$

Eqn. (7) will be used very often to interchange the different averages.

Any average of any variable can be related to a point in the medium. The point $\boldsymbol{x}$ that has the average value is determined by the location of the averaging volume V (Figure 1). The pore average (i.e. intrinsic average) velocity of the fluid $\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle^{\mathrm{f}}$ is given even for a point $\boldsymbol{x}$ that is located in the solid phase.

We can now investigate the question of the choice and the size of the averaging volume V. The averaged values should vary continuously and smoothly with position. This will be achieved if V is large enough so that a small change in the location of V and its attendant reference point $\boldsymbol{x}$ yields a small change in the averaged values. That is, we wish to have

$$\left|\nabla\left\langle B_{\mathrm{f}}\right\rangle\right| << \frac{\left\langle B_{\mathrm{f}}\right\rangle}{l} \tag{8}$$

where l is the characteristic length scale of V. If this condition is satisfied, then the averages of averages obey the following equations as in [59].

$$\left\langle\left\langle B_{\mathrm{f}}\right\rangle\right\rangle^{\mathrm{f}}\approx\left\langle B_{\mathrm{f}}\right\rangle \tag{9}$$

$$\left\langle\left\langle B_{\mathrm{f}}\right\rangle^{\mathrm{f}}\right\rangle^{\mathrm{f}}\approx\left\langle B_{\mathrm{f}}\right\rangle^{\mathrm{f}} \tag{10}$$

Since the averaging process gives the solution of the variables on a scale comparable to l, equations based on local volume averaging technique are only meaningful when the global length scale of the problem is much

larger than *l*. If this is not so, then, rigorously speaking, there is no representative control volume over which the averaging can take place and the approach of local volume averaging cannot be pursued. However, it is not necessary to actually determine an explicit volume *V* in order to derive the governing set of equations based on local volume averages. In practice, one frequently asks the question of whether or not a proper averaging volume exists.

In contrast, one must explicitly determine an averaging volume to make micromechanical representations of the properties of the porous medium, such as permeability. When the porous solid mat has a regular structure, like a woven fabric, then a unit cell like the one shown in Figure 2 (a) can be used as the representative volume. If the medium has a random structure then, the representative volume will contain many particles in an averaged sense (Figure 2 (b)).

2.3 The averaging theorem

When deriving balance equations, one often takes the average of the gradient of some variable, say $\langle \nabla p_{\mathrm{f}} \rangle$. Although one can replace this with the gradient of the average $\nabla \langle p_{\mathrm{f}} \rangle$, but this is incorrect. Instead, the average of the gradient is

$$\langle \nabla p_{\mathrm{f}} \rangle = \nabla \langle p_{\mathrm{f}} \rangle + \frac{1}{V} \int_{S_{\mathrm{fs}}} p_{\mathrm{f}} \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{11}$$

In the above, S_{fs} is the interfacial area between fluid and solid phases within the averaging volume *V* and $\boldsymbol{n}_{\mathrm{fs}}$ is the unit normal to that surface directed from the fluid towards the solid. Because this theorem is central to the development of the conservation equations, its derivation is given in Appendix 3.B of Ref.[10].

2.4 Principals for deriving conservation equations

There are two bases for deriving conservation equations for porous media: postulation and averaging [10]. In the postulation technique, one obtains conservation equations for each phase by stating the conservation laws directly in terms of the averaged quantities. Constitutive relations, including the transport coefficients of the phases and the interactions between them are then determined from experiments. The final form of the mathematical model in this approach is an implicit set of equations in which the dependent variables are the averages of the microscopic field variables such as velocity, pressure, stress, etc.

In the averaging technique, we begin by stating the microscopic conservation equations for each phase in differential form. These are the well-known familiar equations of fluid mechanics. We then take the phase average of each equation to produce an averaged conservation equation. This process is mathematically rigorous, and, at this first stage, the averaged equation is exact.

2.5 Main assumptions

The derivations that follow assumes:

- The solid phase is not moving (i.e. stationary solid phase)
- There is mass transfer between the solid and the fluid
- The densities of the solid and fluid are constant

Other assumptions will be introduced as needed. All assumptions will be indicated with bullets. (•)

3.3 The conservation of mass: Continuity equation

The continuity equation or conservation of mass equation states that matter is conserved. Since the solid phase does not move, it automatically satisfies the continuity, and we need to develop an equation for the fluid only.
To show the process for developing average conservation equations, we describe several versions of the continuity equation for the fluid. The starting point is to write down the microscopic fluid continuity equation. If we make additional assumptions about compressibility, that is we have

$$\frac{\partial \rho_{\mathrm{f}}}{\partial t}+\nabla\cdot\left(\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\right)=0 \tag{12}$$

Now take the phase average of this equation, to obtain

$$\left\langle\frac{\partial \rho_{\mathrm{f}}}{\partial t}\right\rangle+\left\langle\nabla\cdot\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\right\rangle=0 \tag{13}$$

In the first term the order of integration (averaging) and differentiation with respect to time can be exchanged, while the averaging theorem 11 is applied to the second term. The resulting equation is

$$\frac{\partial}{\partial t}\left\langle\rho_{\mathrm{f}}\right\rangle+\nabla\cdot\left\langle\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\right\rangle+\frac{1}{V}\int_{S_{\mathrm{fs}}}\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S=0 \tag{14}$$

The integral will equal zero when the solid is stationary, i.e. when the porous medium does not move with flowing fluid. Then, the averaged continuity equation becomes

$$\frac{\partial}{\partial t}\left\langle\rho_{\mathrm{f}}\right\rangle+\nabla\cdot\left\langle\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\right\rangle=0 \tag{15}$$

This equation is still not a useful equation, since it includes the average of the product $\rho_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}$. One common assumption [10] is to presume that variations in ρ_{f} are small compared to variations in $\boldsymbol{v}_{\mathrm{f}}$, which simplifies Eq. 15 to

$$\frac{\partial}{\partial t}\left(\rho_{\mathrm{f}}\varepsilon_{\mathrm{f}}\right)+\nabla\cdot\left(\rho_{\mathrm{f}}\left\langle\boldsymbol{v}_{\mathrm{f}}\right\rangle\right)=0 \tag{16}$$

This is a useful form of the continuity equation for the analysis of flow problems in porous media. It has a similar form for a homogenous material except for the presence of ε_{f} in the time derivative.

If we now further assume that ρ_{f} is not constant, then the time derivative of ρ_{f} equals zero and ρ_{f} can be factored out of the remaining terms. Some manipulation of the integral term in Eqn. 14 (details are given in Appendix 3.C of Ref. [10] shows it as the partial time derivative of the fluid volume fraction ε_{f}. The continuity equation for a constant density fluid flowing through a porous media:

$$\frac{\partial \varepsilon_{\mathrm{f}}}{\partial t}+\nabla\cdot\left\langle\boldsymbol{v}_{\mathrm{f}}\right\rangle=0 \tag{17}$$

If the solid is stationary then ε_{f} is constant and Eqn. 17 reduces to a familiar form for an incompressible fluid,

$$\nabla\cdot\left\langle\boldsymbol{v}_{\mathrm{f}}\right\rangle=0 \tag{18}$$

This is the continuity equation used for calculations of fluid flow through porous medium.

3.4 The Conservation of momentum equations

3.4.1 Basic forms

The equation of motion (conservation of momentum) balances the forces applied to each material particle against the particle's acceleration. For each point in a homogenous material the equation of motion (microscopic conservation of momentum equation) is

$$\frac{\partial\left(\rho\boldsymbol{v}\right)}{\partial t}+\nabla\cdot\left(\rho\boldsymbol{v}\boldsymbol{v}\right)=\nabla\cdot\boldsymbol{\sigma}+\rho\boldsymbol{g} \tag{19}$$

where $\boldsymbol{\sigma}$ is the total fluid stress tensor which is comprised of the pressure and viscous and polymeric stresses, and $\boldsymbol{g}$ is the gravitational body force per unit mass. We take the phase average of this equation for the fluid phase together with the assumptions that the solid does not move and there is no interchange of matter between the

fluid and the solid. Based on these two assumptions, the term $\left\langle \nabla \cdot \rho_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle$ will be equal to $\nabla \cdot \left\langle \rho_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle$, then, Eqn. 19 becomes

$$\frac{\partial \left\langle \rho_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle}{\partial t} + \nabla \cdot \left\langle \rho_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle = \nabla \cdot \left\langle \boldsymbol{\sigma}_{\mathrm{f}} \right\rangle + \left\langle \rho_{\mathrm{f}} \boldsymbol{g} \right\rangle + \frac{\boldsymbol{1}}{V} \int_{S_{\mathrm{fs}}} \sigma_{\mathrm{f}} \cdot \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{20}$$

The left-hand side of this equation includes the bulk inertial terms, namely $\nabla \cdot \left\langle \rho_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle$, while the right hand side represents the surface terms due to the fluid stress tensor and the gravitational forces.

We now additionally assume that

- The fluid has a constant density, and
- The body force is due to gravity only

As a result of the first assumption, ρ_{f} can be factored from the terms on the left-hand side while the second assumption allows $\boldsymbol{g}$ to be expressed as the gradient of a potential function:

$$\boldsymbol{g} = -\nabla(gh) \tag{21}$$

where $\boldsymbol{g}$ is the acceleration of gravity and h is the height above the reference level. By making use of the assumption of constant density and the averaging theorem given by Eqn. (3.B.9) of Ref. [10], we can obtain the following equation:

$$\left\langle \rho_{\mathrm{f}} \boldsymbol{g} \right\rangle = -\nabla \left\langle \rho_{\mathrm{f}} gh \right\rangle - \frac{1}{V} \int_{S_{\mathrm{fs}}} \rho_{\mathrm{f}} gh \, \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{22}$$

As the next step, we decompose the total fluid stress tensor $\boldsymbol{\sigma}_{\mathrm{f}}$ into an extra-stress tensor $\boldsymbol{\tau}_{\mathrm{f}}$ which will be given by a constitutive model (i.e. Newtonian or non-Newtonian model) and an isotropic contribution due to the pressure p_{f}:

$$\boldsymbol{\sigma}_{\mathrm{f}} = \boldsymbol{\tau}_{\mathrm{f}} - p_{\mathrm{f}} \boldsymbol{\delta} \tag{23}$$

For illustration, in two-dimensional Cartesian representation, Eqn. 23 is given by

$$\begin{bmatrix} \boldsymbol{\sigma}_{xx} & \boldsymbol{\sigma}_{xy} \\ \boldsymbol{\sigma}_{yx} & \boldsymbol{\sigma}_{yy} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\tau}_{xx} & \boldsymbol{\tau}_{xy} \\ \boldsymbol{\tau}_{yx} & \boldsymbol{\tau}_{yy} \end{bmatrix} - p \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{24}$$

In Eqn. 23, $\boldsymbol{\delta}$ is the unit tensor. Now we can define a modified pressure P_{f} since nabla ,∇, operates on the similar scalars for both the pressure and gravity:

$$P_{\mathrm{f}} \equiv p_{\mathrm{f}} + \rho_{\mathrm{f}} gh \tag{25}$$

Then, the averaged conservation of momentum equation can be written as

$$\rho_{\mathrm{f}} \frac{\partial \left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle}{\partial t} + \rho_{\mathrm{f}} \nabla \cdot \left\langle \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle = -\nabla \left\langle P_{\mathrm{f}} \right\rangle + \nabla \cdot \left\langle \boldsymbol{\tau}_{\mathrm{f}} \right\rangle + \frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{\sigma}_{\mathrm{f}} \cdot \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S - \frac{1}{V} \int_{S_{\mathrm{fs}}} \rho_{\mathrm{f}} gh \, \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{26}$$

The fluid pressure obtained in most experiments is the intrinsic phase average $\left\langle p_{\mathrm{f}} \right\rangle^{\mathrm{f}}$, (i.e. the pore pressure). If one introduces $\left\langle p_{\mathrm{f}} \right\rangle^{\mathrm{f}}$ in place of $\left\langle p_{\mathrm{f}} \right\rangle$, the gradient of the average pressure expands to

$$\nabla\langle p_{\mathrm{f}}\rangle = \nabla\left(\varepsilon_{\mathrm{f}}\left\langle p_{\mathrm{f}}\right\rangle^{\mathrm{f}}\right) = \varepsilon_{\mathrm{f}}\nabla\left\langle p_{\mathrm{f}}\right\rangle^{\mathrm{f}} + \left\langle p_{\mathrm{f}}\right\rangle^{\mathrm{f}}\nabla\varepsilon_{\mathrm{f}} \tag{27}$$

By relating the phase average to the intrinsic average, one can also say that

$$\langle\rho_{\mathrm{f}} gh\rangle = \varepsilon_{\mathrm{f}}\left\langle\rho_{\mathrm{f}} gh\right\rangle^{\mathrm{f}} \tag{28}$$

so that the gradient of the modified pressure can be expressed as

$$\nabla\langle P_{\mathrm{f}}\rangle = \varepsilon_{\mathrm{f}}\nabla\left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}} + \left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}}\nabla\varepsilon_{\mathrm{f}} \tag{29}$$

As given in Appendix 3.D, Eqn.(3.D.2) of Ref. [10], the gradient of the porosity (i.e. volume fraction) $\nabla\varepsilon_{\mathrm{f}}$ can be expressed as an integral over the solid fluid interface S_{fs}:

$$\nabla\varepsilon_{\mathrm{f}} = -\frac{1}{V}\int_{S_{\mathrm{fs}}}\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S \tag{30}$$

Now, the volume-averaged conservation of momentum equation can be written as

$$\rho_{\mathrm{f}}\frac{\partial\langle\boldsymbol{v}_{\mathrm{f}}\rangle}{\partial t} + \rho_{\mathrm{f}}\nabla\cdot\langle\boldsymbol{v}_{\mathrm{f}}\boldsymbol{v}_{\mathrm{f}}\rangle = -\varepsilon_{\mathrm{f}}\nabla\left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}} + \nabla\cdot\langle\boldsymbol{\tau}_{\mathrm{f}}\rangle + \frac{1}{V}\int_{S_{\mathrm{fs}}}\boldsymbol{\sigma}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S$$
$$-\frac{1}{V}\int_{S_{\mathrm{fs}}}\rho_{\mathrm{f}} gh\,\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S - \left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}}\frac{1}{V}\int_{S_{\mathrm{fs}}}\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S \tag{31}$$

3.4.1.1 Solid-fluid interactions

We now focus on the integral terms on the right hand side of the Eqn. 31. All three integrals are taken over the fluid-solid interface S_{fs}. Hence, these integrals represent fluid-solid interactions that take place within the representative control volume V. As a result, they are different than the surface forces present in the $\nabla\cdot\langle\boldsymbol{\tau}_{\mathrm{f}}\rangle$ term, and instead can be regarded as distributed force terms. We note that $-(\boldsymbol{\sigma}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{fs}})$ is the traction exerted by the fluid on the solid. Next, $\boldsymbol{f}_{\mathrm{T}}$ is defined as the total fluid-solid interaction force per-unit volume:

$$\boldsymbol{f}_{\mathrm{T}} \equiv -\frac{1}{V}\int_{S_{\mathrm{fs}}}\boldsymbol{\sigma}_{\mathrm{f}}\cdot\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S \tag{32}$$

where the negative sign means the force is acting against the flow direction.
The second integral on the right-hand side of Eqn.31 is to represent the force per-unit volume exerted on the solid as a result of a gravity-induced pressure gradient in the fluid. This integral term will be denoted by $\boldsymbol{f}_{\mathrm{gf}}$ and given by,

$$\boldsymbol{f}_{\mathrm{gf}} \equiv -\frac{1}{V}\int_{S_{\mathrm{fs}}}\rho_{\mathrm{f}} gh\,\boldsymbol{n}_{\mathrm{fs}}\,\mathrm{d}S \tag{33}$$

This force exists only when the gravity is not ignored, and is not dependent on the ambient pressure. Physically, it may be regarded as the fluid contribution to the buoyancy of the solid. The third integral on the right-hand side of Eqn. 31 is to represent the force exerted on the solid due to the average fluid pressure, and will be denoted by $\boldsymbol{f}_{\mathrm{p}}$:

$$\boldsymbol{f}_{\mathrm{P}} \equiv \left\langle P_{\mathrm{f}} \right\rangle^{\mathrm{f}} \frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S = -\left\langle P_{\mathrm{f}} \right\rangle^{\mathrm{f}} \nabla \varepsilon_{\mathrm{f}} \tag{34}$$

We recall that the effective pressure P_{f} is written by definition as the fluid pressure p_{f} minus the hydrostatic contribution of the gravity, which means the fluid can create a force on the solid due to a uniform pressurization. At first, this term may seem unrealistic, but its physical meaning from a fluid flow point of view is clear. Because of the gradient of the volume fraction in the porous medium, a uniform pressure in the fluid flowing over the internal surface creates a net force on the solid surface. Its direction is anti-parallel to the vector $\nabla\varepsilon_{\mathrm{f}}$ and its magnitude is proportional to the intrinsic averaged pressure $\left\langle P_{\mathrm{f}} \right\rangle^{\mathrm{f}}$. In a homogenous porous medium, $\boldsymbol{f}_{\mathrm{p}}$ will be zero, because $\nabla\varepsilon_{\mathrm{f}}$ is equal to zero.

We now introduce the fluid-solid drag force $\boldsymbol{f}_{\mathrm{d}}$ and it is defined as the difference between the total interaction force and the gravitational and average-pressure contributions:

$$\boldsymbol{f}_{\mathrm{d}} \equiv -\frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{\sigma}_{\mathrm{f}} \cdot \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S + \frac{1}{V} \int_{S_{\mathrm{fs}}} \rho_{\mathrm{f}} g h \, \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S + \left\langle P_{\mathrm{f}} \right\rangle^{\mathrm{f}} \frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{35}$$

so that the total interaction force per unit volume is

$$\boldsymbol{f}_{\mathrm{T}} \equiv \boldsymbol{f}_{\mathrm{d}} + \boldsymbol{f}_{\mathrm{gf}} + \boldsymbol{f}_{\mathrm{P}} \tag{36}$$

We expect that $\boldsymbol{f}_{\mathrm{d}}$ will be associated with motion of fluid as it flows through the solid. When both the fluid and solid move, $\boldsymbol{f}_{\mathrm{T}}$ still represents the relevant fluid solid interaction force. However, because the fluid is assumed to flow through a stationary medium in the current research, we focus our attention on the drag force $\boldsymbol{f}_{\mathrm{d}}$.

3.4.1.2 Volume-averaged conservation of momentum equations

Substitution of Eqn.(35) into Eqn.(26) gives a general form of the phase-averaged conservation equation of motion for fluid with constant density as:

$$\rho_{\mathrm{f}} \frac{\partial \left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle}{\partial t} + \rho_{\mathrm{f}} \nabla \cdot \left\langle \boldsymbol{v}_{\mathrm{f}} \boldsymbol{v}_{\mathrm{f}} \right\rangle = -\varepsilon_{\mathrm{f}} \nabla \left\langle P_{\mathrm{f}} \right\rangle^{\mathrm{f}} + \nabla \cdot \left\langle \boldsymbol{\tau}_{\mathrm{f}} \right\rangle - f_{\mathrm{d}} \tag{37}$$

The minus signs indicate the resistance that the fluid is experiencing as it flows through the fiber network.

3.4.1.3 Equilibrium equations for the solid phase

In this section, we consider the momentum equation for the solid. If the solid fiber network does not move with fluid, then the microscopic momentum balance equation becomes

$$\nabla \cdot \boldsymbol{\sigma}_{\mathrm{s}} + \rho_{\mathrm{s}} \boldsymbol{g} = 0 \tag{38}$$

which is the equilibrium equation for the solid phase. Taking the volume average of this equation over the solid phase on the assumption that gravity is the body force yields

$$\nabla \cdot \left\langle \boldsymbol{\sigma}_{\mathrm{s}} \right\rangle + \frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{\sigma}_{\mathrm{s}} \cdot \boldsymbol{n}_{\mathrm{sf}} \, \mathrm{d}S - \nabla \left\langle \rho_{\mathrm{s}} g h \right\rangle - \frac{1}{V} \int_{S_{\mathrm{fs}}} \rho_{\mathrm{s}} g h \, \boldsymbol{n}_{\mathrm{sf}} \, \mathrm{d}S \tag{39}$$

If it is further assumed that:

- Surface tension between the fluid and the solid may be ignored

Then the normal components of the fluid and solid stress tensor should be the same at the solid-fluid interface:

$$\boldsymbol{\sigma}_{\mathrm{s}} \cdot \boldsymbol{n}_{\mathrm{sf}} + \boldsymbol{\sigma}_{\mathrm{f}} \cdot \boldsymbol{n}_{\mathrm{fs}} = 0 \qquad \text{on } S_{\mathrm{fs}} \tag{40}$$

($\boldsymbol{n}_{\mathrm{sf}} = -\boldsymbol{n}_{\mathrm{fs}}$). The second term in Eqn.(39) is the total fluid-solid interaction force $\boldsymbol{f}_{\mathrm{T}}$. The fourth integral denoted by $\boldsymbol{f}_{\mathrm{gs}}$ is

$$\boldsymbol{f}_{\mathrm{gs}} \equiv \frac{1}{V} \int_{S_{\mathrm{fs}}} \rho_{\mathrm{s}} g h \, \boldsymbol{n}_{\mathrm{sf}} \, \mathrm{d}S \tag{41}$$

This term may be regarded as the solid contribution to buoyancy. The volume averaged solid equilibrium equation over the solid phase now becomes

$$\nabla \cdot \langle \boldsymbol{\sigma}_{\mathrm{s}} \rangle - \nabla \langle \rho_{\mathrm{s}} g h \rangle + \boldsymbol{f}_{\mathrm{T}} - \boldsymbol{f}_{\mathrm{gs}} = 0 \tag{42}$$

The remaining body force term simplifies if it is assumed that

- The solid has a constant density over the scale of V, and
- The solid volume fraction ε_{s} is uniform inside the porous media.

Then making use of $\langle h \rangle = \varepsilon_{\mathrm{s}} h$, we can obtain

$$-\nabla \langle \rho_{\mathrm{s}} g h \rangle = \varepsilon_{\mathrm{s}} \rho_{\mathrm{s}} \boldsymbol{g} \tag{43}$$

which is simply the averaged gravity effect on the solid fibers. In the field of flow through porous media, Eqn.(43) can be used to calculate and report the stresses in the solid induced by the fluid motion. Also, a constitutive equation for the solid stress $\langle \boldsymbol{\sigma}_{\mathrm{s}} \rangle$ (i.e. Hook's law) will be used, together with an experimental description of the friction forces between the solid and the mold surface.

3.4.2 Fluid equations without inertia

In this section, in order to obtain a relationship for the drag term, $\boldsymbol{f}_{\mathrm{d}}$, we ignore the inertial terms included in the averaged momentum equations so that scaling arguments can be seen. In the following sections, we will revisit $\boldsymbol{f}_{\mathrm{d}}$ when the inertia is included in the flow field. If is assumed that

- Inertia is negligible

Then, this assumption sets the left-hand side of Eqn.(3.4.18) to be zero so that we can now focus on the divergence of the averaged viscous stress and the solid fluid drag term $\boldsymbol{f}_{\mathrm{d}}$. Inertial effects will be discussed later.

3.4.2.1 Divergence of the average fluid stress tensor

The extra stress can be decomposed as a Newtonian part and a polymeric part in order to derive a relationship for $\boldsymbol{f}_{\mathrm{d}}$:

$$\boldsymbol{\tau}_{\mathrm{f}} = \eta \left[\nabla \boldsymbol{v}_{\mathrm{f}} + \left(\nabla \boldsymbol{v}_{\mathrm{f}} \right)^{\mathrm{T}} \right] + \boldsymbol{T}_{\mathrm{f}} \tag{44}$$

where the superscript $(\,)^{\mathrm{T}}$ indicates a tensor transpose, and $\boldsymbol{T}_{\mathrm{f}}$ is the polymeric contribution to the extra stress and is given by an appropriate constitutive model. The averaging theorem yields

$$\langle \nabla \boldsymbol{v}_{\mathrm{f}} \rangle = \nabla \langle \boldsymbol{v}_{\mathrm{f}} \rangle + \frac{1}{V} \int_{S_{\mathrm{fs}}} \boldsymbol{v}_{\mathrm{f}} \boldsymbol{n}_{\mathrm{fs}} \, \mathrm{d}S \tag{45}$$

The integral on the right hand side will be zero based upon the assumption of a motionless solid. If we make an additional assumption that the fluid has no slip at the pore walls, then the average extra-stress depends only on the averaged rate-of-strain and averaged polymeric stress contribution to the extra-stress. Taking the phase average of Eqn. (44) yields:

$$\langle \boldsymbol{\tau}_{\mathrm{f}} \rangle = \eta \left[\nabla \langle \boldsymbol{v}_{\mathrm{f}} \rangle + \left(\nabla \langle \boldsymbol{v}_{\mathrm{f}} \rangle \right)^{\mathrm{T}} \right] + \langle \boldsymbol{T}_{\mathrm{f}} \rangle \tag{46}$$

3.4.2.2 Viscous drag force for isotropic porous media

The drag term variable $\boldsymbol{f}_{\mathrm{d}}$ must be expressed by some correlation that includes only the averaged field variables and the other material parameters. In derivations of porous media theory, the selection of a function for $\boldsymbol{f}_{\mathrm{d}}$ is considered as a problem in constitutive modeling. The procedure is to assume a vector valued function for $\boldsymbol{f}_{\mathrm{d}}$, to use representative theorems to form this function, and finally impose dimensional analysis to indicate how the various material parameters may appear in the function for $\boldsymbol{f}_{\mathrm{d}}$. For instance, we assume that:

- Any function for $\boldsymbol{f}_{\mathrm{d}}$ will be independent of frame (since $\boldsymbol{f}_{\mathrm{d}}$ is independent of frame)
- The function for $\boldsymbol{f}_{\mathrm{d}}$ should include the difference between the intrinsic phase average velocities of the fluid and solid
- $\boldsymbol{f}_{\mathrm{d}}$ should be independent of density of fluid (consistent with the assumption that inertia is negligible); and
- the dependence of $\boldsymbol{f}_{\mathrm{d}}$ on other flow parameters must be obtained by performing dimensional analysis

Symbolically, these assumptions can be expressed as

$$\boldsymbol{f}_{\mathrm{d}} = \mathbf{F}\left(\left[\left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle^{\mathrm{f}} - \left\langle \boldsymbol{v}_{\mathrm{s}} \right\rangle^{\mathrm{s}} \right], \eta, \varepsilon_{\mathrm{f}}, l_{\mathrm{o}} \right) \tag{47}$$

where l_{o} is a local length scale of the porous medium and **F** is a vector valued function.
Now we additionally assume that

- the porous medium is isotropic

Imposing a representation theorem for a vector valued isotropic function implies that Eqn.(47) must produce

$$\boldsymbol{f}_{\mathrm{d}} = R\left[\left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle^{\mathrm{f}} - \left\langle \boldsymbol{v}_{\mathrm{s}} \right\rangle^{\mathrm{s}} \right] \tag{48}$$

where R is a scalar resistance coefficient and may depend on flow and solid parameters such as η, ε_{f}, l_{o} and on the scalar magnitude $\left| \left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle^{\mathrm{f}} - \left\langle \boldsymbol{v}_{\mathrm{s}} \right\rangle^{\mathrm{s}} \right|$. Performing dimensional analysis gives the desired result as

$$\boldsymbol{f}_{\mathrm{d}} = \frac{\varepsilon_{\mathrm{f}} \eta}{l_{\mathrm{o}}^2 k^*} \left[\left\langle \boldsymbol{v}_{\mathrm{f}} \right\rangle^{\mathrm{f}} - \left\langle \boldsymbol{v}_{\mathrm{s}} \right\rangle^{\mathrm{s}} \right] \tag{49}$$

where k^* is a dimensionless constant. When the solid does not move, Eqn.(3.4.31) can be written as

$$\boldsymbol{f}_{\mathrm{d}} = \frac{\varepsilon_{\mathrm{f}} \eta}{S} \langle \boldsymbol{v}_{\mathrm{f}} \rangle \tag{50}$$

In Eqn. (50), the quantity S ($\equiv \varepsilon_{\mathrm{f}} \, l_{\mathrm{o}}^2 \, k^*$) is called the permeability. A number of experimental investigation implies that permeability is determined primarily by the geometry and dimensions of the solid, and not by any fluid properties, if assumptions made are valid and reasonable.
Comparing Eqn.(49) and Eqn.(50) shows that the local length scale of the volume l_{o} has the same order of magnitude as the square root of permeability. (e.i. $l_{\mathrm{o}} \sim \sqrt{S}$)

3.4.3 Flow Inertia Effects

We now return to the volume averaged conservation of momentum equations and include the inertia. The assumption of constant fluid density will be retained.

3.4.3.1 Bulk Inertia and inertial dispersion of the flow

The time derivative on the left hand side of momentum equation does not have to be simplified any further. Therefore, we need to provide an interpretation of the term $\rho_f \nabla \cdot \langle \boldsymbol{v}_f \boldsymbol{v}_f \rangle$. In Appendix 3.E of Ref. [10], the procedure of how to expand the averages of products is explained as

$$\langle \boldsymbol{v}_f \boldsymbol{v}_f \rangle = \frac{1}{\varepsilon_f} \langle \boldsymbol{v}_f \rangle \langle \boldsymbol{v}_f \rangle + \langle \hat{\boldsymbol{v}}_f \hat{\boldsymbol{v}}_f \rangle \tag{51}$$

where $\hat{\boldsymbol{v}}_f$ is the point-wise deviation from $\boldsymbol{v}_f$ from its intrinsic averaged value,

$$\boldsymbol{v}_f = \langle \boldsymbol{v}_f \rangle^f + \hat{\boldsymbol{v}}_f \tag{52}$$

The inertia term then becomes

$$\rho_f \nabla \cdot \langle \boldsymbol{v}_f \boldsymbol{v}_f \rangle = \rho_f \nabla \cdot \left[\frac{1}{\varepsilon_f} \langle \boldsymbol{v}_f \rangle \langle \boldsymbol{v}_f \rangle \right] + \rho_f \nabla \cdot \langle \hat{\boldsymbol{v}}_f \hat{\boldsymbol{v}}_f \rangle \tag{53}$$

The convection of momentum by the average velocity is represented by the first term on the right hand side of Eqn.(53); and is called the bulk inertia term. Local differences between point-wise and averaged velocity are represented by the second term on the right hand side of Eqn.(53), and is called the inertial dispersion term. The inertial dispersion term is similar to the Reynolds stress terms in turbulence theory; it represents the transport of momentum by velocity fluctuations. We note that a fluid in a porous medium experiences velocity fluctuations even if the flow is laminar, due to the heterogeneous character of the porous medium.

Since $\hat{\boldsymbol{v}}_f$ is not among the flow variables, we must use some constitutive equation or closure approximation for the dispersion term. As an example, one might assume that the average $\langle \hat{\boldsymbol{v}}_f \hat{\boldsymbol{v}}_f \rangle$ is related to the gradient in averaged velocity and make the approximation

$$\rho_f \nabla \cdot \langle \hat{\boldsymbol{v}}_f \hat{\boldsymbol{v}}_f \rangle = \rho_f \nabla \cdot \left[\boldsymbol{M}_D : \nabla \langle \boldsymbol{v}_f \rangle \right] \tag{54}$$

where $\boldsymbol{M}_D$ is a fourth-order "dispersive viscosity" tensor and its value would depend on the average velocity $\langle \boldsymbol{v}_f \rangle$ and other flow parameters such as viscosity, porosity, particle shape and size, etc.

There are no explicit expressions of inertial dispersion tensor $\boldsymbol{M}_D$ in the literature. Some researchers claim that inertial dispersion can be ignored when compared to bulk inertia term because inertial dispersion is of higher order than bulk inertia. However, an order of magnitude analysis can prove that dispersion and bulk inertia term can have the same magnitude. In this work, because the inertial effects are less significant compared to viscous and elastic effects, this term will be ignored.

3.4.3.2 Inertial drag for isotropic media

As a result of phase averaging of the inertial terms, in addition to the bulk inertia and inertial dispersion terms appearing in the averaged momentum equations, inertial effects become important on the micro scale, where they become a contributor to the drag force $\boldsymbol{f}_{\mathrm{d}}$ in addition to the stress terms.
Forchheimer was the first to propose a modification to Darcy's Law to account for inertial effects. Forchheimer's equation for steady state one-dimensional flow in the x direction in a sand bed with a very large cross section is written as

$$-\frac{\partial}{\partial x}\left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}}=\frac{\eta}{S}\left\langle \boldsymbol{v}_{x}\right\rangle+\frac{\rho_{\mathrm{f}}\, b}{\sqrt{S}}\left\langle \boldsymbol{v}_{x}\right\rangle^{2} \tag{55}$$

where v_{x} is the x component of $\boldsymbol{v}_{\mathrm{f}}$ and b is a dimensionless material parameter.
Forchheimer's equation can be generalized by rewriting it as an expression for the solid-fluid drag force $\boldsymbol{f}_{\mathrm{d}}$ as

$$\boldsymbol{f}_{\mathrm{d}}=\frac{\varepsilon_{\mathrm{f}}\,\eta}{S}\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle+\frac{\varepsilon_{\mathrm{f}}\,\rho_{\mathrm{f}}\, b}{\sqrt{S}}\left|\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle\right|\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle \tag{56}$$

In the literature, researchers used arguments based on capillary flow to produce a one-dimensional drag law as:

$$-\frac{\partial}{\partial x}\left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}}=\frac{k_{1}\left(1-\varepsilon_{\mathrm{f}}\right)^{2}}{\varepsilon_{\mathrm{f}}^{3} d_{\mathrm{p}}^{2}}\left\langle v_{x}\right\rangle+\frac{k_{2}\ \rho_{\mathrm{f}}\left(1-\varepsilon_{\mathrm{f}}\right)}{\varepsilon_{\mathrm{f}}^{3} d_{\mathrm{p}}}\left\langle v_{x}\right\rangle^{2} \tag{57}$$

where d_{p} is the particle diameter, while k_1 and k_2 are constants. When Eqn.(57) is compared to Eqn.(55), we deduce that the permeability is

$$S=\frac{\varepsilon_{\mathrm{f}}^{3} d_{\mathrm{p}}^{2}}{k_{1}\left(1-\varepsilon_{\mathrm{f}}\right)^{2}} \tag{58}$$

while the inertial term contains a dependence on porosity,

$$b=\frac{k_{2}}{\sqrt{k_{1}}}\,\varepsilon_{\mathrm{f}}^{-3/2} \tag{59}$$

where k_1 =150 and k_2 =1.75 to fit a variety of experimental data.

In our research, we used the generalized Forchheimer's equation to incorporate the drag into the momentum equations. Also, the permeability given by Eqn. (58) together with Eqn. (59) is used in modeling.

3.4.3.3 Equation of motion with inertia

The purpose of neglecting inertia previously was solely to be able to obtain an expression for the drag term, $\boldsymbol{f}_{\mathrm{d}}$. We can now write down the equation of motion for flow thorough porous media when inertia and elastic effects of the fluid are important. The desired result is given as

$$\begin{aligned}\rho_{\mathrm{f}}\frac{\partial\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle}{\partial t}+\rho_{\mathrm{f}}\nabla\cdot\left[\frac{1}{\varepsilon_{\mathrm{f}}}\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle\right]+\rho_{\mathrm{f}}\nabla\cdot\left[\boldsymbol{M}_{\mathrm{D}}:\nabla\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle\right]&\\ =-\varepsilon_{\mathrm{f}}\nabla\left\langle P_{\mathrm{f}}\right\rangle^{\mathrm{f}}+\nabla\cdot\left\langle \boldsymbol{\tau}_{\mathrm{f}}\right\rangle-\varepsilon_{\mathrm{f}}\left[\frac{\eta}{S}+\frac{\rho_{\mathrm{f}}\, b\left|\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle\right|}{\sqrt{S}}\right]\left\langle \boldsymbol{v}_{\mathrm{f}}\right\rangle&\end{aligned} \tag{60}$$

This equation assumes a general non-Newtonian fluid with constant viscosity and density flowing in an isotropic porous medium. It reduces to the isotropic version of the Brinkman equation for slow Newtonian flows. To use

Eqn.(60), one must either have an expression for the inertial dispersion tensor $\boldsymbol{M}_{\mathrm{D}}$ or else ignore dispersion entirely. In this work, the inertial dispersion tensor $\boldsymbol{M}_{\mathrm{D}}$ is neglected.

In summary, the final form of the two-dimensional governing conservation equations of mass and momentum after applying the local volume averaging theorems for the flow through an isotropic porous medium are given below:

3.4.4.1 Continuity Equation

$$\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}=0 \tag{61}$$

3.4.4.2 Momentum Equations

$$\rho_{\mathrm{f}}\frac{\partial\langle \boldsymbol{v}_{\mathrm{f}}\rangle}{\partial t}+\rho_{\mathrm{f}}\frac{1}{\varepsilon_{\mathrm{f}}}\langle \boldsymbol{v}_{\mathrm{f}}\rangle\cdot\nabla\langle \boldsymbol{v}_{\mathrm{f}}\rangle=-\varepsilon_{\mathrm{f}}\nabla\langle P\rangle^{\mathrm{f}}+\nabla\langle \boldsymbol{\tau}_{\mathrm{f}}\rangle-$$
$$\frac{\eta\varepsilon_{\mathrm{f}}}{S}\langle \boldsymbol{v}_{\mathrm{f}}\rangle-\frac{b\rho_{\mathrm{f}}\left|\langle \boldsymbol{v}_{\mathrm{f}}\rangle\right|}{\sqrt{S}}\langle \boldsymbol{v}_{\mathrm{f}}\rangle \tag{62}$$

3.4.4.2.1 x- Momentum Equation

$$\rho_{\mathrm{f}}\frac{\partial\langle u\rangle}{\partial t}+\rho_{\mathrm{f}}\frac{1}{\varepsilon_{\mathrm{f}}}\left(\langle u\rangle\cdot\frac{\partial\langle u\rangle}{\partial x}+\langle v\rangle\cdot\frac{\partial\langle u\rangle}{\partial y}\right)=-\varepsilon_{\mathrm{f}}\frac{\partial\langle P\rangle^{f}}{\partial x}+\left[\frac{\partial}{\partial x}\langle \boldsymbol{\tau}_{\mathrm{f}xx}\rangle+\frac{\partial}{\partial y}\langle \boldsymbol{\tau}_{\mathrm{f}xy}\rangle\right]-$$
$$\frac{\eta\varepsilon_{\mathrm{f}}}{S}\langle u\rangle-\frac{b\rho_{\mathrm{f}}\left(\langle u\rangle^{2}+\langle v\rangle\right)^{1/2}}{\sqrt{S}}\langle u\rangle \tag{62.a}$$

3.4.4.2.2 y- Momentum Equation

$$\rho_{\mathrm{f}}\frac{\partial\langle v\rangle}{\partial t}+\rho_{\mathrm{f}}\frac{1}{\varepsilon_{\mathrm{f}}}\left(\langle u\rangle\cdot\frac{\partial\langle v\rangle}{\partial x}+\langle v\rangle\cdot\frac{\partial\langle v\rangle}{\partial y}\right)=-\varepsilon_{\mathrm{f}}\frac{\partial\langle P\rangle^{\mathrm{f}}}{\partial y}+\left[\frac{\partial}{\partial x}\langle \boldsymbol{\tau}_{\mathrm{f}xy}\rangle+\frac{\partial}{\partial y}\langle \boldsymbol{\tau}_{\mathrm{f}yy}\rangle\right]-$$
$$\frac{\eta\varepsilon_{\mathrm{f}}}{S}\langle v\rangle-\frac{b\rho_{\mathrm{f}}\left(\langle u\rangle^{2}+\langle v\rangle\right)^{1/2}}{\sqrt{S}}\langle v\rangle \tag{62.b}$$

4. Conclusion

We have adapted the volume averaging technique to derive the mass and momentum balance equations for the flow of resin through the woven fiber mats in LCM which are modeled as dual-scale porous media and where the gaps and fiber tows are treated as two different phases.

The general form of the momentum balance equation is given in Eqn. (60). This equation is a general equation where one can incorporate non-Newtonian flow effects which is the case for generalized Newtonian or viscoelastic polymeric fluid flows.

These general averaged equations contain more than just the averaged variables. Various constitutive equations representing the physics of resin flow are then substituted to replace those terms with functions of the averaged variables using some assumptions and guidance from experiments. These constitutive equations are substituted back into the general averaged equations to obtain a useful set of equations in terms of the averaged variables. The averaging approach has two important advantages. Firstly, it ensures that all of the terms are accounted for. Therefore, averaging reveals those effects that might not have been predicted. Second, averaging puts useable

physical insight into the relationship between microscopic and macroscopic behavior, and provides a basis for micromechanical calculations of the transport coefficients such as permeability. The averaging approach is used here to obtain the model.

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Figures

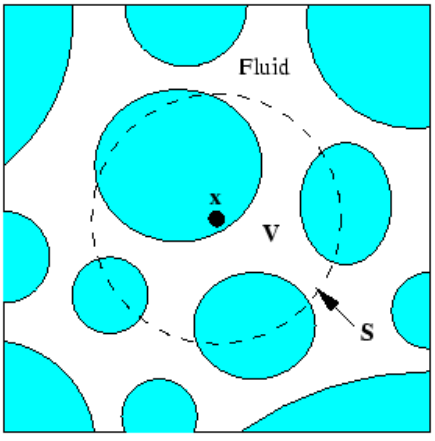

Figure 1 Microscopic View of a Porous Medium,
showing the averaging volume *V* and its surface S associated with a point

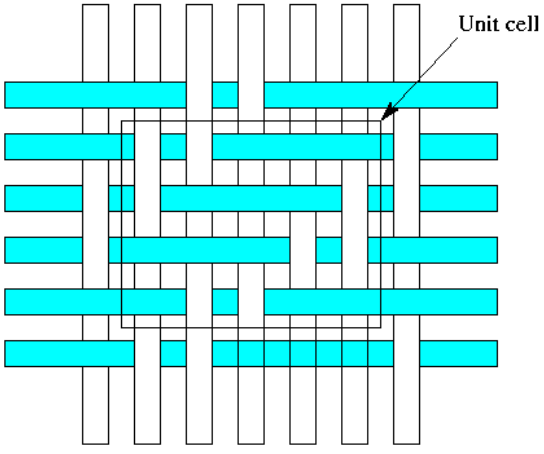

Figure 2 (a) A unit cell (heavy line)
in a 2 : 2 twill weave fabric

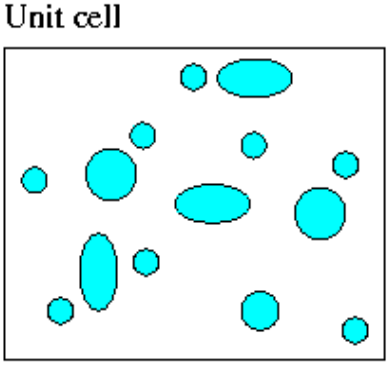

Figure 2 (b) A unit cell (heavy line)
for many particles in an averaged sense