Univers. J. Math. Appl., *8 (1) (2025) 30-40* https://doi.org/10.32323/ujma.1592877

UJMA

Universal Journal of Mathematics and Applications

ISSN (Online) 2619-9653
https://dergipark.org.tr/en/pub/ujma

Research Article

Solvability of Infinite Systems of Third Order Differential Equations in a Sequence Space $n(\phi)$ via Measures of Non-Compactness

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Article Info

Keywords: *Darbo's fixed point theorem, Green's function, Infinite system of third order differential equation, Measures of non-compactness, Meir-Keeler condensing operator*
2020 AMS: *34A34, 46B45, 46B99, 47H08*
Received: *28 November 2024*
Accepted: *6 March 2025*
Available online: *11 March 2025*

Abstract

This paper establishes the necessary conditions for the existence of ω-periodic solutions in the sequence space $n(\phi)$ for an infinite system of third-order differential equations. The analysis utilizes the system's Green's function, the Meir-Keeler condensing operator, and measures of non-compactness. To illustrate our results, we provide relevant examples.

1. Introduction and Preliminaries

Measures of non-compactness refers to a function that determines to what extent a set is non-compact. This concept was pioneered by Kuratowski [1] in 1930, who introduced the function α. Other researchers used it as a base to come up with more measures of non compactness (see, [2–4]). Measures of non-compactness in combination with some fixed point theorem has been widely used to show the existence of solutions to various infinite system of equations. Banaś and Lecko [5] presented the existence theorems for infinite systems of first order differential equations by using the concept of measures of non compactness on Banach sequence spaces c_0, c and ℓ_1. On extension of this result, Mursaleen and Mohiuddine [6] gave conditions for existence of solutions to a similar system in a sequence space ℓ_p using techniques related with measures of non-compactness. Mursaleen [7] introduced the Hausdorff measure of non-compactness on the sequence space $n(\phi)$ and proved the theorem that validates there exist solutions to infinite systems of first order differential equations in this space. A theorem to support the existence of solutions to an infinite system of second order differential equations in the sequence spaces c_0 and ℓ_1 was developed by Mursaleen and Rizvi [8] . The authors established the existence theorems for an infinite system of second order differential equations [9] and [10] in $n(\phi)$. Green's function for third-order differential equations with constant coefficients was established by Chen *et al.* [11]. This result motivated Saadat *et al.* [12] to investigate wheather infinite system of third order differential equations are solvable. Using the obtained Green's function, measures of non-compactness, and Meir-Keeler condensing operators, they demonstrated that an infinite system of third-order differential equations can have ω-periodic solutions in the Banach sequence space c_0. This conclusion was expanded to another sequence space ℓ_p, by Pourhadi *et al.* [13]. Inspired by these results, the focal point of this study is to examine the necessary conditions for the ω-periodic solutions to exist in an infinite system of third order differential equations within a sequence space $n(\phi)$. Recently in [14–18], the solvability of infinite systems of fractional differential equations has been studied in tempered sequence spaces. One can see more results in [19–23] and the references therein.

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Cite as: P. Malaki, S. Kumar, M. Mursaleen, *Solvability of infinite systems of third order differential equations in a sequence space $n(\phi)$ via measures of non-compactness*, Univers. J. Math. Appl., 8(1) (2025), 30-40.

We consider a Banach space E with the norm $\|.\|$. We assume that $B(x,r)$ is the closed ball centred at x with a radius r and B_r represents the ball $B(\theta,r)$ where as θ is the zero element of the Banach space E. Let $\mathfrak{M}$ be a non-empty subset of a set E. In this context, the closure of $\mathfrak{M}$ is represented by $\bar{\mathfrak{M}}$, while the convex closure is denoted as Conv$\mathfrak{M}$. Further, we define M_E as the collection of all non-empty and bounded subsets of E, and N_E as its subset consisting of sets that are relatively compact. The set of real numbers is denoted by $\mathbb{R}$, the interval $[0,+\infty)$ is represented by $\mathbb{R}_+$ and $\mathbb{N}$ stands for the set of natural numbers. The axiomatic measures of non-compactness proposed by Banas and Goebel [24] is defined as follows:

Definition 1.1. *[24] If a mapping $\gamma : M_E \to \mathbb{R}_+$ satisfies the following conditions is reffered to be a measure of non-compactness in E*

i. *The family ker $\gamma = \{\mathfrak{M} \in M_E : \gamma(\mathfrak{M}) = 0\}$ is non-empty and ker $\gamma \subset N_E$.*
ii. $\mathfrak{M}_1 \subset \mathfrak{M}_2 \Rightarrow \gamma(\mathfrak{M}_1) \subset \gamma(\mathfrak{M}_2$.
iii. $\gamma(\bar{\mathfrak{M}}) = \gamma(\mathfrak{M})$.
iv. γ *(Conv $\mathfrak{M}$)* $= \gamma(\mathfrak{M})$.
v. *For all $\lambda \in [0,1]$*
$\gamma(\lambda\mathfrak{M}_1 + (1-\lambda)\mathfrak{M}_2) \leq \lambda\gamma(\mathfrak{M}_1) + (1-\lambda)\gamma(\mathfrak{M}_2)$.
vi. *Suppose $\mathfrak{M}_n$ is a sequence of closed sets taken from M_E such that $\mathfrak{M}_{n+1} \subset \mathfrak{M}_n$ $\forall n \in \mathbb{N}$. If the limit as n approaches infinity of the measure of non-compactness, denoted by $\gamma(\mathfrak{M}_n)$, equals zero, then the intersection set $\mathfrak{M}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{M}_n$ is guaranteed to be non-empty. A measure of non-compactness is classified as a regular measure if it satisfies the following additional conditions.*
vii. $\gamma(\mathfrak{M}_1 \cup \mathfrak{M}_2) = max\{\gamma(\mathfrak{M}_1), \gamma(\mathfrak{M}_2)\}$
viii. $\gamma(\mathfrak{M}_1 + \mathfrak{M}_2) \leq \gamma(\mathfrak{M}_1) + \gamma(\mathfrak{M}_2)$
ix. $\gamma(\lambda\mathfrak{M}) = |\lambda|\gamma(\mathfrak{M})$
x. *ker $\gamma = N_E$.*

The Haursdoff measure of non-compactness developed by Goldenstian *et al.* [2] and further researched by Goldenstian and Markus [3] is the most beneficial and convenient in terms of application among all measures of non-compactness.

Definition 1.2. *[25] Consider $(\mathscr{X},d)$ be a metric space and let $\mathfrak{M}$ be a bounded subset of $\mathscr{X}$. The Hausdorff measure of non-compactness of $\mathfrak{M}$, denoted as $(\chi(\mathfrak{M}))$, is the infimum of all real numbers $\varepsilon > 0$ such that $\mathfrak{M}$ can be covered by a finite number of balls with radii $< \varepsilon$. In other words,*
$\chi(\mathfrak{M})$ = *inf* $\{\varepsilon > 0 : \mathfrak{M}$ *has a finite* $\varepsilon -$ *net in* $\mathscr{X}\}$.

Definition 1.3. *[25] Let F_1 and F_2 be Banach spaces and γ_1 and γ_2 be arbitrary measures of non-compactness on F_1 and F_2 respectively. An operator $\mathfrak{T}$ mapping from F_1 to F_2 is referred to as $(\gamma_1$-$\gamma_2)$ condensing operator if it satisfies two conditions*

i. *continuity and*
ii. *for every bounded non-compact set $\mathfrak{M}$ in F_1, the measure of non-compactness of the image set $\mathfrak{T}(\mathfrak{M})$ under $\mathfrak{T}$, denoted as $\gamma_2(\mathfrak{T}(\mathfrak{M}))$, is strictly smaller than the measure of non-compactness of $\mathfrak{M}$, denoted as $\gamma_1(\mathfrak{M})$.*

Remark: *If $F_1 = F_2$ and $\gamma_1 = \gamma_2 = \gamma$, then $\mathfrak{T}$ is known as γ-condensing operator.*

Darbo [26] developed fixed point theorem based on the idea of measures of non-compactness. The existence of solutions to numerous types of differential equations and integral equations has been proven using this theorem.

Theorem 1.4. *[26] Let H be a non-empty, closed, bounded, and convex subset of a Banach space F. Suppose $\mathfrak{T} : H \to H$ is a continuous mapping such that for any set $E \subset H$, $\gamma(\mathfrak{T}(H)) \leq k\gamma(H)$, where k is a constant in the range $[0,1)$. Then, the mapping $\mathfrak{T}$ has a fixed point in H.*

Meir and Keeler in 1969 [27], developed another contraction known as Meir-Keeler contraction with it's corresponding fixed point theorem .

Definition 1.5. *[27] Let $(\mathscr{X},d)$ be a complete metric space. A mapping $\mathfrak{T} : \mathscr{X} \to \mathscr{X}$ is said to be Meir-Keeler contraction if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following conditions holds,*
$\varepsilon \leq d(u,v) < \varepsilon + \delta \Rightarrow d(\mathfrak{T}u, \mathfrak{T}v) < \varepsilon$, $\forall u, v \in \mathscr{X}$.

Theorem 1.6. *[27] Let $(\mathscr{X},d)$ be a complete metric space. If $\mathfrak{T} : \mathscr{X} \to \mathscr{X}$ is a Meir-Keeler contraction, then $\mathfrak{T}$ has a unique fixed point.*

Aghajan *et al.* [28] generalized Darbo's fixed point theorems unto Meir-Keeler condensing operators fixed point theorem. This attracted numerous researchers as they turned their mathematical interest on this topic, due to the fact that the imposed conditions are significantly weakened. Aghajan *et al.* [28] extended Darbo's fixed point theorem to fixed point theorems for Meir-Keeler condensing operators.

Definition 1.7. *[28] Let H be a non empty subset of a Banach space F, and let γ be an arbitrary measure of non-compactness of F. An operator $\mathfrak{T} : H \to H$ is called a Meir Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following condition is satisfied,*
$\varepsilon \leq \gamma(Y) < \varepsilon + \delta$ *implies* $\gamma(\mathfrak{T}(Y)) < \varepsilon$ *for any bounded subset Y of H.*

This theorem will be helpful in demonstrating our key finding.

Theorem 1.8. *[28] Let H be a non empty, bounded, closed and convex subset of a Banach space F and let γ be an arbitrary measure of non-compactness on F. If $\mathfrak{T} : H \to H$ is a continuous and Meir-Keeler condensing operator, then $\mathfrak{T}$ has at least one fixed point. Furthermore, the set of all fixed points of $\mathfrak{T}$ in H is compact.*

Definition 1.9. *[29] Let β stands for the space of finite sets of distinct positive integers. For any $\nu \in \beta$, we define the sequence $b(\nu)$ with*

$$b_n(\nu) = \begin{cases} 1 & if \quad n \in \nu, \\ 0 & if \quad n \notin \nu, \end{cases}$$

and

$$\beta_r = \left\{ \nu \in \beta : \sum_{n=1}^{\infty} b_n(\nu) \leq r \right\},$$

such that β_r is the set of ν whose support has cardinality at most r. The set Φ contains all sequences $(\phi_i)_{i=1}^{\infty}$ such that;
$\phi_1 > 0$, $\Delta\phi_i \geq 0$ and $\Delta(\frac{\phi_i}{i}) \leq 0$, for $i = (1,2,...)$.
For $\phi \in \Phi$, we have the following sequences

$$m(\phi) = \left\{ x = x_i : \|x\|_{m(\phi)} = \sup_{r\geq 1} \sup_{\nu \in \beta_r} \left(\frac{1}{\phi_r} \sum_{i\in\nu} |x_i| \right) < \infty \right\},$$

$$n(\phi) = \left\{ x = x_i : \|x\|_{n(\phi)} = \sup_{u\in S_x} \left(\sum_{i=1}^{\infty} |u_i| \Delta\phi_i \right) < \infty \right\},$$

where $\Delta\phi_i = \phi_i - \phi_{i-1}$, $\phi_0 = 0$ and $S(x)$ denotes the set of all sequences that are rearrangements of x.
***Remark**: For all $n \in \mathbb{N}$, if $\phi_n = 1$ then $m(\phi) = \ell_1$ and $n(\phi) = \ell_\infty$;*
and if $\phi_n = n$ then $m(\phi) = \ell_\infty$ and $n(\phi) = \ell_1$.

Mursaleen [7] introduced the Hausdorff measure of non-compactness on the sequence space $n(\phi)$. But this formula does not define a measure of non-compactness in $n(\phi)$ for the case $\phi_n = 1$. We redefine it as follows:

Theorem 1.10. *For any bounded subset M of $n(\phi)$, the Hausdorff measure of non-compactness of the set M is given by*

$$\chi(M) = \lim_{k\to\infty} \sup_{x\in M} \left(\sup_{u\in S(x)} \left(\sum_{n=k}^{\infty} |u_n| \Delta\phi_n \right) \right), \text{ for } \phi_n \neq 1;$$

$$= \lim_{k\to\infty} \left\{ \sup_{x\in M} \{\sup[| x_n - x_m |: n,m \geq k]\} \right\}, \text{ for } \phi_n = 1.$$

Throughout this paper, we study the following infinite system of third order differential equations:

$$y_i''' + py_i'' + qy_i' + ry_i = h_i(\psi, y_1(\psi), y_2(\psi), ...) \tag{1.1}$$

where $h_i \in C(\mathbb{R} \times \mathbb{R}^\infty, \mathbb{R})$ with regard to the first coordinate ψ is ω- periodic and $p,q,r \in \mathbb{R}$ are constants.

Based on the theory of ordinary differential equations, the corresponding homogeneous equation of (1.1) is

$$y_i''' + py_i'' + qy_i' + ry_i = 0, i \in \mathbb{N}$$

as well as the corresponding characteristic equation is

$$\xi^3 + p\xi^2 + q\xi + r = 0. \tag{1.2}$$

The roots of the polynomial Equation of (1.2) take one of the following cases:

1. $\xi_1 \neq \xi_2 \neq \xi_3$
2. $\xi_1 = \xi_2 \neq \xi_3$
3. $\xi_1 = \xi_2 = \xi_3$
4. $\xi_1 = a+ib, \xi_2 = a-ib, \xi_3 = \xi$, where a, b, and ξ are real numbers.

The case $r = 0$ is not considered since the results can be easily extended to cover this special case. Therefore, the roots are assumed to be non-zero.
The main novelty of this work is to establish the necessary conditions for the existence of ω-periodic solutions in the sequence space $n(\phi)$ for an infinite system of third-order differential equations. The advantage of our results are that the space inhand $n(\phi)$ is more general than the classical seuence spaces c_0, c and ℓ_p.
This paper is organized into four sections. Section 1 provides an introduction and covers the necessary preliminaries and background for establishing the main results. Section 2 is divided into five subsections, presents four distinct cases for the theorem proved as the main result. Section 3 discusses two examples that validate the results of Section 2. Finally, Section 4 concludes the study with the suggestion for future study.

2. Main Results

2.1. Solvability in a Banach sequence space $n(\phi)$

In this section we provide the required conditions for existence of ω- periodic solution to the system (1.1).
Firstly, we recall the Fréchet space $\mathbb{R}^\infty$ which is the linear space of all real sequences equipped with the distance

$$d_{\mathbb{R}^\infty}(x,y) = \sup\left\{ \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} : i \in \mathbb{N} \right\},$$

for $x = (x_i), y = (y_i) \in \mathbb{R}^\infty$.

The space of all continuous real functions on $\mathbb{R}$ is represented by $C(\mathbb{R},\mathbb{R}^\infty)$, and $C^3(\mathbb{R},\mathbb{R}^\infty)$ stands for the group of functions on $\mathbb{R}$ that have a third continuous derivative. A function $y \in C^3(\mathbb{R},\mathbb{R}^\infty)$ is known to be a solution of Equation (1.1) if and only if $y \in C(\mathbb{R},\mathbb{R}^\infty)$ is a solution of the following infinite system:

$$y_i(\psi) = \int_{\psi}^{\psi+\omega} G(\psi,\varsigma)h_i(\varsigma,y(\varsigma))d(\varsigma), (i \in \mathbb{N}),$$

where the Green's function will be specified in corresponding to different cases.

i. The functions $h_i : \mathbb{R}\times\mathbb{R}^\infty \to \mathbb{R}$ are supposed to be ω-periodic with regard to the first coordinate. The operator $h_i : \mathbb{R}\times n(\phi) \to n(\phi)$ is defined below

$$(\psi,y) \to (hy)(\psi) = (h_1(\psi,y),h_2(\psi,y),...)$$

is such that the class of all functions $\{(hy)(\psi)\}_{\psi\in\mathbb{R}}$ is equicontinuous at each point of the space $n(\phi)$.

ii. The following inequality is true for any $i \in \mathbb{N}$

$$|h_n(\psi,y_1,y_2,...)| \leq u_n(\psi)+v_n(\psi)|y_i(\psi)|,$$

for $\psi \in \mathbb{R}$ and $y = y_i$ in $n(\phi)$. It is assumed that the functions $u_n(\psi)$ and $v_n(\psi)$ are continuous on $\mathbb{R}$, such that the mapping series

$$\sum_{k\geq 1} |u_k|\Delta\phi_k$$

converges uniformly on $\mathbb{R}$, while the sequence $v_n(\psi)$ is equibounded on $\mathbb{R}$.

Suppose,

$$U = \sup_{\psi\in\mathbb{R}} \left\{ \sum_{k\geq 1} u_k\Delta\phi_k \right\},$$

$$V = \sup_{n\in\mathbb{N}} \{ v_n(\psi)\},$$

and assume L is given as seen in [12] i.e

$$L = \frac{e^{(\omega|\xi_1|)}}{\left|(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})\right|} + \frac{e^{(\omega|\xi_2|)}}{\left|(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})\right|} + \frac{e^{(\omega|\xi_3|)}}{\left|(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})\right|}.$$

2.2. Solvability for case 1

In this part, we demonstrate the system's (1.1) solvability by assuming that the roots of Equation (1.2) are $\xi_1 \neq \xi_2 \neq \xi_3$. According to Chen *et al.* [11], the appropriate Green's function in this instance is as follows:

$$G_1(\psi,\varsigma) = \frac{e^{(\xi_1(\psi+\omega-\varsigma))}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})} + \frac{e^{(\xi_2(\psi+\omega-\varsigma))}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})} + \frac{e^{(\xi_3(\psi+\omega-\varsigma))}}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}. \tag{2.1}$$

Theorem 2.1. *If the assumptions (i) and (ii) are true, the system (1.1) has at least one ω-periodic solution, $y(\psi) = y_i(\psi)$ whenever 0 ¡ ωLV ¡ 1, such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. The set of all solutions is also compact.*

Proof. Assume that $S(y(\psi))$ is the collection of all sequences that are rearrangements of $y(\psi)$, and let assumption (ii) hold. Using relation (2.1) for any $\psi \in \mathbb{R}$,

$$\begin{aligned}
\|y(\psi)\|_{n(\phi)} &= \sup_{p\in S(y(\psi))} \left(\sum_{k=1}^{\infty} \left| \int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)h_k(\varsigma,p(\varsigma))d(\varsigma) \right| \Delta\phi_k \right) \\
&\leq \sup_{p\in S(y(\psi))} \left(\sum_{k=1}^{\infty} \int_{\psi}^{\psi+\omega} |G_1(\psi,\varsigma)h_k(\varsigma,p(\varsigma))|\, d(\varsigma)\Delta\phi_k \right) \\
&\leq \sup_{p\in S(y(\psi))} \left(\sum_{k=1}^{\infty} \int_{\psi}^{\psi+\omega} |G_1(\psi,\varsigma)| \left(u_k(\psi)+v_k(\psi)|p_k(\psi)|\right) d(\varsigma)\Delta\phi_k \right) \\
&= \sup_{p\in S(y(\psi))} \left(\sum_{k=1}^{\infty} \int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)u_k(\psi)\Delta\phi_k d(\varsigma) + \sum_{k=1}^{\infty} \int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)v_k(\psi)|p_k(\psi)|\Delta\phi_k d(\varsigma) \right)
\end{aligned}$$

$$\leq \sup_{p\in S(y(\psi))}\left(\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}u_k(\psi)\Delta\phi_k d(\varsigma)+V\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_k(\psi)|\Delta\phi_k d(\varsigma)\right)$$

$$\leq \sup_{p\in S(y(\psi))} U\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)d(\varsigma)+V\sup_{p\in S(y(\psi))}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_k(\psi)|\Delta\phi_k d(\varsigma)$$

$$\leq \omega LU+V\omega L\|y(\psi)\|_{n(\phi)}$$

$$\|y(\psi)\|_{n(\phi)}\leq \frac{\omega LU}{1-\omega LV},=r.$$

This implies that y is a member of B_r where B_r denotes the closed ball with radius r centred at zero. So B_r is non empty, bounded, closed and convex subset of $n(\phi)$.
Here, we define the operator $\mathscr{J}=\mathscr{J}_i$ on $C(\mathbb{R},B_r)$ as:

$$(\mathscr{J}y)(\psi)=(\mathscr{J}_iy)(\psi)=\left\{\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)h_i(\varsigma,y(\varsigma))d(\varsigma)\right\},\psi\in\mathbb{R}, \tag{2.2}$$

where $y(\psi)=y_i(\psi)\in B_r$ and $y_i(\psi)\in C(\mathbb{R},\mathbb{R}),\psi\in\mathbb{R}$. It is plainly known from the presumption (i) that $\mathscr{J}$ is continuous on $C((\mathbb{R},n(\phi))$. Obviously since $y(\psi)=y_i(\psi)\in n(\phi)$, also $(\mathscr{J}y)(\psi)\in n(\phi)$ and $\mathscr{J}y$ is continuous function. Moreover, the function $(\mathscr{J}_iy)(\psi)$ is ω-periodic function whenever $y_i(\psi)$ is ω-periodic function.
Since $\|\mathscr{J}y(\psi)\|_{n(\phi)}\leq r$, thus $\mathscr{J}$ is a self mapping on B_r. We will now demonstrate that $\mathscr{J}$ is a Meir-Keeler condensing operator. Finding δ ¿ 0 such that for any given ε ¿ 0, $\varepsilon\leq\chi(B_r)<\varepsilon+\delta$ implies $\chi(\mathscr{J}(B_r))<\varepsilon$. Assumption (ii) and Theorem 1.10 allow us to arrive at,

$$\chi(\mathscr{J}B_r)=\lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\left|\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)h_i(\varsigma,p(\varsigma))d(\varsigma)\right|\Delta\phi_i\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)h_i(\varsigma,p(\varsigma))|d(\varsigma)\Delta\phi_i\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{i=k}^{\infty}\int_{\psi}^{\psi+\omega}|G_1(\psi,\varsigma)|\left(u_i(\psi)+v_i(\psi)|p_i(\psi)|\right)d(\varsigma)\Delta\phi_i\right)\right)\right\}$$

$$=\lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)u_i(\psi)\Delta\phi_i d(\varsigma)\ +\ \sum_{k=1}^{\infty}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)v_i(\psi)|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right)\right\}$$

$$\leq \lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\left(\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}u_i(\psi)\Delta\phi_i d(\varsigma)\ +\ V\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right)\right\}$$

$$\leq V\lim_{k\to\infty}\left\{\sup_{y(\psi)\in B}\left(\sup_{p\in S(y(\psi))}\int_{\psi}^{\psi+\omega} G_1(\psi,\varsigma)\sum_{k=1}^{\infty}|p_i(\psi)|\Delta\phi_i d(\varsigma)\right)\right\}$$

$$\leq V\omega L\chi(B_r).$$

Therefore, for a given $\varepsilon>0$, if we take $0<\delta\leq\frac{(1-\omega LV)\varepsilon}{\omega LV}$ we get the following $\varepsilon\leq\chi(B_r)<\varepsilon+\delta\implies\chi(\mathscr{J}(B_r))<\varepsilon$. Thus, $\mathscr{J}$ is a Meir-Keeler condensing operator on the set $B_r\subset n(\phi)$. As a result, according to Theorem 1.8, $\mathscr{J}$ has a fixed point in B_r that is a part ker χ. This is the needed solution for the system (1.1). □

Chen *et al.* ([11]) on their work introduced some bounds for Green's function $G_1(\psi,\varsigma)$ which may be used to restate the Theorem 2.1 by exchanging L by obtained upper bounds.
For more simplicity of notations let us consider,

$$f_1:=(\xi_2-\xi_3)e^{(\xi_3\omega)}+2(\xi_1-\xi_3)e^{(\xi_2\omega)}+(\xi_1-\xi_2)e^{(\xi_1\omega)}+(\xi_1-\xi_3)e^{((\xi_1+\xi_2+\xi_3)\omega)},$$
$$g_1:=(\xi_1-\xi_3)+(\xi_1+\xi_2-2\xi_3)e^{((\xi_2+\xi_3)\omega)}+(2\xi_1-\xi_2-\xi_3)e^{((\xi_1+\xi_2)\omega)},$$
$$f_2:=(\xi_1+\xi_2-2\xi_3)e^{(\xi_1\omega)}+(2\xi_1-\xi_2-\xi_3)e^{(\xi_3\omega)}+(\xi_1-\xi_3)e^{((\xi_1+\xi_2+\xi_3)\omega)},$$

$$g_2:=(\xi_1-\xi_3)+(\xi_1-\xi_2)e^{((\xi_2+\xi_3)\omega)}+(\xi_2-\xi_3)e^{((\xi_1+\xi_2)\omega)}+2(\xi_1-\xi_3)e^{((\xi_1+\xi_3)\omega)}$$

$$\mathscr{A}_3=\frac{e^{(\omega\xi_1)}}{(\xi_1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})}+\frac{1}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})}+\frac{e^{(\omega\xi_3)}}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}$$

$$\mathscr{B}_3=\frac{1}{(\xi 1-\xi_2)(\xi_1-\xi_3)(1-e^{(\xi_1\omega)})}+\frac{e^{(\xi_2\omega)}}{(\xi_2-\xi_1)(\xi_2-\xi_3)(1-e^{(\xi_2\omega)})}+\frac{1}{(\xi_3-\xi_2)(\xi_3-\xi_1)(1-e^{(\xi_3\omega)})}. \tag{2.3}$$

Then
$(\mathscr{C}_1)$ If $f_1 \leq g_1$, and one of the following conditions holds:

i. $0 < \xi_3 < \xi_2 < \xi_1$
ii. $\xi_1 > 0 > \xi_2 > \xi_3$

then $\mathscr{A}_3 \leq G_1(\psi,\varsigma) \leq \mathscr{B}_3 < 0$.
$(\mathscr{C}_2)$ If $f_2 > g_2$, and one of the following conditions holds:

i. $\xi_3 < \xi_2 < \xi_1 < 0$
ii. $\xi_3 < 0 < \xi_2 < \xi_1$

then $0 < \mathscr{A}_3 \leq G_1(\psi,\varsigma) \leq \mathscr{B}_3$.
Thus, one can quickly infer the following direct implication of Theorem 2.1 replacing L by the new boundaries by utilizing the recent findings made by Chen *et al.* [11] by using the above bounds, Theorem 2.1 may be changed to Theorem 2.1 and stated as follows:

Theorem 2.2. *Suppose that hypothesis $\mathscr{C}_1$ and the assumptions (i) - (ii) are true and $\omega V|\mathscr{A}_3| < 1$. Additionally, assume that $\mathscr{C}_2$ and the assumptions (i) - (ii) are true and $\omega V\mathscr{B}_3 < 1$. Then the infinite system (1.1) has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. The set of all solutions is compact.*

Proof. On replacing L by $|\mathscr{A}_3|$ or $\mathscr{B}_3$ which are upper bounds of the Green's function $G_1(\psi,\varsigma)$ in the proof of Theorem 2.1, yields the intended outcome. □

2.3. Solvability for case 2

We shall present the solvability for the system (1.1), considering the roots corresponding to the homogenous part of the equation to be
$\xi_1 = \xi_2 \neq \xi_3$
in this section. The following is the Green's function for this instance:

$$G_2(\psi,\varsigma) = \frac{e^{(\xi_1(\psi+\omega-\varsigma))}\left[(1-e^{(\xi_1\omega)})((\psi-\varsigma)(\xi_3-\xi_1)-1)-(\xi_3-\xi_1)\omega\right]}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3(\psi+\omega-\varsigma))}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})}, \varsigma \in \{\psi, \psi+\omega\}.$$

We shall use the bounds concluded by Chen *et al.* [11] on proving the existence of solution for this case. Consider the following:

$$\mathscr{A}_4 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{A}_5 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(2\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{A}_6 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega e^{(\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{e^{(\xi_3\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_4 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(2\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_5 := \frac{e^{(\xi_1\omega)}-1+(\xi_1-\xi_3)\omega}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-e^{(\xi_3\omega)})},$$

$$\mathscr{B}_6 := \frac{e^{(2\xi_1\omega)}-e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega e^{(\xi_1\omega)}}{(\xi_1-\xi_3)^2(1-e^{(\xi_1\omega)})^2} + \frac{1}{(\xi_1-\xi_3)^2(1-\exp(\xi_3\omega))},$$

$$g_3 = e^{(\xi_1\omega)}+(\xi_1-\xi_3)\omega+(e^{(\xi_1\omega)}-3)e^{((\xi_1+\xi_3)\omega)}+(2+(\xi_3-\xi_1)\omega)e^{(\xi_3\omega)},$$

$$g_4 = (3-(\xi_1-\xi_3)\omega)e^{(\xi_1\omega)}+((\xi_1-\xi_3)\omega-1)e^{((\xi_1+\xi_3)\omega)}+(e^{(\xi_3\omega)}-2)e^{(2\xi_1\omega)}.$$

Then from the results obtained by Chen *et al.* [11] we have that
$(\mathscr{C}_3)$ $0 < \mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4$ whenever $\xi_3 < 0 < \xi_1 = \xi_2$,
$(\mathscr{C}_4)$ $\mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4 < 0$ whenever $\xi_1 = \xi_2 < 0 < \xi_3$,
$(\mathscr{C}_5)$ $\mathscr{A}_5 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_5 < 0$ whenever $0 < \xi_1 = \xi_2 < \xi_3$, and $e^{(\xi_1\omega)} < 1+(\xi_3-\xi_1)\omega$,
$(\mathscr{C}_6)$ $0 < \mathscr{A}_4 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_4$ whenever $\xi_1 = \xi_2 < \xi_3 < 0$ and $g_3 > 1$,
$(\mathscr{C}_7)$ $\mathscr{A}_6 \leq G_2(\psi,\varsigma) \leq \mathscr{B}_6 < 0$ whenever $0 < \xi_3 < \xi_2 = \xi_1$, and $g_4 < 1$.

Theorem 2.3. *Assume that (i)-(ii) and hypothesis $\mathscr{C}_3$ (respcively $\mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6$, and $\mathscr{C}_7$) hold. Consider the case when $\omega V\mathscr{B}_4 < 1$ (respecively $\omega V|\mathscr{A}_4| < 1, \omega V|\mathscr{A}_5| < 1, \omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_6| < 1$). The infinite system (1.1) then has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi), \psi \in \mathbb{R}$. Also, the set of all solutions is compact.*

Proof. We get desired results by changing the Green's function from $G_1(\psi,\varsigma)$ to $G_2(\psi,\varsigma)$ from Theorem 2.1, since we are in case 2 where the associated Green's function is $G_2(\psi,\varsigma)$. And also replacing $L\omega V < 1$ from Theorem 2.1 by $\omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_4| < 1, \omega V|\mathscr{A}_5| < 1, \omega V\mathscr{B}_4 < 1, \omega V|\mathscr{A}_6| < 1$ respectively, which are upper bounds of the Green's function $G_2(\psi,\varsigma)$. □

2.4. Solvability for case 3

In this section, we present the theorem for existence of ω-periodic solution to the system (1.1) considering the roots of (1.2) to be $\xi_1 = \xi_2 = \xi_3$. From [11] the associated Green's function for this case is shown to be:

$$G_3(\psi,\varsigma) = \frac{\left[(\varsigma-\psi)e^{(\xi\omega)}+\omega-\varsigma+\psi\right]^2+\omega^2 e^{(\xi\omega)}}{2(1-e^{(\xi\omega)})^3} e^{(\xi(\psi+\omega-\varsigma))}, \varsigma\in\{\psi,\psi+\omega\}.$$

In this case,we establish the existence theorem based on the upper bounds given by Chen *et al.* [11]
For more simplicity denote

$$\mathscr{A}_7 = \frac{\omega^2 e^{(2\xi\omega)}(1+e^{(\xi\omega)})}{2(1-e^{(\xi\omega)})^3}, \mathscr{B}_7 = \frac{\omega^2(1+e^{(\xi\omega)})}{2(1-e^{(\xi\omega)})^3}.$$

($\mathscr{C}_8$) $\mathscr{A}_7 \le G_3(\psi,\varsigma) \le \mathscr{B}_7 < 0$ whenever $\xi > 0$,
($\mathscr{C}_9$) $0 < \mathscr{A}_7 \le G_3(\psi,\varsigma) \le \mathscr{B}_7$ whenever $\xi < 0$.

Theorem 2.4. *Suppose that the presumptions (i)-(ii) and $\mathscr{C}_8$ ($\mathscr{C}_9$ respectively) are true. Consider $\omega V|\mathscr{A}_7| < 1$ ($\omega V\mathscr{B}_7 < 1$ respectively). Then the infinite system (1.1) has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi)\in n(\phi), \psi\in\mathbb{R}$. Moreover, the set of all solutions is compact.*

Proof. Considering $G_3(\psi,\varsigma)$ as the Green's function and exchanging L from the proof of Theorem 2.1 by $|\mathscr{A}_7|$ and $\mathscr{B}_7$ which are the upper bounds of the Green's function $G_3(\psi,\varsigma)$, we are able to achieve the required result. □

2.5. Solvability for case 4

In this section, we present the solvability of system (1.1) by considering the roots of equation (1.2) as $\xi_1 = a+ib, \xi_2 = a-ib, \xi_3 = \xi$. From [11], for this case the Green's function is as follows:

$$G_4(\psi,\varsigma) = \frac{e^{(a(\psi+\omega-\varsigma))}\left[(a-\xi)\mathscr{B}_2(\psi)-b\mathscr{A}_2(\psi)\right]}{b\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})} + \frac{e^{(\xi(\psi+\omega-\varsigma))}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]}, \varsigma\in\{\psi,\psi+\omega\}.$$

where,

$$\mathscr{A}_2(\psi) := \cos b(\psi+\omega-\varsigma) - e^{(a\omega)}\cos b(\psi-\varsigma),$$

$$\mathscr{B}_2(\psi) := \sin b(\psi+\omega-\varsigma) - e^{(a\omega)}\sin b(\psi-\varsigma).$$

We simplify the notations to

$$\mathscr{A}_8 = \frac{-e^{(a\omega)}}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{e^{(\xi\omega)}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{B}_8 = \frac{e^{(a\omega)}}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{1}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{A}_9 = \frac{-1}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{e^{(\xi\omega)}}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]},$$

$$\mathscr{B}_9 = \frac{1}{b\sqrt{\left[(a-\xi)^2+b^2\right](1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)})}} + \frac{1}{(1-e^{(\xi\omega)})\left[(a-\xi)^2+b^2\right]}.$$

From [11] we have,

($\mathscr{C}_{10}$) $0 < \mathscr{A}_8 \le G_4(\psi,\varsigma) \le \mathscr{B}_8$ whenever $\xi < 0 < a, b$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2 e^{(2a\omega)}} < \frac{1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}}{e^{(2a\omega)}},$$

($\mathscr{C}_{11}$) $0 < \mathscr{A}_9 \le G_4(\psi,\varsigma) \le \mathscr{B}_9$ whenever $a, \xi < 0 < b$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2 e^{(2a\omega)}} < 1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)},$$

$(\mathscr{C}_{12})$ $0 < \mathscr{A}_8 \leq G_4(\psi,\varsigma) \leq \mathscr{B}_8$ whenever $a,b,\xi > 0$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2} < \frac{1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}}{e^{(2a\omega)}},$$

$(\mathscr{C}_{13})$ $0 < \mathscr{A}_9 \leq G_4(\psi,\varsigma) \leq \mathscr{B}_9$ whenever $a < 0 < b,\xi$ and

$$\frac{\left[(a-\xi)^2+b^2\right](1-e^{(\xi\omega)})^2}{b^2} < 1+e^{(2a\omega)}-2\cos(b\omega)e^{(a\omega)}.$$

Theorem 2.5. *Suppose that the assumptions (i)-(ii) and hypothesis $\mathscr{C}_{10}$ ($\mathscr{C}_{11}$,$\mathscr{C}_{12}$ and $\mathscr{C}_{13}$ respectively) hold. Let $\omega V\mathscr{B}_8 < 1$ ($\omega V\mathscr{B}_9 < 1$, $\omega V|\mathscr{A}_8| < 1$ and $\omega V|\mathscr{A}_9| < 1$ respectively). Then the infinite system has at least one ω-periodic solution $y(\psi) = y_k(\psi)$ such that $y(\psi) \in n(\phi)$, $\psi \in \mathbb{R}$. Besides, the set of all solutions is compact.*

Proof. In order to achieve the desired conclusion, we replace the Green's function from $G_1(\psi,\varsigma)$ to $G_4(\psi,\varsigma)$ from Theorem 2.1 and $L\omega V < 1$ by $\omega V\mathscr{B}_8 < 1$ ($\omega V\mathscr{B}_9 < 1$, $\omega V|\mathscr{A}_8| < 1$ and $\omega V|\mathscr{A}_9| < 1$ respectively). □

3. Examples

We present two examples in this section for cases 1 and 3 to validate the aforementioned theorems.

3.1. Example 1

Take into account of the following infinite system of differential equation of third order:

$$y_n'''(\psi)+2.9y_n''(\psi)+1.7y_n'(\psi)+0.2y_n(\psi) = \frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}. \tag{3.1}$$

Consider:

$$h_n(\psi,y_k(\psi)) = \frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6}.$$

For $n \in \mathbb{N}$, the function $h_n(\psi,y_k(\psi))$ is seen to be continuous at every point on $\mathbb{R}$ and is 2π - periodic. Additionally, whenever $y(\psi) = y_n(\psi) \in n(\phi)$, $h_n(\psi,y_k(\psi)) \in n(\phi)$.

$$\begin{aligned}
\sum_{n=1}^{\infty}|h_n(\psi,y_k(\psi))| &= \sum_{n=1}^{\infty}\left|\frac{1}{n^4}+\sum_{k=n}^{\infty}\frac{\%y_k(\psi)\cos\psi}{100n^2(k+1)^6}\right| \\
&\leq \sum_{n=1}^{\infty}\frac{1}{n^4}+\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\left|\frac{y_k(\psi)\cos\psi}{\%100n^2(k+1)^6}\right| \\
&\leq \frac{\pi^4}{90}+\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{1}{\%100n^2(k+1)^6}|y_k(\psi)| \\
&\leq \frac{\pi^4}{90}+\frac{1}{100}\sum_{n=1}^{\infty}\sum_{k=n}^{\%\infty}\frac{1}{n^2(1+k)^6}|y_k(\psi)| \\
&\leq \frac{\pi^4}{90}+\frac{\pi^6}{95400}\|y_k(\psi)\|_{n(\phi)} < \infty.
\end{aligned}$$

Now let us prove that the assumption (i) holds. Choose an arbitrary ε ¿ 0 and $y(\psi) = y_n(\psi), z(\psi) = z_n(\psi) \in n(\phi)$ such that,

$$\|y(\psi)-z(\psi)\|_{n(\phi)} < \delta(\varepsilon) := \frac{95400\varepsilon}{\pi^6}.$$

Then,

$$|h_n(\psi,y(\psi))-h_n(\psi,z(\psi))| = \sum_{k=n}^{\infty}\left|\frac{(y_k(\psi)-z_k(\psi))\cos\psi}{100n^2(k+1)^6}\right|$$

$$\leq \sum_{k=n}^{\infty} \frac{|y_k(\psi) - z_k(\psi)|}{100n^2(k+1)^6}$$

$$\leq \frac{\pi^6}{95400} \|y(\psi) - z(\psi)\|_{n(\phi)}$$

$$\leq \frac{\pi^6}{95400} \delta < \varepsilon.$$

This guarantee that, the function is continuous as assumed in (i). We now show the assumption (ii) hold

$$\begin{aligned}|h_n(\psi, y_k(\psi))| &= \left| \frac{1}{n^4} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6} \right| \\ &\leq \frac{1}{n^4} + \sum_{k=n}^{\infty} \left| \frac{y_k(\psi)\cos\psi}{100n^2(k+1)^6} \right| \\ &\leq \frac{1}{n^4} + \sum_{k=n}^{\infty} \frac{1}{100n^2(k+1)^6} |y_k(\psi)| \\ &:= u_n(\psi) + v_n(\psi)\,|y_k(\psi)|\,.\end{aligned}$$

The function $u_n(\psi)$ is continuous on $\mathbb{R}$ with $n \in \mathbb{N}$ and $\sum_{n\geq 1} u_n(\psi)$ converges uniformly to $\frac{\pi^4}{90}$. More also, the sequence $v_n(\psi)$ is equibonded on $\mathbb{R}$. Thus the assumption (ii) is fulfilled.
The roots of homogeneous equations which correspond to (3.1) are $\xi_1 = 2, \xi_2 = 1, \xi_3 = -0.1$. This demonstrates that the Green's function associated with (3.1) is a form of $G_1(\psi, \varsigma)$ and $f_2 = 1.7295 \times 1^8 > g_2 = 1.6955 \times 10^8$. Applying the formula (2.3) ,we have $0 < \mathscr{A}_3 = 0.0206 \leq G_1(\psi, \varsigma) \leq \mathscr{B}_3 = 1.8387$. Thus, the condition in $\mathscr{C}_2$ is satisfied. The value $\omega V \mathscr{B}_3 \approx 0.1164 < 1$, for $\omega = 2\pi$. This indicates that the infinite system (3.1) has atleast one 2π-periodic solution $y(\psi) = (y_n(\psi)) \in n(\phi)$ as all criteria of Theorem 2.1 are met.

3.2. Example 2

We now provide a further illustrative example to further elucidate our conclusion for the case 3.
Consider the infinite system of differential equation of third order below:

$$y_n'''(\psi) - 3y_n''(\psi) - y_n'(\psi) - 1 = \frac{\cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}. \tag{3.2}$$

Consider:

$$h_n(\psi, y_k(\psi)) = \frac{\cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}.$$

We observe that, the function $h_n(\psi, y_k(\psi))$ is continuous at every points on $\mathbb{R}$ and is 2π - periodic for $n \in \mathbb{N}$.
The system (3.2) is a particular case of the considered system (1.1). Moreover, $h_n(\psi, y_k(\psi)) \in n(\phi)$ whenever $y(\psi) = y_n(\psi) \in n(\phi)$ as we have

$$\begin{aligned}\sum_{n=1}^{\infty} |h_n(\psi, y_k(\psi))| &= \sum_{n=1}^{\infty} \left| \frac{cos(\psi)}{n^2} + \sum_{k=n}^{\infty} \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2} \right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \left| \frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2} \right| \\ &\leq \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{1}{512(1+n^2)(k+1)^2} |y_k(\psi)| \\ &\leq \frac{\pi^2}{6} + \frac{1}{512}\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{1}{(1+n^2)(k+1)^2} |y_k(\psi)| \\ &\leq \frac{\pi^2}{6} + \frac{1}{512} \times \frac{\pi^2}{6} \|y_k(\psi)\|_{n(\phi)} < \infty.\end{aligned}$$

Now let us prove that the assumption (i) is satisfied. Consider any ε ¿ 0 and $y(\psi) = y_n(\psi), z(\psi) = z_n(\psi) \in n(\phi)$ such that,

$$\|y(\psi) - z(\psi)\|_{n(\phi)} < \delta(\varepsilon) := \frac{3072\varepsilon}{\pi^2}.$$

We have that

$$|h(\psi,y(\psi))-h(\psi,z(\psi))| = \sum_{k=n}^{\infty}\left|\frac{(y_k(\psi)-z_k(\psi))\sin\psi}{512(1+n^2)(k+1)^2}\right|$$

$$\leq \sum_{k=n}^{\infty}\frac{|y_k(\psi)-z_k(\psi)|}{512(1+n^2)(k+1)^2}$$

$$\leq \frac{\pi^2}{6}\frac{1}{512}\|y(\psi)-z(\psi)\|_{n(\phi)}$$

$$\leq \frac{\pi^2}{3072}\delta < \varepsilon,$$

which ensures the desired continuity as assumed in (i). We now show the assumption (ii) hold

$$|h_n(\psi,y_k(\psi))| = \left|\frac{\cos(\psi)}{n^2}+\sum_{k=n}^{\infty}\frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right|$$

$$\leq \frac{1}{n^2}+\sum_{k=n}^{\infty}\left|\frac{y_k(\psi)\sin\psi}{512(1+n^2)(k+1)^2}\right|$$

$$\leq \frac{1}{n^2}+\frac{1}{512}\sum_{k=n}^{\infty}\frac{1}{(1+n^2)(k+1)^2}|y_k(\psi)|$$

$$:= u_n(\psi)+v_n(\psi)|y_k(\psi)|.$$

The function $u_n(\psi)$ is continuous on $\mathbb{R}$ with $n\in\mathbb{N}$ and $\sum_{n\geq 1}u_n(\psi)$ converges uniformly to $\frac{\pi^2}{6}$. Furthermore, the sequence $v_n(\psi)$ is equibonded on $\mathbb{R}$. Thus the assumption (ii) is satisfied.
Using the notations from the preceding section, we can observe that the roots of the related homogeneous equation of (3.2) are $\xi_1=\xi_2=\xi_3=1$. Using the concept of $\mathscr{C}_8$ and the aforementioned roots, we find, $\mathscr{A}_7=-19.8873\leq G_3(\psi,\varsigma)\leq\mathscr{B}_7=-6.9354\times 10^{-5}<0$, for $\omega=2\pi$ and $\omega V|\mathscr{A}_7|\approx 0.40145<1$.
All the hypothesis of Theorem 2.3 are satisfied, because for $n\in\mathbb{N}$, the function $h_n(\psi)$ is 2π- periodic with regard to first coordinate. The infinite system (3.2) therefore has a 2π-periodic, $y(\psi)=(y_n(\psi))\in n(\phi)$.

4. Conclusion

In our work, we have presented the conditions for existence of ω-periodic solution to an infinite system of third order differential equations in a sequence space $n(\phi)$ are given. Our conclusion was supported by the Meir-Keeler condensing operator and the notion of measures of non-compactness. To help illustrate the outcome, we have also included examples. More investigations is still needed to determine the required conditions for the existence of solutions to an infinite system of similar type in different Banach spaces.
For some related future work, we suggest that such type of differential equations of order higher than three can be studied in different sequence spaces, like c_0, c, ℓ_p, $m^{\beta}(\phi)$, $m^{\beta}(\phi,p)$, etc..

Article information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: All authors contributed equally to the writing of this paper.

Conflict of Interest Disclosure: No potential conflict of interest was declared by authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

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