Mathematical Sciences and Applications E-Notes
https://doi.org/10.36753/mathenot.1592896
13 (2) 72-83 (2025) - Research Article
ISSN: 2147-6268

Hankel Determinants of Logarithmic Coefficients for the Class of Bounded Turning Functions Associated with Lune Domain

Bilal Şeker*, Bilal Çekiç, Sevtap Sümer and Onur Akçiçek

Abstract
In this paper, we first obtained some initial logarithmic coefficient bounds on a subclass of bounded turning functions $\mathcal{R}_{\mathcal{L}}$ related to a lune domain. For functions belonging to this class, we determined the sharp bounds for the second Hankel determinant of logarithmic coefficients $H_{2,1}(F_f/2)$ of bounded turning functions related to a lune domain. Finally, we calculated the bounds of third Hankel determinant of logarithmic coefficients $H_{3,1}(F_f/2)$ of bounded turning functions associated with a lune domain.

Keywords: *Bounded turning functions, Hankel determinant, Logarithmic coefficients*

AMS Subject Classification (2020): *30C45; 30C50*

1. Introduction

The study of analytic and univalent functions in the unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ remains a foundational and dynamic area within geometric function theory. Denoted typically by $\mathcal{A}$, the class of analytic functions in $\mathbb{U}$ can be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

where the normalization conditions $f(0) = f'(0) - 1 = 0$ are satisfied. A function $f \in \mathcal{A}$ that is also injective in $\mathbb{U}$ belongs to the distinguished class $\mathcal{S}$ of univalent functions. The exploration of univalent functions plays a central role in complex analysis due to their rich structural properties and applications in diverse mathematical and physical theories.

Received : *29-11-2024,* ***Accepted :*** *04-05-2025,* ***Available online :*** *19-05-2025*

*(**Cite as** "B.Şeker, B.Çekiç, S. Sümer, O. Akçiçek, Hankel Determinants of Logarithmic Coefficients for the Class of Bounded Turning Functions Associated with Lune Domain, Math. Sci. Appl. E-Notes, 13(2) (2025), 72-83")*

A fundamental concept in this domain is the notion of subordination: a function $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ on $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$. Subordination has been instrumental in defining various significant subclasses of $\mathcal{S}$, such as starlike, convex, and bounded turning functions.

Of particular interest is the Carathéodory class $\mathcal{P}$, consisting of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad (z \in \mathbb{U}) \tag{1.2}$$

which satisfy $Rep(z) > 0$ in $\mathbb{U}$ and $p(0) = 1$. Functions in $\mathcal{P}$ often serve as comparison functions in subordination relations.

In their seminal work, Ma and Minda [1] introduced generalized subclasses of $\mathcal{S}$ through the use of subordination with a function φ mapping $\mathbb{U}$ onto domains symmetric about the real axis and starlike with respect to $\varphi(0) = 1$. These generalized classes include:

- The Ma-Minda starlike functions $\mathcal{S}^*(\varphi)$,
- The Ma-Minda convex functions $\mathcal{C}(\varphi)$, and
- The Ma-Minda bounded turning functions $\mathcal{R}(\varphi)$,

with respective subordination conditions involving $\frac{zf'(z)}{f(z)}$, $\frac{(zf'(z))'}{f'(z)}$, and $f'(z)$.

Recently, many authors [2, 3] have described the Ma-Minda type functions or subordinations with different regions such as cardioid, lune and nephroid [4–6].

The Logarithmic coefficients, introduced through the expansion

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k z^k, \qquad (z \in \mathbb{U}) \tag{1.3}$$

have attracted considerable attention for their role in characterizing the geometric properties of univalent functions. γ_k are called logarithmic coefficients [7]. Although for the Koebe function $k(z) = z(1-z)^{-2}$ the coefficients satisfy $\gamma_k = \frac{1}{k}$, it is known that for general $f \in \mathcal{S}$, sharp bounds for γ_k are not completely determined beyond γ_1, γ_2 and γ_3. Over the past few years, numerous researchers [8–10] have sought to establish upper bounds for the logarithmic coefficients associated with select subclasses within the class of univalent functions.

The Hankel determinant of $f \in \mathcal{A}$ the function for $q, n \in \mathbb{N}$, denoted by $H_{q,n}(f)$, is defined as follows:

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}$$

have emerged as a significant object of study due to their connections to the coefficient problem and geometric function properties. In particular, the second and third-order Hankel determinants, $H_{2,1}(f)$ and $H_{3,1}(f)$, are closely related to coefficient functionals that measure the deviation from extremal functions.

The Hankel determinant

$$H_{2,1}(f) = a_3 - a_2^2$$

is the recognized as the Fekete-Szegö functional [11]. The second Hankel determinant $H_{2,2}(f)$ is represented by

$$H_{2,2}(f) = a_2 a_4 - a_3^2.$$

Determining the upper bound of $|H_{q,n}(f)|$ for various subclasses of $\mathcal{A}$ is a fascinating and well-studied problem in the field of Geometric Function Theory. Several authors have successfully derived sharp upper bounds for $|H_{2,2}(f)|$ and $|H_{3,1}(f)|$ within specific subclasses of analytic functions, as the referenced in [12–23].

The logarithmic coefficients of analytic univalent functions are crucial for understanding the behaviour of a function in the boundary region where it is defined. The logarithmic coefficients of analytic univalent functions are employed in the investigation of a range of properties, including the sharp bounds of the determinants of Hankel matrices, growth estimates for the moduli of these functions and their derivatives.

Several authors have contributed to this growing field. Duren and Leung[7] initially investigated logarithmic coefficients and their impact on univalent function theory. Subsequent studies, such as those by Girela[24], Obradović [25] , and Ponnusamy [26], refined the understanding of logarithmic coefficients and established bounds in various contexts. More recent work by Kowalczyk and Lecko [27, 28] has expanded the theory to include subclasses associated with special geometric domains, such as the cardioid and nephroid, thereby providing broader applicability and sharper estimates for Hankel determinants. The question of computing sharp bounds for strongly starlike and strongly convex functions was addressed by Eker et al.[29]. Additionally, upper bounds for the second Hankel determinant of logarithmic coefficients for various subclasses of $\mathcal{S}$ were obtained by et al. [30], Eker et al. [31], Shi et al.[32] and Mandal et al. [6].

For a function $f \in \mathcal{S}$, as defined in equation (1.1), differentiating equation (1.3) allows the logarithmic coefficients to be determined.

$$\gamma_1 = \frac{1}{2}a_2, \tag{1.4}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{1.5}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{1.6}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \tag{1.7}$$

$$\gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right). \tag{1.8}$$

In light of the ideas presented above, we propose the study of the Hankel determinant, whose entries are logarithmic coefficients of f, namely

$$H_{q,n}(f) := \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix}.$$

From qth-order Hankel determinant $H_{q,n}(F_f/2)$ whose entries are the logarithmic coefficients of f, one can easily deduce that

$$H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2, \tag{1.9}$$
$$H_{2,2}(F_f/2) = \gamma_2\gamma_4 - \gamma_3^2,$$

and

$$H_{3,1}(F_f/2) = \gamma_3\left(\gamma_2\gamma_4 - \gamma_3^2\right) - \gamma_4\left(\gamma_1\gamma_4 - \gamma_2\gamma_3\right) + \gamma_5\left(\gamma_1\gamma_3 - \gamma_2^2\right). \tag{1.10}$$

Morever, if $f \in \mathcal{S}$, then for $f_\theta \in \mathcal{S}$, $\theta \in \mathbb{R}$, defined as

$$f_\theta(z) := e^{-i\theta}f(e^{i\theta}z) \qquad (z \in \mathbb{U}),$$

we find that

$$H_{2,1}\left(F_{f_\theta}/2\right) = e^{4i\theta}H_{2,1}\left(F_f/2\right)$$

and

$$H_{2,2}\left(F_{f_\theta}/2\right) = e^{6i\theta}H_{2,2}\left(F_f/2\right).$$

Given the importance of bounded turning functions, a particular subclass $\mathcal{R}_\mathcal{L}$ has been defined, wherein functions satisfy

$$\mathcal{R}_\mathcal{L} := \left\{f \in \mathcal{A} : f'(z) \prec z + \sqrt{1+z^2}, \quad z \in \mathbb{U}\right\}, \tag{1.11}$$

where branch of the square root is selected such that $\wp(0) = 1$. Geometrically, $\mathcal{R}_\mathcal{L}$ functions are associated with a domain bounded by a lune shape, and they naturally generalize the notion of bounded turning to a setting influenced by complex geometric regions.

The motivation of this paper is twofold. First, we aim to establish new sharp bounds for the second Hankel determinant $|H_{2,1}(F_f/2)|$ of logarithmic coefficients for functions belonging to $\mathcal{R}_{\mathcal{L}}$. Second, we endeavor to determine precise estimates for the third Hankel determinant $|H_{3,1}(F_f/2)|$ within the same class. Such results contribute both to the ongoing exploration of Hankel determinants and to the broader understanding of logarithmic coefficients in function theory.

Moreover, we build upon existing lemmas related to the structure of $\mathcal{P}$, utilize inequalities concerning the coefficients c_k , and employ techniques from complex analysis to develop our main theorems. By enriching the analytical framework and refining the known bounds, this study adds a novel perspective to the intricate relationship between geometric function theory and coefficient-based functionals. For our consideration we need the next lemmas.

For our consideration we need the next lemmas.

Lemma 1.1. *[33] If $p \in \mathcal{P}$ is of the form (1.2) with $c_1 \geq 0$, then*

$$\begin{aligned} c_1 &= 2d_1, \\ c_2 &= 2d_1^2 + 2\left(1-d_1^2\right)d_2, \\ c_3 &= 2d_1^3 + 4\left(1-d_1^2\right)d_1d_2 - 2\left(1-d_1^2\right)d_1d_2^2 + 2\left(1-d_1^2\right)\left(1-|d_2|^2\right)d_3 \end{aligned} \tag{1.12}$$

for some $d_1 \in [0,1]$ and $d_2, d_3 \in \overline{\mathbb{U}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

For $d_1 \in \mathbb{U}$ and $d_2 \in \partial\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ there is a unique function $p \in \mathcal{P}$ with c_1 and c_2 as in (1.12), namely

$$p(z) = \frac{1 + \left(\overline{d_1}d_2 + d_1\right)z + d_2z^2}{1 + \left(\overline{d_1}d_2 - d_1\right)z + d_2z^2}, \qquad (z \in \mathbb{U}).$$

Lemma 1.2. *If $p \in \mathcal{P}$ is of the form (1.2) then the following inequalities hold*

$$|c_n| \leq 2 \quad \textit{for} \quad n \geq 1, \tag{1.13}$$

$$|c_{n+k} - \mu c_n c_k| < 2 \quad \textit{for} \quad 0 \leq \mu \leq 1, \tag{1.14}$$

$$|c_m c_n - c_k c_l| \leq 4 \quad \textit{for} \quad m+n = k+l, \tag{1.15}$$

$$|c_{n+2k} - \mu c_n c_k^2| \leq 2(1+2\mu) \quad \textit{for} \quad \mu \in \mathbb{R}, \tag{1.16}$$

and for complex number λ, we have

$$|c_2 - \lambda c_1^2| \leq 2\max(1, |\lambda - 1|). \tag{1.17}$$

For the inequalities in (1.13), (1.14), (1.15) and (1.16), we refer to [34]. Also, see [35] for the inequality (1.17).

Lemma 1.3. *[36] Let $p \in \mathcal{P}$ and has the form (1.2), then*

$$|Kc_1^3 - Lc_1c_2 + Mc_3| \leq 2|K| + 2|L-2K| + 2|K-L+M|.$$

Lemma 1.4. *[37] Given real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{\left|A + Bz + Cz^2\right| + 1 - |z|^2 : z \in \overline{\mathbb{U}}\right\}.$$

I. *If $AC \geq 0$, then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. *If $AC < 0$, then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \dfrac{B^2}{4(1-|C|)}, & -4AC(\frac{1}{C^2} - 1) \leq B^2 \wedge |B| < 2(1-|C|), \\ 1 + |A| + \dfrac{B^2}{4(1+|C|)}, & B^2 < \min\{4(1+|C|)^2, -4AC(\frac{1}{C^2} - 1)\}, \\ R(A,B,C), & \text{otherwise}, \end{cases}$$

where

$$R(A,B,C) := \begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|) \leq |AB|, \\ -|A|+|B|+|C|, & |AB| \leq |C|(|B|-4|A|), \\ (|A|+|C|)\sqrt{1-\dfrac{B^2}{4AC}}, & \text{otherwise.} \end{cases}$$

The objective of this paper to provide the sharp bounds for $|H_{2,1}(F_f/2)|$ for bounded turning functions in the open unit disk. In addition, we intended to find the sharp bound of $|H_{3,1}(F_f/2)|$ for the class bounded turning functions.

The present work seeks to address a gap in the literature concerning bounded turning functions associated with non-circular symmetric domains, particularly the lune domain. The lune, characterized by its asymmetry and curvature variation, offers a richer geometric environment than simple circular or cardioidal regions, necessitating a more nuanced analysis of subordination and coefficient behavior. The determination of sharp bounds for second and third-order Hankel determinants within this framework not only enhances theoretical understanding but also supports practical computations in applied sciences. In engineering fields such as signal processing, the stability and distortion behavior of systems modeled by analytic functions can benefit directly from tight coefficient bounds. Similarly, in quantum mechanics and potential theory, conformal mappings associated with bounded domains are critical, and accurate coefficient bounds ensure better physical modeling. Future directions include the study of higher-order determinants, variations under perturbations of the domain, and extension to more complex multi-connected regions, paving the way for deeper theoretical insights and cross-disciplinary applications.

2. Logarithmic coefficients for bounded turning function associated with a lune domain

Theorem 2.1. *If $f \in \mathcal{R}_{\mathcal{L}}$ and it has the form given in (1.1), then*

$$|\gamma_1| \leq \frac{1}{4},$$

$$|\gamma_2| \leq \frac{1}{6},$$

$$|\gamma_3| \leq \frac{1}{8}, \tag{2.1}$$

$$|\gamma_4| \leq \frac{607}{2304}, \tag{2.2}$$

$$|\gamma_5| \leq \frac{1973}{5760}. \tag{2.3}$$

The functions listed below illustrate the sharpness of the aforementioned first three inequalities:

$$f_1(z) = \int_0^z \left(t+\sqrt{1+t^2}\right)dt$$
$$f_2(z) = \int_0^z \left(t^2+\sqrt{1+t^4}\right)dt$$
$$f_3(z) = \int_0^z \left(t^3+\sqrt{1+t^6}\right)dt.$$

Proof. Let $f \in \mathcal{R}_{\mathcal{L}}$ and then, by the definitions of subordination, there exists a Schwarz function $w(z)$ with the properties that

$$w(0)=0 \quad and \quad |w(z)| \leq 1$$

such that

$$f'(z) = w(z) + \sqrt{1+w^2(z)}. \tag{2.4}$$

Define the function

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots.$$

It is clear that $p(z) \in \mathcal{P}$. This implies that

$$\begin{aligned} w(z) =& \frac{p(z)-1}{p(z)+1} = \frac{1+c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots} \\ =& \frac{1}{2}c_1z + \Big(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\Big)z^2 + \Big(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\Big)z^3 + \cdots . \end{aligned}$$

Now, from (2.4), we have

$$f'(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + \cdots \tag{2.5}$$

and

$$w(z)+\sqrt{1+w(z)^2} = 1+\frac{1}{2}c_1z+\frac{1}{2}\Big(c_2-\frac{1}{4}c_1^2\Big)z^2+\frac{1}{2}\Big(c_3-\frac{1}{2}c_1c_2\Big)z^3+\frac{1}{128}\Big(64c_4-16c_2^2+3c_1^4-32c_1c_3\Big)z^4+\cdots . \tag{2.6}$$

Comparing (2.5) and (2.6), we achieve

$$a_2 = \frac{1}{4}c_1, \tag{2.7}$$

$$a_3 = \frac{1}{6}\Big(c_2 - \frac{1}{4}c_1^2\Big),$$

$$a_4 = \frac{1}{8}\Big(c_3 - \frac{1}{2}c_1c_2\Big),$$

$$a_5 = \frac{1}{640}\Big(64c_4 - 16c_2^2 + 3c_1^4 - 32c_1c_3\Big),$$

$$a_6 = \frac{1}{384}\left(32c_5 + 6c_1^3c_2 - c_1^5 - 16c_1c_4 - 16c_2c_3\right). \tag{2.8}$$

Now, from (1.4) to (1.8) and (2.7) to (2.8), we obtain

$$\gamma_1 = \frac{1}{8}c_1, \tag{2.9}$$

$$\gamma_2 = \frac{1}{192}\left(16c_2 - 7c_1^2\right), \tag{2.10}$$

$$\gamma_3 = \frac{1}{384}\left(24c_3 + 3c_1^3 - 20c_1c_2\right), \tag{2.11}$$

$$\gamma_4 = \frac{1}{92160}\left(4608c_4 + 1520c_1^2c_2 - 1792c_2^2 + 11c_1^4 - 3744c_1c_3\right), \tag{2.12}$$

$$\gamma_5 = \frac{1}{92160}\Big(1176c_1^2c_3 - 115c_1^5 - 3072c_1c_4 + 3840c_5 + 140c_1^3c_2 + 1088c_1c_2^2 - 2880c_2c_3\Big). \tag{2.13}$$

Applying (1.13) to (2.9), we get

$$|\gamma_1| \le \frac{1}{4}.$$

From (2.10) and using (1.17), we have

$$|\gamma_2| = \frac{1}{12}\Big|c_2 - \frac{7}{16}c_1^2\Big| \le \frac{1}{6}\max\Big\{1, \Big|\frac{7}{16} - 1\Big|\Big\} = \frac{1}{6}.$$

Applying Lemma 1.3 to the equation (2.11), we get

$$|\gamma_3| \le \frac{1}{8}.$$

From (2.12), it follows that

$$\gamma_4 = \frac{1}{20}\Big(c_4 - \frac{1792}{4608}c_2^2\Big) + \frac{c_1}{92160}\Big(11c_1^3 + 1520c_1c_2 - 3744c_3\Big)$$

By making use of (1.14) and Lemma (1.3), along with the triangle inequality, we get

$$\left|\gamma_4\right| \leq \frac{607}{2304}.$$

If we revise the equation in (2.13), we obtain

$$\gamma_5 = \frac{1}{92160}\Big(1176c_1^2\big(c_3 - \frac{115}{1176}c_1c_1^2\big) + 3840\big(c_5 - \frac{3072}{3840}c_1c_4\big) + c_2\big(140c_1^3 + 1088c_1c_2 - 2880c_3\big)\Big).$$

Using triangle inequality along with (1.13), (1.14), (1.16) and Lemma (1.3), we get

$$\left|\gamma_5\right| \leq \frac{1973}{5760}.$$

Since

$$f_1(z) = \int_0^z \big(t + \sqrt{1+t^2}\big)dt = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots,$$

$$f_2(z) = \int_0^z \big(t^2 + \sqrt{1+t^4}\big)dt = z + \frac{1}{3}z^3 + \cdots$$

and

$$f_3(z) = \int_0^z \big(t^3 + \sqrt{1+t^6}\big)dt = z + \frac{1}{4}z^4 + \cdots,$$

from the equations (1.4) and (1.5) and (1.6) , it is easily obtained that the first three results given in the theorem are sharp. □

3. Second Hankel determinant of logarithmic coefficients for bounded turning with a lune domain

Theorem 3.1. *If $f \in \mathcal{R}_{\mathcal{L}}$, then*

$$|H_{2,1}(F_f/2)| \leq \frac{1}{36}. \tag{3.1}$$

The inequality in (3.1) is sharp.

Proof. Let $f \in \mathcal{R}_{\mathcal{L}}$ be of the form (1.1). Then by (1.11) we have

$$f'(z) = w(z) + \sqrt{1 + w^2(z)}, \qquad (z \in \mathbb{U}). \tag{3.2}$$

for some function $p \in \mathcal{P}$ of the form (1.2). So equating coefficients we obtain

$$\begin{aligned} a_2 &= \frac{1}{4}c_1, \\ a_3 &= \frac{1}{6}\Big(c_2 - \frac{1}{4}c_1^2\Big), \\ a_4 &= \frac{1}{8}\Big(c_3 - \frac{1}{2}c_1c_2\Big). \end{aligned} \tag{3.3}$$

Since the class $\mathcal{R}_{\mathcal{L}}$ and $|H_{2,1}(F_f/2)|$ are rotationally invariant, without loss of generality we may assume that $a_2 \geq 0$, so $c = c_1 \in [0, 2]$ (i.e., in view of (1.12) that $d_1 \in [0, 1]$). By using (1.5)-(1.7) and (1.9) we obtain

$$\begin{aligned} \gamma_1\gamma_3 - \gamma_2^2 &= \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) \\ &= -\frac{1}{2304}\Big[13d_1^4 - 8\left(1-d_1^2\right)d_1^2d_2 + 8\left(8+d_1^2\right)\left(1-d_1^2\right)d_2^2 - 72d_1\left(1-d_1^2\right)\Big(1-|d_2|^2\Big)d_3\Big]. \end{aligned} \tag{3.4}$$

Now, we may have the following cases on d_1.

Case 1. Suppose that $d_1 = 1$. Then by (3.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{13}{2304}.$$

Case 2. Suppose that $d_1 = 0$. Then by (3.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{1}{36}\left|d_2\right|^2 \le \frac{1}{36}.$$

Case 3. Suppose that $d_1 \in (0,1)$. By the fact that $|d_3| \le 1$, applying the triangle inequality to (3.4) we can write

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \frac{1}{32}\left(1-d_1^2\right)d_1\left(\left|A + Bd_2 + Cd_2^2\right| + 1 - \left|d_2\right|^2\right)$$

where

$$A := \frac{13d_1^3}{72\left(1-d_1^2\right)} > 0, \qquad B := -\frac{d_1}{9} < 0 \quad \text{and} \quad C := \frac{8+d_1^2}{9d_1} > 0.$$

Since $AC > 0$, we apply the part I of Lemma 1.3.

We consider the following sub-case. Note that

$$\begin{aligned} |B| - 2\left(1-|C|\right) &= \frac{d_1}{9} - 2\left(1 - \frac{8+d_1^2}{9d_1}\right) \\ &= \frac{3d_1^2 - 18d_1 + 16}{9d_1} \\ &\ge \frac{2\left(d_1-8\right)\left(d_1-1\right)}{9d_1} > 0. \end{aligned}$$

Applying Lemma 1.3, we obtain

$$\begin{aligned} \left|\gamma_1\gamma_3 - \gamma_2^2\right| &\le \frac{1}{32}(1-d_1^2)d_1\left(|A| + |B| + |C|\right) \\ &= \frac{1}{32}(1-d_1^2)d_1\left(\frac{13d_1^3}{72(1-d_1^2)} + \frac{2d_1}{9} + \frac{d_1^2+8}{9d_1}\right) \\ &= \frac{1}{2304}\left(64 - 48d_1^2 - 3d_1^4\right) \\ &\le \frac{1}{2304}64 = \frac{1}{36}. \end{aligned}$$

Summarizing parts from Case 1-3, it follows that the inequality (3.1) is true.

To show the sharpness, consider the function as follows

$$p\left(z\right) := \frac{1+z^2}{1-z^2}.$$

It is obvious that the function p is in $\mathcal{R}_{\mathcal{L}}$ with $c_1 = c_3 = 0$ and $c_2 = 2$. The corresponding function $f \in \mathcal{R}_{\mathcal{L}}$ is described by (3.2). Hence by (3.3) it follows that $a_2 = a_4 = 0$ and $a_3 = \frac{1}{3}$. From (3.4) we obtain

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| = \frac{1}{36}.$$

This completes the proof. □

4. Third Hankel determinant of logarithmic coefficients for bounded turning with a lune domain

Theorem 4.1. *If $f \in \mathcal{R}_{\mathcal{L}}$, then*

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \le \frac{4247}{69120}. \tag{4.1}$$

Proof. From (2.10), (2.11) and (2.12)

$$\gamma_2\gamma_4-\gamma_3^2=\frac{1}{17694720}\Big(-3936c_1^4c_2+1157c_1^6+28672c_2^3+11136c_1^2c_2^2 73728c_2c_4+32256c_1^2c_4-8928c_1^3c_3-55296c_1c_2c_3+69120c_3^2\Big)$$

Rearranging the above and applying triangle inequality, we get

$$\begin{aligned}\left|\gamma_2\gamma_4-\gamma_3^2\right|\leq\frac{1}{17694720}\Big(3936|c_1|^4\Big|c_2-\frac{1157}{3936}c_1^2\Big|+28672|c_2|^2\Big|c_2+\frac{11136}{28672}c_1^2\Big|\\+73728|c_4|\Big|c_2-\frac{32256}{73728}c_1^2\Big|+|c_3|\Big|-8928c_1^3-55296c_1c_2+69120c_3\Big|\Big)\end{aligned}$$

Using (1.13), (1.17) and Lemma 1.3, we get the required result.

□

Theorem 4.2. *If $f\in\mathcal{R}_{\mathcal{L}}$, then*

$$\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|<\frac{103}{1920}. \tag{4.2}$$

Proof. From (2.10), (2.11) and (2.12)

$$\gamma_1\gamma_4-\gamma_2\gamma_3=\frac{1}{737280}\Big(-360c_1^3c_2+221c_1^5+4608c_1c_4-2064c_1^2c_3-3840c_2c_3+1408c_1c_2^2\Big)$$

Rearranging the above and applying triangle inequality, we get

$$\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|\leq\frac{1}{737280}\Big(360|c_1|^3\Big|c_2-\frac{221}{360}c_1^2\Big|+4608|c_1|\Big|c_4-\frac{2064}{4608}c_1c_3\Big|+3840|c_2|\Big|c_3-\frac{1408}{3840}c_1c_2\Big|\Big)$$

Using (1.13), (1.14) (1.17) and Lemma 1.3, we get the required result.

□

Theorem 4.3. *If $f\in\mathcal{R}_{\mathcal{L}}$, then*

$$\left|H_{3,1}(f)\right|\leq\frac{415763}{13271040}.$$

Proof. Since (1.10), it follows that

$$\left|H_{3,1}(f)\right|\leq|\gamma_3|\left|\gamma_2\gamma_4-\gamma_3^2\right|+|\gamma_4|\left|\gamma_1\gamma_4-\gamma_2\gamma_3\right|+|\gamma_5|\left|\gamma_1\gamma_3-\gamma_2^2\right|.$$

From the values of (2.1-2.3), (3.1), (4.1) and (4.2), we achieve the required result.

□

5. Conclusion

In this study, we have examined the Hankel determinants of logarithmic coefficients for a specific subclass of bounded turning functions associated with the lune domain. By deriving sharp upper bounds for $|H_{2,1}(F_f/2)|$ and $|H_{3,1}(F_f/2)|$, we contribute meaningful advancements to the field of geometric function theory. Our analysis extends the methodologies established by previous scholars and offers novel insights into the behavior of logarithmic coefficients within this structured context.

The results presented reinforce the intricate relationship between geometric properties of univalent functions and their associated coefficient functionals. Furthermore, the techniques and lemmas utilized here pave the way for broader applications, suggesting that similar methods could be adapted for more complex domains or higher-order Hankel determinants.

Future research directions may include extending these results to subclasses defined by different geometric constraints, such as nephroid or symmetric cardioid domains, and exploring applications of these bounds in related fields such as signal processing or complex dynamical systems. The interplay between geometry, coefficient behavior, and function theory remains a fertile ground for ongoing and impactful mathematical discovery.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's Contributions: Conceptualization, B.Ş, B.Ç., S.S. and O.A.; methodology, B.Ş, B.Ç., S.S. and O.A.; writing—original draft preparation, B.Ş, B.Ç., S.S. and O.A.; investigation, B.Ş, B.Ç., S.S. and O.A. All authors have read and agreed to the published version of the manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the authors.

Plagiarism Statement: This article was scanned by the plagiarism program.

References

[1] Ma, W. C., Minda, D.: *A unified treatment of some special classes of univalent functions.* In: Proceedings of the Proceedings of the Conference on Complex Analysis. Tianjin, China, 157-169 (19–23 June 1992).

[2] Sharma, P., Raina, R. K., Sokół, J.: *Certain Ma–Minda type classes of analytic functions associated with the crescent-shaped region.* Analysis and Mathematical Physics. **9**, 1–17 (2019).

[3] Wani, L. A., Swaminathan, A.: *Starlike and convex functions associated with a Nephroid domain.* Bulletin of the Malaysian Mathematical Sciences Society. **44**, 79–104 (2021).

[4] Kumar, S. S., Kamaljeet, G.: *A cardioid domain and starlike functions.* Analysis and Mathematical Physics. **11**, 54 (2021).

[5] Malik, S. N, Raza, M., Xin, Q., Sokol, J., Manzoor, R., Zainab, S. : *On convex functions associated with symmetric cardioid domain.* Symmetry. **13**(12), 2321 (2021).

[6] Mandal S, Ahamed MB: Second Hankel determinant of logarithmic coefficients for starlike and convex functions associated with lune. arXiv preprint, arXiv:2307.02741.(2023).

[7] Duren, P. L., Leung, Y. J.: *Logarithmic coefficients of univalent functions.* Journal d'Analyse Mathématique. **36**(1), 36-43(1979).

[8] Cho, N. E., Kowalczyk, B., Kwon, O.S., Lecko, A., Sim, Y.J.: *On the third logarithmic coefficient in some subclasses of close-to-convex functions.* Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. **114**(2), 1-14 (2020).

[9] Thomas, D. K.: *On logarithmic coefficients of close to convex functions.* Proceedings of the American Mathematical Society. **144**, 1681-1687 (2016).

[10] Zaprawa, P.: *Initial logarithmic coefficients for functions starlike with respect to symmetric points.* Boletin De La Sociedad Matematica Mexicana. **27**(3), 1-13 (2021).

[11] Fekete, M., Szegö, G.: *Eine Bemerkung Über ungerade schlichte Funktionen.* Journal of the London Mathematical Society. **8**, 85–89 (1933).

[12] Cho, N. E., Kowalczyk, B., Kwon, O. S., Lecko, A., Sim, Y. J.: *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha.* Journal of Mathematical Inequalities. **11**(2), 429-439 (2017).

[13] Deniz, E., Özkan, Y., Kazımoglu, S.: *Logarithmic coefficients for starlike functions associated with generalized telephone numbers.* Filomat. **38**(20), 7041-7050 (2024).

[14] Deniz, E., Budak, L.: *Second Hankel determinant for certain analytic functions satisfying subordinate condition.* Mathematica Slovaca. **68**(2), 463-471(2018).

[15] Janteng, A., Halim, S. A., Darus, M.: *Hankel determinant for starlike and convex functions.* International Journal of Mathematical Analysis. **1**(1), 619-625 (2007).

[16] Kazımoglu, S., Deniz, E., Srivastava, H. M.: *Sharp coefficients bounds for starlike functions associated with Gregory coefficients.* Complex Analysis and Operator Theory. **18**(1), 6 (2024).

[17] Kowalczyk, B., Lecko, A., Sim, Y. J.: *The sharp bound for the Hankel determinant of the third kind for convex functions.* Bulletin of the Australian Mathematical Society. **97**, 435-445 (2018).

[18] Krishna, D. V., Ramreddy, T.: *Hankel determinant for starlike and convex functions of order alpha.* Tbilisi Mathematical Journal. **5**(1), 65-76 (2012).

[19] Lee, S. K., Ravichandran, V., Supramaniam, S.: *Bounds for the second Hankel determinant of certain univalent functions.* Journal of Inequalities and Applications. **281**, (2013).

[20] Shi, L., Srivastava, H. M., Arif, M., Hussain, S., Khan, H.: *An investigation of the third Hankel determinant problem for certain subfamilies of univalent functions involving the exponential function.* Symmetry. **11**, 1-14 (2019).

[21] Sim, Y.J., Lecko, A., Thomas, D. K.: *The second Hankel determinant for strongly convex and Ozaki close-to-convex functions.* Annals of Mathematics. **200**, 2515-2533 (2021).

[22] Sokol, J., Thomas, D. K.: *The second Hankel determinant for alpha-convex functions.* Lithuanian Mathematical Journal. **58**(2), 212-218 (2018).

[23] Srivastava, H. M., Ahmad, Q. Z., Darus, M., Khan, N., Khan, B., Zaman, N., Shah, H. H.: *Upper bound of the third Hankel determinant for a subclass of close-to-convex functions associated with the lemniscate of Bernoulli.* Mathematics. **7**, 1-10 (2019).

[24] Girela, D.: *Logarithmic coefficients of univalent functions.* Annales Academiæ Scientiarum Fennicæ Mathematica. **25** (2), 337–350 (2000).

[25] Obradović, M., Ponnusamy, S., Wirths, K. J.: *Logarithmic coefficients and a coefficient conjecture for univalent functions.* Monatshefte für Mathematik. **185**, 489-501 (2018).

[26] Ponnusamy, S., Sharma, N. L., Wirths, K. J.: *Logarithmic coefficients problems in families related to starlike and convex functions.* The Journal of the Australian Mathematical Society. 109(2), 230-249, (2020).

[27] Kowalczyk, B., Lecko, A.: *Second Hankel determinant of logarithmic coefficients of convex and starlike functions.* Bulletin of the Australian Mathematical Society. **105**(3), 458-467 (2022).

[28] Kowalczyk, B., Lecko, A.: *Second Hankel determinant of logarithmic coefficients of convex and starlike functions of order alpha.* Bulletin of the Malaysian Mathematical Sciences Society. **45**, 727-740 (2022).

[29] Sümer Eker S., Şeker B., Çekiç B., Acu M.: *Sharp bounds for the second Hankel determinant of logarithmic coefficients for strongly starlike and strongly convex functions.* Axioms, **11**(8), 369 (2022).

[30] Srivastava H. M., Sümer Eker S., Şeker B., Çekiç B.: *Second Hankel determinant of logarithmic coefficients for a subclass of univalent functions.* Miskolc Mathematical Notes. **25**, 479-488 (2024).

[31] Sümer Eker S., Lecko A., Çekiç B., Şeker B.: *The second Hankel determinant of logarithmic coefficients for strongly Ozaki close-to-convex functions.* Bulletin of the Malaysian Mathematical Sciences Society. **46**(6), 183, (2023).

[32] Shi, L., Arif, M., Iqbal, J., Ullah, K., Ghufran, S.M.: *Sharp bounds of Hankel determinant on logarithmic coefficients for functions starlike with exponential function.* Fractal and Fractional. **6** (11), 645 (2022).

[33] Cho, N., Kowalczyk, B., Lecko, A.: *Sharp bounds of some coefficient functionals over the class of functions convex in the direction of the imaginary axis.* Bulletin of the Australian Mathematical Society. **100**(1), 86-96 (2019).

[34] Pommerenke, C.: Univalent Functions. Vanderhoeck, Ruprecht, Gottingen, Germany, (1975).

[35] Keogh, F. R., Merkes, E. P.: *A coefficient inequality for certain classes of analytic functions.* Proceedings of the American Mathematical Society. **20**, 8–12 (1969).

[36] Arif, M., Raza, M., Tang, H., Hussain, S., Khan, H.: *Hankel determinant of order three for familiar subsets of analytic functions related with sine function.* Open Mathematics. **17**, 1615–1630 (2019).

[37] Choi, J.H., Kim, Y.C., Sugawa, T.: *A general approach to the Fekete-Szegö problem.* The Journal of the Mathematical Society of Japan. **59**, 707-727 (2007).

Affiliations

BILAL ŞEKER
ADDRESS: Batman University, Department of Mathematics, Faculty of Art and Science, Batman, Türkiye
E-MAIL: bilal.seker@batman.edu.tr
ORCID ID: 0000-0003-1777-8145

BILAL ÇEKİÇ
ADDRESS: Dicle University, Department of Mathematics, Faculty of Science, Diyarbakır, Türkiye
E-MAIL: bilalcekic@gmail.com
ORCID ID: 0000-0002-6408-0455

SEVTAP SÜMER
ADDRESS: Dicle University, Department of Mathematics, Faculty of Science, Diyarbakır, Türkiye
E-MAIL: sevtaps@dicle.edu.tr
ORCID ID: 0000-0002-2573-0726

ONUR AKÇİÇEK
ADDRESS: Dicle University, Institute of Natural and Applied Science, Diyarbakır, Türkiye
E-MAIL: onurakcc@gmail.com
ORCID ID: 0009-0008-9721-5345