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**Research Article**

# An explicit solution of linear conformable systems with variable coefficients

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## ABSTRACT

This paper is mainly devoted to exact solutions to the initial value problem for linear conformable systems with variable coefficients. The famous method known as the generalized Peano-Baker series, which inholds the conformable integral, is exploited to acquire the state-transition matrix. A representation of an exact solution in a closed interval for linear confromable systems with variable coefficients is determined with the help of this matrix. It is verified by showing that the determined exact solution satisfies the systems step by step. Moreover, another exact solution in the same closed interval is identified thanks to the method of variation of parameters. The existence and uniqueness of the second exact solution to the systems are provided by the Banach contraction mapping principle. This provides that the representations of the two solutions coincide although they are obtained by completely different approaches and they have completely different structures. A couple of examples are presented to exmplify the use of the findings.

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## INTRODUCTION

For many right reasons, such as being a generalization of the traditional derivative and better representing scientific and social problems, the fractional order derivative [1-11], which can be obtained by replacing the integer order with the fractional order, has become a fascinating subject in the theory of functional spaces for a couple of decades. [12-17]. So far, effots have been made to define so many distinct fractional derivatives by many succesful researchers. There is no doubt that the most prevailing employed ones are Riemann-Liouville and Caputo fractional derivatives. They are both introduced by means of fractional integrals. This gives them nonlocal behaviors such as future dependence and historical memory. They satisfy the linearity which is the only feature inherited from the traditional 1st derivative. But including both the above-mentioned ones, the available fractional derivatives in the literatüre have many setbacks. For instance, most of them do not fulfilll that fractional derivatives of one are equal to zero except Caputo-type derivatives. All of them do not satisfy the corresponding product rule, the corresponding quotient rule, the corresponding chain rule, the corresponding Rolle theorem, the corresponding mean value theorem, and generally the

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corresponding semigroup property. To deal with some of these flaws, Khalil et al. [18] in 2014 defined the conformable derivative which is a novel fractional derivative and can be seen as an extension of the classical 1st derivative of a function. Dazhi and Maokang [19] in 2017 managed to describe the physical and geometrical interpretations of the conformable derivatives fort he first time.

Various social and scientific phenomena are formulated by means of linear fractional differential systems with variable coefficients, such as linearized aircraft systems, linearized population growth, linearized diffusion of the batteries, and linearized parameters' distribution in the charge transfer. Although there are lots of papers about linear fractional differential systems with constant coefficients and almost all of their aspects are investigatedi only a few studies are devoted to linear fractional differential systems with variable coefficients and their explicit solutions. To the best of our knowledge, I can find no paper about such systems and their solutions in the conformable sense where are obtained via generalized Peano-Baker series. [20,21]. Finding current in an electrical circut [22], falling objects with air resistance, or determinig the motion of a rising [23] are included in applications of linear conformable systems with variable coefficients given in (1). In order to obtain an exact solution of a system, there are distinct approaches and methods such as the first integral method [24] which is applied succesfully for solving the comformable Wu-Zhang system with the time-fraction, the new extended direct algebraic method [25] which is used to find the new solitons solutions of the complex Ginzburg-Landau equation with Kerr law nonlinearity, the sine-Gordon expansion approach and the generalized Kudryashov approach [26] which are applied to get exact solitary wave solutions to te Boussinesq model. This paper mainly provides an exact solution to the system by applying the generalized Peano-Baker series approach and the variation of constants method seperately.

In light of the above-cited works, the following linear coformable systems with variable coefficients are taken into consideration

$$\begin{cases} \mathbb{D}_0^{\beta}\rho(\varsigma) = A(\varsigma)\rho(\varsigma) + \daleth(\varsigma), & (\varsigma) \in [0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{1}$$

where $\mathbb{D}^{\beta}$ represents the conformable derivative of fractional order $0 < \beta < 1$, $\rho$: $[0, T] \to \mathbb{R}^n$, which is the well-known n-dimensional Euclidean space, is a $\mathbb{R}^n -$ valued function , both the matrix function $A$: $[0, T] \to \mathbb{R}^{n\times n}$ and the function $\daleth$: $[0, T] \to \mathbb{R}^n$ are continuous.

The results in this paper are presented below.

(i) A representation of the exact solution of the problem (1) is given in terms of determining the state-transition (matrix) function obtained from the generalized Peano-Baker series.

(ii) Another representation of the exact solution, which is different from the first one, of the problem (1) is offered based on the variation of constants method.

(iii) The existence uniqueness of the global solution of the nonlinear system (9) into a fixed point problem is transferred, which allows us to use the Banach fixed point to prove our main results.

## PRELIMINARIES

In this section, a couple of necessary paraphernalia to be available in the literatüre are remembered in order to allow a better understanding of the content af the paper.

**Definition 1.** [27] A fractional derivative in the conformable sense of order $0 < \beta < 1$ with a lower bound $\tau$ of a function $\mu$: $[\tau, \infty) \to \mathbb{R}$ is given by

$$\mathbb{D}_\tau^{\beta}\mu(\varsigma) = \lim_{\sigma\to 0}\frac{\mu\left(\varsigma + \sigma(\varsigma-\tau)^{1-\beta}\right) - \mu(\varsigma)}{\sigma}, \quad \varsigma > \tau, 0 < \beta \leq 1.$$

In addition, if $\mu(.)$ is differentiable and $\lim_{\varsigma\to\tau^+}\mathbb{D}_\tau^{\beta}\mu(\varsigma)$ exists, $\lim_{\varsigma\to\tau^+}\mathbb{D}_\tau^{\beta}\mu(\varsigma) = \mathbb{D}_\tau^{\beta}\mu(\tau)$.

**Definition 2.** [27] The conformable integral of fractional order $0 < \beta < 1$ with a lower bound $\tau$ of a function $\mu$: $[\tau, \infty) \to \mathbb{R}$ is given by

$$\mathbb{J}_\tau^{\beta}\mu(\varsigma) = \int_\tau^{\varsigma}\mu(s)(s-\tau)^{\beta-1}\,ds, \quad \varsigma > 0.$$

**Theorem 3.** [27] Let μ be continuous in the domain of $\mathbb{J}_\tau^{\beta}$. Then $\mathbb{D}_\tau^{\beta}\left(\mathbb{J}_\tau^{\beta}\right)(\varsigma) = \mu(\varsigma), \varsigma \geq \tau$.

**Lemma 4.** [28] The conformable derivative of fractional order $0 < \beta < 1$ for a function μ: $[\tau, \infty) \to \mathbb{R}$ exist iff it is differentiable at ς and also $\mathbb{D}_\tau^{\beta}\mu(\varsigma) = (\varsigma-\tau)^{1-\beta}\mu'(\varsigma)$ is satisfied.

**Lemma 5.** [29] The conformable derivative of fractional order $0 < \beta < 1$ of an integral sign is as noted below

$$\mathbb{D}_0^{\beta}\left(\int_{-\tau}^{\alpha(\varsigma)}\rho(\varsigma,s)ds\right) = \left(\int_{-\tau}^{\alpha(\varsigma)}\mathbb{D}_0^{\beta}\rho(\varsigma,s)ds\right) + \rho\left(\varsigma,\alpha(\varsigma)\right)\mathbb{D}_0^{\beta}\alpha(\varsigma),$$

here, ρ is differentiable w.r.t the first component, and also α and α′ are continuous in a finite closed interval.

### Homogeneous Linear Systems with Variable Coefficients

In this section, an explicit solution to the homogeneous version of the linear conformable systems with variable coefficients is investigated

$$\begin{cases} \mathbb{D}_0^{\beta}\rho(\varsigma) = A(\varsigma)\rho(\varsigma), & (\varsigma) \in [0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{2}$$

here, all of the information is introduced in (1).

Now, the state-transition (matrix) function will be offered to costruct the fundamental structure of the explicit solution to the system (2).

**Definition 6.** The state-transition (matrix) function of system (2) is defined as noted below

$$\mathcal{X}(\varsigma,\tau) = \sum_{k=0}^{\infty} \mathfrak{I}_{\tau}^{ko\beta} A(\varsigma),$$

where

$$\mathfrak{I}_{\tau}^{0o\beta} A(\varsigma) = I,$$

and

$$\mathfrak{I}_{\tau}^{(k+1)o\beta} A(\varsigma) = \mathbb{J}_{\tau}^{\beta} \left( A(\varsigma) \mathfrak{I}_{\tau}^{ko\beta} A(\varsigma) \right) \; k = 0,1,2, \dots,$$

here, $I$ and $\Theta$ are the n-by-n unit and zero matrices. The series in Definition 6 can be seen and named as the generalized Peano-Baker series. [20,30].

**Theorem 7.** The state-transition function satisfies the equation (2) with the initial circumstance $\mathcal{X}(0,0) = I$ provided that it is uniformly convergent.

*Proof.* The mathematical meaning of the statement of this theorem is as follows:

$$\begin{cases} \mathbb{D}_0^{\beta} \mathcal{X}(\varsigma,0) = A(\varsigma)\mathcal{X}(\varsigma,0), & (\varsigma) \in [0,T], \\ \mathcal{X}(0,0) = I. \end{cases}$$

Then let's start showing the satisfaction of the first equation

$$\begin{aligned} \mathbb{D}_0^{\beta} \mathcal{X}(\varsigma,0) &= \mathbb{D}_0^{\beta} \sum_{k=0}^{\infty} \mathfrak{I}_0^{ko\beta} A(\varsigma) \\ &= \sum_{k=1}^{\infty} \mathbb{D}_0^{\beta} \mathfrak{I}_0^{ko\beta} A(\varsigma) \\ &= \sum_{k=1}^{\infty} \mathbb{D}_0^{\beta} \mathfrak{I}_0^{\beta} \left( A(\varsigma) \mathfrak{I}_0^{(k-1)o\beta} A(\varsigma) \right). \end{aligned}$$

Theorem 3 gives that

$$\begin{aligned} \mathbb{D}_0^{\beta} \mathcal{X}(\varsigma,0) &= A(\varsigma) \sum_{k=1}^{\infty} \mathfrak{I}_0^{(k-1)o\beta} A(\varsigma) \\ &= A(\varsigma)\mathcal{X}(\varsigma,0). \end{aligned}$$

One can verify the initial circumstance as follows

$$\mathcal{X}(\varsigma,0) = \mathfrak{I}_0^{0o\beta} A(\varsigma)\big|_{\varsigma=0} + \sum_{k=1}^{\infty} \mathfrak{I}_0^{\beta} \left( A(\varsigma) \mathfrak{I}_0^{(k-1)o\beta} A(\varsigma) \right)\big|_{\varsigma=0} = I.$$

**Theorem 8.** $\rho(\varsigma) = \mathcal{X}(\varsigma,0)\,\rho_0$ fulfills the equation (2), which means that it is a solution to the given system, provided that it is uniformly convergent.

*Proof.* In the light of Theorem 7, one can easily get the following equalities

$$\mathbb{D}_0^{\beta} \rho(\varsigma) = \mathbb{D}_0^{\beta} \mathcal{X}(\varsigma,0)\rho_0 = A(\varsigma)\mathcal{X}(\varsigma,0)\rho_0 = A(\varsigma)\rho(\varsigma),$$

and

$$\rho(0) = \mathcal{X}(0,0)\rho_0 = I\rho_0 = \rho_0.$$

**Corollary 9.** When the special case of $A(\varsigma) = A$ (constant) in the system (2) is considered, an explicit solution of the following system

$$\begin{cases} \mathbb{D}_0^{\beta} \rho(\varsigma) = A\rho(\varsigma), & \varsigma \in [0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{3}$$

is given by the equation $\rho(\varsigma) = e^{\frac{\varsigma^{\beta}}{\beta}Y} \rho_0$, which corresponds to that of [5].

**Corollary 10.** For the case of $A(\varsigma) = \varsigma^p$, $\rho \in \mathbb{R}^+$, an eplicit solution of the following system

$$\begin{cases} \mathbb{D}_0^{\beta} \rho(\varsigma) = \varsigma^p \rho(\varsigma), & \varsigma \in [0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{4}$$

is offered by

$$\rho(\varsigma) = e^{\frac{\varsigma^{\beta+p}}{\beta+p}Y} \rho_0.$$

*Proof.* One can begin with the following calculation of the state-transition function as follows:

$$\mathfrak{I}_0^{0o\beta} \varsigma^p = 1$$

$$\mathfrak{I}_0^{1o\beta} \varsigma^p = \mathfrak{I}_0^{\beta} (\varsigma^p \mathfrak{I}_0^{0o\beta} \varsigma^p) = \mathfrak{I}_0^{\beta} \varsigma^p = \frac{\varsigma^{\beta+p}}{\beta+p},$$

$$\mathfrak{I}_0^{2o\beta} \varsigma^p = \mathfrak{I}_0^{\beta} (\varsigma^p \mathfrak{I}_0^{1o\beta} \varsigma^p) = \mathfrak{I}_0^{\beta} \varsigma^p \frac{\varsigma^{\beta+p}}{\beta+p} = \frac{\varsigma^{2(\beta+p)}}{2(\beta+p)^2},$$

$$\mathfrak{I}_0^{3o\beta} \varsigma^p = \mathfrak{I}_0^{\beta} (\varsigma^p \mathfrak{I}_0^{2o\beta} \varsigma^p) = \mathfrak{I}_0^{\beta} \varsigma^p \frac{\varsigma^{2(\beta+p)}}{2!\,(\beta+p)^2} = \frac{\varsigma^{3(\beta+p)}}{3!\,(\beta+p)^3},$$

$$\vdots$$

$$\begin{aligned} \mathfrak{I}_0^{no\beta} \varsigma^p &= \mathfrak{I}_0^{\beta} (\varsigma^p \mathfrak{I}_0^{(n-1)o\beta} \varsigma^p) = \mathfrak{I}_0^{\beta} \varsigma^p \frac{\varsigma^{(n-1)(\beta+p)}}{(n-1)(\beta+p)^{(n-1)!}} \\ &= \frac{\varsigma^{n(\beta+p)}}{n!\,(\beta+p)^n}. \end{aligned}$$

It follows from Definition 6 and the just-above information that one writes the corresponding state-transition function as noted below

$$\mathcal{X}(\varsigma,0) = \sum_{n=0}^{\infty} \mathfrak{I}_0^{no\beta} \varsigma^p = \sum_{n=0}^{\infty} \frac{\varsigma^{n(\beta+p)}}{n!\,(\beta+p)^n} = e^{\frac{\varsigma^{\beta+p}}{\beta+p}}.$$

According to Theorem 8, the solution to the system 4 is given by $\rho(\varsigma) = e^{\frac{\varsigma^{\beta+p}}{\beta+p}} \rho_0$. This concludes the proof.

**Remark 11.** The solution in Corollary 10 reduces to that of [31] for $p = 0$.

## NONHOMOGENEOUS LINEAR SYSTEMS WITH VARIABLE COEFFICIENTS

In this section, an explicit solution to the nonhomogeneous version of the linear conformable systems with variable coefficients is investigated

$$\begin{cases} \mathbb{D}_0^\beta \rho(\varsigma) = A(\varsigma)\rho(\varsigma) + \daleth(\varsigma), & \varsigma \in [0,T], \\ \rho(0) = \rho_0, \end{cases} \tag{5}$$

here, all of the information is introduced in (1).

**Theorem 12.** The solution $\rho(\varsigma)$ of (5) fulfilling zero initial circumstance $\rho(0) = 0$ has the following integral form

$$\rho(\varsigma) = \int_0^\varsigma \mathcal{X}(\varsigma, s)\,\daleth(s) s^{\beta-1} ds.$$

*Proof.* By employing Theorem 3 and Lemma 5, one acquires

$$\begin{aligned} \mathbb{D}_0^\beta \rho(\varsigma) &= \mathbb{D}_0^\beta \int_0^\varsigma \mathcal{X}(\varsigma, s)\,\daleth(s) s^{\beta-1} ds \\ &= \int_0^\varsigma \mathbb{D}_0^\beta \mathcal{X}(\varsigma, s)\,\daleth(s) s^{\beta-1} ds + \daleth(\varsigma) \\ &= \sum_{n=1}^{\infty} \int_0^\varsigma \mathbb{D}_0^\beta \mathbb{J}_s^\beta \left(A(\varsigma)\mathbb{J}_s^{(n-1)o\beta} A(\varsigma)\right) \daleth(s) s^{\beta-1} ds + \daleth(\varsigma) \\ &= A(\varsigma) \int_0^\varsigma \sum_{n=1}^{\infty} \mathfrak{J}_s^{(n-1)o\beta} A(\varsigma) \daleth(s) s^{\beta-1} ds + \daleth(\varsigma) \\ &= A(\varsigma) \int_0^\varsigma \mathcal{X}(\varsigma, s) \daleth(s) s^{\beta-1} ds + \daleth(\varsigma). \end{aligned}$$

**Theorem 13.** The solution $\rho(\varsigma)$ of (5) has the following integral form

$$\rho(\varsigma) = \mathcal{X}(\varsigma, 0)\rho_0 + \int_0^\varsigma \mathcal{X}(\varsigma, s)\daleth(s) s^{\beta-1} ds. \tag{6}$$

*Proof.* The proof is an immediate result of Theorem 8 with Theorems 12.

**Remark 14.** For $\beta = 1$, and $A(\varsigma) = A$, the integral equation in Theorem 13 makes into the following integral equation

$$\rho(\varsigma) = e^{A\varsigma}\rho_0 + \int_0^\varsigma e^{A(\varsigma - s)} \daleth(s) ds$$

which is, as it is well known, the analytical solution to the following first order Cauchy system

$$\begin{cases} \rho(\varsigma) = A\rho(\varsigma) + \daleth(\varsigma), & \varsigma \in [0,T], \\ \rho(0) = \rho_0. \end{cases}$$

**Example 15.** The below nonhomogeneous linear conformable system with variable coefficients is examined

$$\begin{cases} \mathbb{D}_0^\beta \rho(\varsigma) = \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} \rho(\varsigma) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \varsigma \in [0,T], \\ \rho(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{cases} \tag{7}$$

One firstly determines the state-transition matrix of the system (7) step by step by using its definition

$$\mathfrak{J}_0^{0o\beta} A(\varsigma) = I$$

$$\mathfrak{J}_0^{1o\beta} A(\varsigma) = \mathbb{J}_0^\beta \left(A(\varsigma)\mathfrak{J}_0^{0o\beta} A(\varsigma)\right) = \mathbb{J}_0^\beta A(\varsigma) = \mathbb{J}_0^\beta \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 0 \end{pmatrix}$$

$$\mathfrak{J}_0^{2o\beta} A(\varsigma) = \mathbb{J}_0^\beta \left(A(\varsigma)\mathfrak{J}_0^{1o\beta} A(\varsigma)\right) = \mathbb{J}_0^\beta \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 0 \end{pmatrix} = \mathbb{J}_0^\beta \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix}$$

$$\mathfrak{J}_0^{3o\beta} A(\varsigma) = \mathbb{J}_0^\beta \left(A(\varsigma)\mathfrak{J}_0^{2o\beta} A(\varsigma)\right) = \mathbb{J}_0^\beta \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\vdots$$

$$\mathfrak{J}_0^{no\beta} A(\varsigma) = \mathbb{J}_0^\beta \left(A(\varsigma)\mathfrak{J}_0^{(n-1)o\beta} A(\varsigma)\right) = \mathbb{J}_0^\beta \begin{pmatrix} 0 & 0 \\ \varsigma & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, n = 2,3, \ldots.$$

In brief, one has

$$\mathfrak{J}_0^{0o\beta} A(\varsigma) = I, \mathfrak{J}_0^{1o\beta} A(\varsigma) = \begin{pmatrix} 0 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 0 \end{pmatrix}, \mathfrak{J}_0^{no\beta} A(\varsigma) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, n \geq 2.$$

Then the system's state-transition matrix is given by

$$\mathcal{X}(\varsigma, 0) = \sum_{k=0}^{\infty} \mathfrak{J}_0^{ko\beta} A(\varsigma) = \mathfrak{J}_0^{0o\beta} A(\varsigma) + \mathfrak{J}_0^{1o\beta} A(\varsigma) = \begin{pmatrix} 1 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 1 \end{pmatrix}.$$

Then the analytical solution of the system (7) is given by

$$\begin{aligned} \rho(\varsigma) &= \mathcal{X}(\varsigma, 0)\rho_0 + \int_0^\varsigma \mathcal{X}(\varsigma, s)\daleth(s) s^{\beta-1} ds \\ &= \begin{pmatrix} 1 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^\varsigma \begin{pmatrix} 1 & 0 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} s^{\beta-1} ds \\ &= \begin{pmatrix} 1 \\ \dfrac{\varsigma^{\beta+1}}{\beta+1} + \dfrac{\varsigma^\beta}{\beta} \end{pmatrix}. \end{aligned}$$

**Remark 16.** Until here, uniformly convergent infinite series involving nested compositions of fractional integrals in the conformable sense to represent the solutions is used. In a way, this can be seen as a fractional approach and it has a setback such as being uniform convergence.

Now, an explicit solution of the same system (5), which can be also called as a conformable linear differential equation of fractional order $0 < \beta < 1$, will be investigated by variation of constants technique. Firstly, one looks for a solution of its linear case.

$$\begin{aligned} \mathbb{D}_0^\beta \rho(\varsigma) = A(\varsigma)\rho(\varsigma) &\Rightarrow \varsigma^{1-\beta}\rho'(\varsigma) = A(\varsigma)\rho(\varsigma) \\ &\Rightarrow \frac{\rho'(\varsigma)}{\rho(\varsigma)} = A(\varsigma)\varsigma^{\beta-1} \\ &\Rightarrow \ln\left(\frac{\rho(\varsigma)}{\rho(\varsigma)}\right) = \int A(\varsigma)\varsigma^{\beta-1} d\varsigma \\ &\Rightarrow \rho(\varsigma) = C(\varsigma) e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma}. \end{aligned}$$

Let's get into the system (5) with the just-above equation to determine $C(\varsigma)$,

$$\mathbb{D}_0^\beta \left(C(\varsigma) e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma}\right) = A(\varsigma)C(\varsigma) e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma} + \daleth(\varsigma),$$

$$C(\varsigma)\mathbb{D}_0^\beta e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma} + e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma} \mathbb{D}_0^\beta C(\varsigma) = A(\varsigma)C(\varsigma) e^{\int A(\varsigma)\varsigma^{\beta-1} d\varsigma} + \daleth(\varsigma),$$

$$\Rightarrow C(\varsigma) = \int e^{-\int A(\varsigma)\varsigma^{\beta-1} d\varsigma} \daleth(\varsigma)\varsigma^{\beta-1} d\varsigma + K.$$

Then, one of explicit solutions of the system (5) is given by the following continuous integral equation

$$\rho(\varsigma) = e^{\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\left[e^{-\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\daleth(\varsigma)\varsigma^{\beta-1}d\varsigma + K\right], \tag{8}$$

where $K \in \mathbb{R}$ is an integration constant.

**Remark 17.** In one respect, this approach appears one of the advantages of conformable derivatives among other available fractional derivatives.

**Remark 18.** Under the variable transform $\rho(\varsigma) = u(\varsigma)v(\varsigma)$, one can easily get the same integral equation (8).

**Corollary 19.** The following continuous integral equation

$$\rho(\varsigma) = e^{\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\left[\int e^{-\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\daleth(\varsigma)\varsigma^{\beta-1}d\varsigma + K\right],$$

is a global solution of the following nonlinear conformable system with variable coefficients

$$\begin{cases}\mathbb{D}_0^\beta\rho(\varsigma) = A(\varsigma)\rho(\varsigma) + \daleth(\varsigma,\rho(\varsigma)), & \varsigma \in [0,T],\\ \rho(0) = \rho_0. \end{cases} \tag{9}$$

**Theorem 20.** Assume that the nonlinear function is a Lipschitzian with a $L_\daleth > 0$ and $\frac{T^\beta}{\beta}L_\daleth < 1$. Then the system (9) has an unique solution on $[0, T]$ with $T > 0$.

*Proof.* Let $C[0, T]$ be the well-known continuous Banach space endowed with the infinity norm $\|.\|_\infty = \sup_{\varsigma\in[0,T]}\|\rho(\varsigma)\|$ for a norm $\|.\|$ on $\mathbb{R}^n$. Define the operator $\mathcal{H}: C[0, T] \to C[0, T]$ by

$$\mathcal{H}\rho(\varsigma) = e^{\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\left[\int e^{-\int A(\varsigma)\varsigma^{\beta-1}d\varsigma}\daleth(\varsigma,\rho(\varsigma))\varsigma^{\beta-1}d\varsigma + K\right].$$

For $\rho, \nu \in \mathbb{R}^n$, one can get the below inequality

$$\|\mathcal{H}\rho(\varsigma) - \mathcal{H}\nu(\varsigma)\| \le \frac{T^\beta}{\beta}L_\daleth\|\rho - \nu(\varsigma)\|_\infty,$$

which implies that $\mathcal{H}$ is a contraction. Based on the Banach fixed point theorem, $\mathcal{H}$ has an unique fixed point on $[0, T]$. So, the system (9) has an unique solution on $[0, T]$.

**Remark 21.** According to Theorem 20, by the uniqueness of solutions, the closed-form solution in (6) coincides with the explicit continuous solution in (8).

**Example 22.** If the system (4) in Corollary 10 with the initial circumstance is reconsidered, based on the representation of the solution in (8), one can get

$$\rho(\varsigma) = e^{\int \varsigma^p\varsigma^{\beta-1}d\varsigma}K = e^{\frac{\varsigma^{p+\beta}}{p+\beta}}K.$$

The initial circumstance $\rho(0) = \rho_0$ provides $\rho(\varsigma) = e^{\frac{\varsigma^{p+\beta}}{p+\beta}}\rho_0$, which is the same solution in Corollary 10.

**Example 23.** If the system (7) in Example 15 with the initial circumstance is reconsidered, based on the representation of the solution in (8), one can get

$$\begin{aligned}\rho(\varsigma) &= e^{\int_0^\varsigma\begin{pmatrix}0 & 0\\ \varsigma & 0\end{pmatrix}\varsigma^{\beta-1}d\varsigma}\left[\int_0^\varsigma e^{-\int_0^\varsigma\begin{pmatrix}0 & 0\\ \varsigma & 0\end{pmatrix}\varsigma^{\beta-1}d\varsigma}\begin{pmatrix}0\\1\end{pmatrix}\varsigma^{\beta-1}d\varsigma + K\right]\\ &= e^{\begin{pmatrix}0 & 0\\ \frac{\varsigma^{\beta+1}}{\beta+1} & 0\end{pmatrix}}\rho_0 + e^{\begin{pmatrix}0 & 0\\ \frac{\varsigma^{\beta+1}}{\beta+1} & 0\end{pmatrix}}\int_0^\varsigma e^{\begin{pmatrix}0 & 0\\ \frac{\varsigma^{\beta+1}}{\beta+1} & 0\end{pmatrix}}\begin{pmatrix}0\\1\end{pmatrix}\varsigma^{\beta-1}d\varsigma\\ &= \begin{pmatrix}1 & 0\\ \frac{\varsigma^{\beta+1}}{\beta+1} & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}1 & 0\\ \frac{\varsigma^{\beta+1}}{\beta+1} & 1\end{pmatrix}\int_0^\varsigma\begin{pmatrix}1 & 0\\ -\frac{\varsigma^{\beta+1}}{\beta+1} & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\varsigma^{\beta-1}d\varsigma\\ &= \begin{pmatrix}1\\ \frac{\varsigma^{\beta+1}}{\beta+1} + \frac{\varsigma^\beta}{\beta}\end{pmatrix},\end{aligned}$$

which is the same solution in Example 15.

## CONCLUSION

The state-transition matrix is obtained from the generalized Peano-Baker series. An explicit solution to the linear homogeneous and nonhomogeneous conformable systems with variable coefficients is derived based on this state-transition matrix. Another explicit solution of the same system is acquired with the help of the variation of constants method. These obtained solutions are shown to coincide on the closed interval by the fixed point theorem. By solving some examples with two different solutions, the results are verified to match. Again, the same examples are used to illustrate the results.

As a future work, one can discuss different kinds of stabilities and distinct sorts of controllability of linear conformable Dynamics with variable coefficients. As another future work, one can also introduce a conformable Riccati-type differential equation and a conformable Bernoulli-type differential equation annd investigate their solutions.

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## AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

## DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

## CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

## ETHICS

There are no ethical issues with the publication of this manuscript.

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